Functions of multivectors in 3D Euclidean geometric algebra via spectral decomposition (for physicists and engineers)

Miroslav Josipović
Faculty of Veterinary Medicine, Zagreb, Croatia
miroslav.josipovic@gmail.com
June, 2015

Geometric algebra is a powerful mathematical tool for description of physical phenomena. The article [3] gives a thorough analyze of functions of multivectors in *Cl*3relaying on involutions, especially Clifford conjugation and complex structure of algebra. Here is discussedanother elegant way to do that, relaying on complex structure and idempotents of algebra.Implementation of *Cl*3 using ordinary complex algebra is briefly discussed.

Keywords: *function ofmultivector, idempotent,nilpotent, spectral decomposition, unipodal numbers, geometric algebra*

1. Numbers

Geometric algebra is a promising platform for mathematical analysis of physical phenomena. The simplicity and naturalness of the initial assumptions and the possibility of formulation of(all?) main physical theories with the same mathematical language imposes the need for a serious study of this beautiful mathematical structure. Many authors have made significant contributions and there is some surprising conclusions. Important one is certainly the possibility of natural defining Minkowski metrics within Euclidean 3D space without the need for the introduction of negative signature, that is, without the need to define time as the fourth dimension ([1, 6]).

In Euclidean 3D space we defineorthogonal unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ with the property

$$\boldsymbol{e}_i^2 = 1,\ \boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i = 0,$$

so one could recognize the rule for multiplication of Pauli matrices. Each element of algebra (*Cl*3)can be expressed as linear combination of elements of 2^3 – dimensional basis (*Clifford basis*)

$$\{1,\ \boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3,\ \boldsymbol{e}_1\boldsymbol{e}_2,\ \boldsymbol{e}_3\boldsymbol{e}_1,\ \boldsymbol{e}_2\boldsymbol{e}_3,\ \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\},$$

where we have a scalar, three vectors, three bivectors and pseudoscalar. According to the number of unit vectors in the product we are talking about odd or even elements. If we define $j = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$ it is easy to show that pseudoscalar *j*has two interesting properties in *Cl*3:1) $j^2 = -1$, 2) $jX = Xj$, for any element X of algebra, and behaves like an *ordinary imaginary unit*, which enables as tostudya rich complex structure of *Cl*3. This property we have for *n* = 3, 7, … [3].Bivectors can be expressed as product of pseudoscalars and vectors, $j\vec{\boldsymbol{v}}$.

We define a general element of algebra (multivector)

$$M = t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb = z + \boldsymbol{F}, \quad z = t + jb, \quad \boldsymbol{F} = \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}}$$

where z is complex scalar and an element of center of algebra, while $\boldsymbol{F}$, by analogy, is a *complex vector*. *Complex conjugation* is $z^* = z^\dagger = t - jb$, $\boldsymbol{F}^* = \boldsymbol{F}^\dagger = \vec{\boldsymbol{x}} - j\vec{\boldsymbol{n}}$. The complex structure allows different ways of expressing multivectors, one is

$$M = t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb = t + j\vec{\boldsymbol{n}} + j\left(b - j\vec{\boldsymbol{x}}\right),$$

where multivector of the form $a + vj\hat{\boldsymbol{v}}$ belongs to even part of algebra and can be associated with rotations, spinors or quaternions. Also we could treat multivector as ([12]) $M = \alpha_0 + \sum_{i=1}^{3} \alpha_i \boldsymbol{e}_i, \quad \alpha_k \in \mathbb{C}$ and implement it relying on ordinary complex numbers.

Main involutions in Clifford algebra are:

1) grade involution: $\hat{M} = t - \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} - jb$
2) reverse (*adjoint*): $M^\dagger = t + \vec{\boldsymbol{x}} - j\vec{\boldsymbol{n}} - jb = z^* + \boldsymbol{F}^*$
3) Clifford conjugation: $\bar{M} = t - \vec{\boldsymbol{x}} - j\vec{\boldsymbol{n}} + jb = \bar{z} + \bar{\boldsymbol{F}} = z - \boldsymbol{F}$,

where asterisk stands for a complex conjugate. Grade involution is transformation $\hat{\vec{\boldsymbol{x}}} = -\vec{\boldsymbol{x}}$ (*space inversion*), while reverse in Cl_3 is like complex conjugation, $\vec{\boldsymbol{x}}^\dagger = \vec{\boldsymbol{x}}$, $j^\dagger = -j$. Clifford conjugation is combination of two involutions $\bar{M} = \hat{M}^\dagger$, $\bar{\vec{\boldsymbol{x}}} = -\vec{\boldsymbol{x}}$, $\bar{j} = j$. Bivectors given as a wedge product could be expressed as $\vec{\boldsymbol{x}} \wedge \vec{\boldsymbol{y}} = j\vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}}$, where $\vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}}$ is a *cross product*. Application of involutions is easy now.

Defining paravector $p = t + \vec{\boldsymbol{x}}$ we have $p\bar{p} = \left|t + \vec{\boldsymbol{x}}\right|^2 = \left(t + \vec{\boldsymbol{x}}\right)\left(t - \vec{\boldsymbol{x}}\right) = t^2 - x^2$ and we have usual metric of special relativity.

From $M = \hat{M} \Rightarrow M = t + j\vec{\boldsymbol{n}}$, even part of algebra (spinors).

From $M = M^\dagger \Rightarrow M = t + \vec{\boldsymbol{x}}$, paravector; Reverse is anti-automorphism $\left(MM^\dagger\right)^\dagger = MM^\dagger$, so $MM^\dagger$ (square of *multivector magnitude*, [2]) is a paravector.

From $M = \bar{M} \Rightarrow M = t + jb = z$, complex scalar. Clifford conjugation is anti-automorphism, $\overline{M\bar{M}} = M\bar{M}$, so $M\bar{M}$ (square of *multivector amplitude*, [2]) is a complex scalar and there is no other “amplitude” with such a property ([1]).

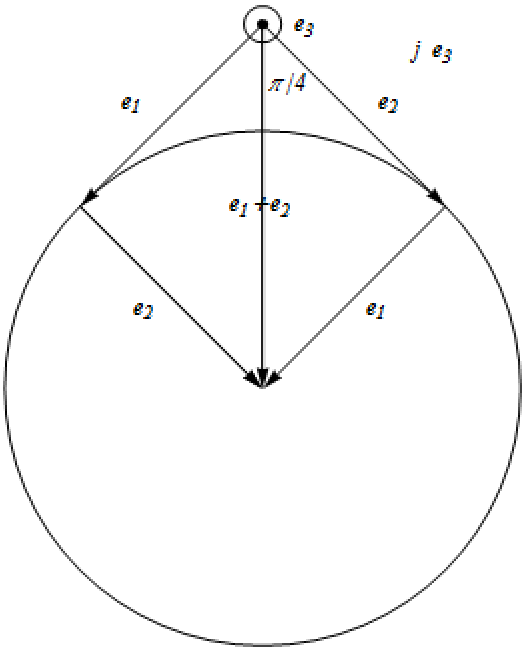
We define a *multivector amplitude* $|M|$ (hereinafter MA)

$$M\bar{M} = |M|^2 = t^2 - x^2 + n^2 - b^2 + 2j\left(tb - \vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{n}}\right), \quad |M| \in \mathbb{C} \quad (3)$$

which one could express as

$$\sqrt{M\bar{M}} = |M| = \sqrt{\left(z + \boldsymbol{F}\right)\left(z - \boldsymbol{F}\right)} = \sqrt{z^2 - \boldsymbol{F}^2}, \quad \boldsymbol{F}^2 = x^2 - n^2 + 2j\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{n}} \in \mathbb{C}.$$

For $\boldsymbol{F} = 0$, or $\boldsymbol{F}^2 = \boldsymbol{N}^2 = 0$ is $|M| = z$ ($\boldsymbol{N}^2 = x^2 - n^2$, $\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{n}} = 0$ is nilpotent in algebra). For $\boldsymbol{F}^2 = c \in \mathbb{R}$ ($\vec{\boldsymbol{x}} \cdot \vec{\boldsymbol{n}} = 0$, *whirl* [1]) here is used designation $\boldsymbol{F}_{(c)}$. From $\boldsymbol{F}^2 \in \mathbb{C}$ we have $\boldsymbol{F}_{(1)} = \boldsymbol{F} / \sqrt{\boldsymbol{F}^2}$, $\boldsymbol{F}_{(1)}^2 = 1$, and $\hat{\boldsymbol{F}} = \boldsymbol{F} / \sqrt{-\boldsymbol{F}^2} = \boldsymbol{F} / |\boldsymbol{F}| = -j\boldsymbol{F}_{(1)}$, $\hat{\boldsymbol{F}}^2 = -1$ (*complex unit vector*). With $\boldsymbol{f} = \boldsymbol{F}_{(1)}$ we also define $u_\pm = \left(1 \pm \boldsymbol{f}\right)/2$, $u_\pm^{\,2} = u_\pm$, $u_+ u_- = 0$, idempotents of algebra ([1, 9]). Every idempotent in Cl_3 can be expressed as $u_\pm = p_\pm + \boldsymbol{N}_\pm, \boldsymbol{N}_\pm^2 = 0$, where $p_\pm$ are simple idempotents, like $p_\pm = (1 \pm \boldsymbol{e}_1)/2$. For example, $u_+ = (1 + \boldsymbol{e}_1 + \boldsymbol{e}_2 + j\boldsymbol{e}_3)/2$, $\boldsymbol{N} = \boldsymbol{e}_2 + j\boldsymbol{e}_3$



2. Implementation

From $M = \alpha_0 + \sum_{i=1}^{3} \alpha_i \boldsymbol{e}_i = \alpha_0 + A, \quad \alpha_k \in \mathbb{C}$, it is easy to implement algebra on computer using ordinary complex numbers only. In [15] are defined products:

$A \circ B = \sum_{i=1}^{3} \alpha_i \beta_i$ (*generalized inner product*),

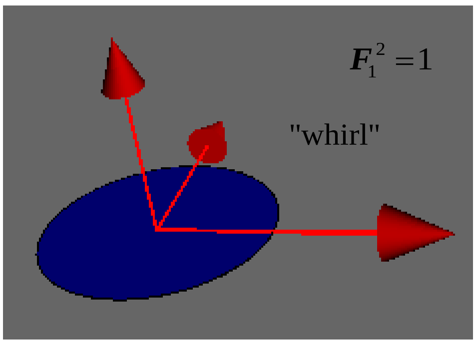
$A \otimes B = \det[\boldsymbol{e}_i,\ \alpha_i,\ \beta_i]$, (*generalized outer product*), and

$AB = A \circ B + A \otimes B$ (*generalized geometric product*).

Now we have $(\alpha_A + A)(\alpha_B + B) = \alpha_A \alpha_B + \alpha_B A + \alpha_A B + AB$. We can find α_i for multivector $M = t + \vec{\boldsymbol{x}} + j\vec{\boldsymbol{n}} + jb$ using linear independency.

.

3. Spectral decomposition



Starting from multivector

$$M = z + \boldsymbol{F} = z + \sqrt{\boldsymbol{F}^2}\,\boldsymbol{f} = x + y\boldsymbol{f}, \quad \boldsymbol{F}^2 \neq 0$$

we see the form of unipodal numbers. Defining $M_\pm = x \pm y$ and recalling easy proofing relation $\boldsymbol{f}u_\pm = \pm u_\pm$ follows

$$Mu_\pm = (x \pm y\boldsymbol{f})u_\pm = (x \pm y)u_\pm = M_\pm u_\pm,$$

so we have projection. Spectral basis $u_\pm$ is very useful because binomial expansion of multivector is very simple

$$M^2 = (M_+u_+ + M_-u_-)^2 = M_+^2 u_+ + M_-^2 u_- \Rightarrow M^n = M_+^n u_+ + M_-^n u_-, \ n \in \mathbb{Z},$$

where $n < 0$ is possible for $|M| \neq 0$ ($M^{-1} = \bar{M}/(M\bar{M})$).

Defining conjugation $(a + b\boldsymbol{f})^- = a - b\boldsymbol{f}$ (obviously the Clifford conjugation) we have

$\left(a+b\boldsymbol{f}\right)^{-}\left(a-b\boldsymbol{f}\right)=a^{2}-b^{2}$, where $\sqrt{a^{2}-b^{2}}=\sqrt{MM^{-}}=\sqrt{M\bar{M}}$ is a *multivector amplitude*. In spectral basis using $u_{\pm}^{-}=u_{\mp}$ we have $MM^{-}=\left(M_{+}u_{+}+M_{-}u_{-}\right)\left(M_{+}u_{-}+M_{-}u_{+}\right)=M_{+}M_{-}$.

Starting from $M=z+\boldsymbol{F}$ and $\rho=\sqrt{x^{2}-y^{2}}$ we have

$$M=z+\sqrt{\boldsymbol{F}^{2}}\,\boldsymbol{f}=x+y\boldsymbol{f}=\rho\left(\frac{x}{\rho}+\frac{y}{\rho}\boldsymbol{f}\right)=\rho\left(\cosh\varphi+\boldsymbol{f}\sinh\varphi\right),$$

obtaining the polar form of multivector. If multivector amplitude is zero we have light-like multivector and there is no a polar form. Now defining $\tanh\varphi=\vartheta$ (“velocity”) we have

$$M=\rho\left(\cosh\varphi+\boldsymbol{f}\sinh\varphi\right)=\rho\gamma\left(1+\vartheta\boldsymbol{f}\right),\quad \gamma^{-1}=\sqrt{1-\vartheta^{2}}$$

and in spectral basis

$$M=\rho\gamma\left(1+\vartheta\boldsymbol{f}\right)=k_{+}u_{+}+k_{-}u_{-}\Rightarrow k_{\pm}=\rho\gamma\left(1\pm\vartheta\right)=\rho K^{\pm1},$$

where $K=\sqrt{\left(1+\vartheta\right)/\left(1-\vartheta\right)}$ is generalized *Bondi factor* ($\varphi=\log K$). Now we have

$$\left(K_{1}u_{+}+K_{1}^{-1}u_{-}\right)\left(K_{1}u_{+}+K_{1}^{-1}u_{-}\right)=K_{1}K_{2}u_{+}+K_{1}^{-1}K_{2}^{-1}u_{-}=Ku_{+}+K^{-1}u_{-}\Rightarrow K=K_{1}K_{2},$$

or

$$\begin{gathered}\gamma_{1}\gamma_{2}\left(1+\vartheta_{1}\boldsymbol{f}\right)\left(1+\vartheta_{2}\boldsymbol{f}\right)=\gamma_{1}\gamma_{2}\left(1+\left(\vartheta_{1}+\vartheta_{2}\right)\boldsymbol{f}+\vartheta_{1}\vartheta_{2}\right)=\\ \gamma_{1}\gamma_{2}\left(1+\vartheta_{1}\vartheta_{2}\right)\left(1+\left(\vartheta_{1}+\vartheta_{2}\right)/\left(1+\vartheta_{1}\vartheta_{2}\right)\boldsymbol{f}\right)\Rightarrow\\ \gamma=\gamma_{1}\gamma_{2}\left(1+\vartheta_{1}\vartheta_{2}\right),\quad \vartheta=\left(\vartheta_{1}+\vartheta_{2}\right)/\left(1+\vartheta_{1}\vartheta_{2}\right),\end{gathered}$$

and we have “velocity addition rule”. So, every multivector could be mathematically treated as an ordinary boost in special relativity. For $\rho=1$ we have a “boost” $\gamma\left(1+\vartheta\boldsymbol{f}\right)=Ku_{+}+K^{-1}u_{-}$ as transformation that preserves multivector amplitude and should be considered as part of Lorentz group ([1, 2]). As a simple example of unit complex vector we already mentioned $\boldsymbol{f}=\boldsymbol{e}_{1}+\boldsymbol{e}_{2}+j\boldsymbol{e}_{3}=\boldsymbol{e}_{1}+\boldsymbol{e}_{2}+\boldsymbol{e}_{1}\boldsymbol{e}_{2}$, completely in *Cl*2, suggesting that one could analyze problem in basis $\left(1,\boldsymbol{F}_{1}\right)$ or related spectral basis and rotate all elements to obtain relations for arbitrary orientation of plane, using powerful apparatus of geometric algebra for rotations.

Mapping basis $(1,\boldsymbol{f})$ to $(e^{\phi f},\boldsymbol{f}e^{\phi f})$ we obtain new orthogonal basis and new components of multivector

$$\begin{aligned}&a+b\boldsymbol{f}\rightarrow a'e^{\phi f}+b'\boldsymbol{f}e^{\phi f}=(a'+b'\boldsymbol{f})e^{\phi f},\\ &a'=ae^{-\phi f},\quad b'=be^{-\phi f},\quad \left|a+b\boldsymbol{f}\right|=\left|a'+b'\boldsymbol{f}\right|.\end{aligned}$$

4. Functions of multivectors

Using series expansion it is straight forward to find closed formulae for (analytic at least) functions. If $\boldsymbol{F}^{2}=0\Rightarrow f\left(M\right)=f\left(z\right)$ we have $f\left(M\right)=f\left(z\right)$ and it is easy to find closed form using theory of functions on complex field. Otherwise, from the series expansion

$$f\left(x\right)=f\left(0\right)+\sum_{n}\frac{f^{(n)}\left(0\right)x^{n}}{n!}$$

using $M^2 = M_+^n u_+ + M_-^n u_-$ we have

$$f(M) = f(M_+)u_+ + f(M_-)u_-$$

and, again, it is easy to find closed form because of $M_\pm \in \mathbb{C}$ .For $M = \boldsymbol{F} = \sqrt{\boldsymbol{F}^2}\boldsymbol{f}$ we have $M_\pm = \pm\sqrt{\boldsymbol{F}^2} \Rightarrow f(M) = f\left(\sqrt{\boldsymbol{F}^2}\right)u_+ + f\left(-\sqrt{\boldsymbol{F}^2}\right)u_-$. If function is even we have $f(\boldsymbol{F}) = f\left(\sqrt{\boldsymbol{F}^2}\right)(u_+ + u_-) = f\left(\sqrt{\boldsymbol{F}^2}\right)$ and $f(\boldsymbol{F}) = f\left(\sqrt{\boldsymbol{F}^2}\right)(u_+ - u_-) = f\left(\sqrt{\boldsymbol{F}^2}\right)\boldsymbol{f}$ for odd functions. For $M = z + \boldsymbol{F}$, $\boldsymbol{F}^2 = \boldsymbol{N}^2 = 0$ there is no spectral decomposition ($\boldsymbol{f}$ is not defined), but we have $M^n = (z + \boldsymbol{N})^n = z^n + nz^{n-1}\boldsymbol{N}$, giving $f(z + \boldsymbol{N}) = f(z) + f'(z)\boldsymbol{N}$. We also have a special cases

$$f(u_\pm) = f(\pm 1)u_\pm,$$
$$f(\boldsymbol{f}) = f(u_+ - u_-) = f(1)u_+ + f(-1)u_-,$$
$$f\left(\hat{\boldsymbol{F}}\right) = f(-ju_+ + ju_-) = f(-j)u_+ + f(j)u_-.$$

Obviously for an odd function is $f(\boldsymbol{f}) = f(1)\boldsymbol{f}$, $f\left(\hat{\boldsymbol{F}}\right) = -f(j)\boldsymbol{f}$, while for even functions is $f(\boldsymbol{f}) = f(1)$, $f\left(\hat{\boldsymbol{F}}\right) = f(j)\boldsymbol{f}$.

For an inverse functions we have

$$f^{-1}(y) = x \Rightarrow f(x) = y \Rightarrow f(x_\pm) = y_\pm \Rightarrow x_\pm = f^{-1}(y_\pm).$$

For a *light-like* multivectors ($M\bar{M} = 0$) we have

$M = z + \sqrt{\boldsymbol{F}}\boldsymbol{f}$ =z+z$_{\boldsymbol{F}}\boldsymbol{f}$, z^2- $\boldsymbol{F}^2 = 0 = $(z-z$_{\boldsymbol{F}}$)(z+z$_{\boldsymbol{F}}$),

and two possibilities:

1) $z = z_{\boldsymbol{F}} \Rightarrow M_+ = 2z_{\boldsymbol{F}}$, $M_- = 0 \Rightarrow f(M) = f(2z_{\boldsymbol{F}})u_+$
2) $z = -z_{\boldsymbol{F}} \Rightarrow M_+ = 0$, $M_- = -2z_{\boldsymbol{F}} \Rightarrow f(M) = f(-2z_{\boldsymbol{F}})u_-$

Once a spectral decomposition of function is analyzed there remains just to use well known properties of functions of complex variables.

5. Examples

Here we define $M = z + \boldsymbol{F} = z + \sqrt{\boldsymbol{F}^2}\boldsymbol{f} = z + z_{\boldsymbol{F}}\boldsymbol{f}$ and $M_+ = z + z_{\boldsymbol{F}}$, $M_- = z - z_{\boldsymbol{F}}$.

For inverse of M we have ($|M| \neq 0$)

$$M^{-1} = \frac{1}{M_+u_+ + M_-u_-} = \frac{M_+u_- + M_-u_+}{(M_+u_+ + M_-u_-)(M_+u_- + M_-u_+)} = \frac{M_+u_- + M_-u_+}{M_+M_-} = \frac{u_+}{M_+} + \frac{u_-}{M_-},$$

as expected. Now it is obvious that

$$M^{-n}=\frac{1}{\left(M_+u_+ + M_-u_-\right)^n}=\frac{u_+}{\left(M_+\right)^n}+\frac{u_-}{\left(M_-\right)^n}.$$

We can find square root using

$$\sqrt{M}=S=S_+u_+ + S_-u_- \Rightarrow M=M_+u_+ + M_-u_- = \left(S_+\right)^2 u_+ + \left(S_-\right)^2 u_- \Rightarrow S_\pm = \pm\sqrt{M_\pm}\ .$$

So, generally we have $M^{\pm 1/n}=S\Rightarrow S_\pm=\left(M_\pm\right)^{\pm 1/n}$, $n\in\mathbb{N}$. As a simple example (interested reader could compare with [3]) $\sqrt{\boldsymbol{e}_i}=\pm\left(j+\boldsymbol{e}_i\right)/\sqrt{2j}$.

Exponential function is easy one, $e^M=e^{M_+}u_+ + e^{M_-}u_-$ and now we have exponentials of complex numbers (just use ordinary $i=\sqrt{-1}$ replacing $i\to j$ at the end). Logarithm is the inverse function to exponential, so we have

$$\log M = X \Rightarrow e^X = M = M_+u_+ + M_-u_- = \exp\left(X_+\right)u_+ + \exp\left(X_-\right)u_- \Rightarrow X_\pm = \log M_\pm .$$

In [3] is derived formula $\log M=\log|M|+\varphi\hat{\boldsymbol{F}}$, $\varphi=\arctan\left(|\boldsymbol{F}|/z\right)$, but those are equivalent:

$$z_{\boldsymbol{F}}=\sqrt{\boldsymbol{F}^2}=-j|\boldsymbol{F}|,\ \ \hat{\boldsymbol{F}}=-j\boldsymbol{f}\Rightarrow$$

$$\log\left(M_+\right)u_+ + \log\left(M_-\right)u_- = \frac{\log\left(M_+\right)+\log\left(M_-\right)}{2}+\frac{\log\left(M_+\right)-\log\left(M_-\right)}{2}\boldsymbol{f}=$$

$$\log|M|-j\hat{\boldsymbol{F}}\log\left(\sqrt{\left[1-j\left(|\boldsymbol{F}|/z\right)\right]/\left[1+j\left(|\boldsymbol{F}|/z\right)\right]}\right)=\log|M|+\hat{\boldsymbol{F}}\arctan\left(|\boldsymbol{F}|/z\right)=\log|M|+\varphi\hat{\boldsymbol{F}}.$$

Now we can find for $a\in\mathbb{R}$ $M^a=X\Rightarrow\log X=a\log M\Rightarrow X=e^{a\log M}$,

but the same appears to be correct for $a=z+\boldsymbol{F}$ and one can find, for example,

$$M^{\vec{v}}=X\Rightarrow\log X=\vec{\boldsymbol{v}}\log M\Rightarrow X=e^{\vec{v}\log M},$$

although here some caution is needed because of possibility $\log X=\left(\log M\right)\vec{\boldsymbol{v}}\Rightarrow$ $X=e^{(\log M)\vec{v}}$. Also, relation $M=e^{\log M}$ is generally not valid and needs some care due to the multivalued nature of the logarithm operation. Nevertheless, expressions like $j^{\boldsymbol{e}_1}=\exp\left(\boldsymbol{e}_1\log j\right)=\exp\left(j\pi\boldsymbol{e}_1/2\right)=j\boldsymbol{e}_1$, or $\left(\boldsymbol{e}_1\boldsymbol{e}_2\right)^{\boldsymbol{e}_3}=j$ are quite possible in Cl_3. Simple examples ($u_\pm=\left(1\pm\boldsymbol{e}_1\right)/2$):

1. $\boldsymbol{e}_1^{\boldsymbol{e}_1}=X\Rightarrow\boldsymbol{e}_1\log\boldsymbol{e}_1=\log X$,
   $\boldsymbol{e}_1=u_+-u_-\Rightarrow\log\boldsymbol{e}_1=u_+\log 1+u_-\log\left(-1\right)=j\pi u_-\Rightarrow$
   $\boldsymbol{e}_1\log\boldsymbol{e}_1=-j\pi u_-\Rightarrow X=\exp\left(-j\pi u_-\right)=\exp\left(-j\pi\right)u_-=-u_-$
   (solution $\boldsymbol{e}_1$ is not valid because of $\boldsymbol{e}_1\log\boldsymbol{e}_1=-j\pi u_-=-\log\boldsymbol{e}_1$).
2. $\boldsymbol{e}_1^{\boldsymbol{e}_2}=X\Rightarrow\log X=\boldsymbol{e}_2\log\boldsymbol{e}_1=j\pi\boldsymbol{e}_2u_-$,
   but $\boldsymbol{e}_2u_-\boldsymbol{e}_2u_-=\boldsymbol{e}_2\boldsymbol{e}_2u_+u_-=0$, so $X=\exp\left(j\pi\boldsymbol{e}_2u_-\right)=1+j\pi\boldsymbol{e}_2u_-$,
   or $\log X=\left(\log\boldsymbol{e}_1\right)\boldsymbol{e}_2=j\pi u_-\boldsymbol{e}_2\Rightarrow X=1+j\pi\boldsymbol{e}_2u_+$,
   and finally $X=1+j\pi\boldsymbol{e}_2u_\pm$ and $X^n=1+jn\pi\boldsymbol{e}_2u_\pm$, $n\in\mathbb{Z}$
   (solution 1 is not valid because of $\boldsymbol{e}_2\log\boldsymbol{e}_1\neq 0$, it is nilpotent).

Trigonometric and hyperbolic trigonometric functions are straightforward and ctg one could obtain as inverse of tan. For example

$$\sin\boldsymbol{F} = \sin\left(\sqrt{\boldsymbol{F}^2}\right)u_+ + \sin\left(-\sqrt{\boldsymbol{F}^2}\right)u_- = \sin\left(\sqrt{\boldsymbol{F}^2}\right)\left(u_+ - u_-\right) = \boldsymbol{F}_1 \sin\left(\sqrt{\boldsymbol{F}^2}\right),$$

$$\cos\boldsymbol{F} = \cos\left(\sqrt{\boldsymbol{F}^2}\right)u_+ + \cos\left(-\sqrt{\boldsymbol{F}^2}\right)u_- = \cos\left(\sqrt{\boldsymbol{F}^2}\right)\left(u_+ + u_-\right) = \cos\left(\sqrt{\boldsymbol{F}^2}\right).$$

For $\boldsymbol{F} = \boldsymbol{N}$ there is no spectral decomposition, but using exp($z\boldsymbol{N}$) = 1+ $z\boldsymbol{N}$ we have

$\sin\boldsymbol{N} = \boldsymbol{N}$, $\cos\boldsymbol{N} = 1$, $\sinh\boldsymbol{N} = \boldsymbol{N}$, $\cosh\boldsymbol{N} = 1$. There is no $\boldsymbol{N}^{-1}$, so ctg$\boldsymbol{N}$ is not defined.

Series in powers of argument are crucial for all analytic functions (in geometric algebra too) and we can use presented spectral decomposition to obtain components of such functions in spectral basis.

Conclusion

Geometric algebra of Euclidean 3D space (Cl_3) is really rich in structure and gives possibility to analyze functions defined on multivectors, extending thus theory of functions of real and complex variables, providing intuitive geometrical interpretation also. From simple fact that for complex vector ($\boldsymbol{F}^2 \neq 0$) we can write $\boldsymbol{F}/\sqrt{\boldsymbol{F}^2} = \boldsymbol{f}$, $\boldsymbol{f}^2 = 1$, $\boldsymbol{F}^2 \in \mathbb{C}$ follows nice possibility to explore idempotent structure $u_\pm = \left(1 \pm \boldsymbol{f}\right)/2$ and spectral decomposition of multivectors. Using orthogonality of spectral basis vectors (idempotents) $u_\pm$ it is shown that all multivectors (even rotation operators, [1]) can be treated as unipodal numbers (i.e. hypercomplex numbers over complex field).Definition of functions is then quite simple and natural and strongly counts on the theory of functions of complex variable. Complex numbers and vectors (bivectors, trivectors) are thus united in one promising system.

Appendix

A1. *Bilinear transformations*

Regarding that bilinear transformations of multivectors do not change some property of multivectors one could ask yourself: what property? In [2] was shown that multivector amplitude, defined using Clifford conjugation which is unique involution that is commutative, belongs to center of algebra.

Now, let T be an operator on multivectors that does not change some property of multivectors. We have $T\left(M'\right) = T\left(XMY\right)$ so

1) X, M, Y generally have different orientations, so we could demand $T\left(X\right) \in \mathbb{C}$ and $T\left(M'\right) = T\left(X\right)T\left(M\right)T\left(Y\right)$
2) T is linear and $T\left(X\right) = 1$
3) $T\left(X\right)$ is well defined quantity for all (interesting) X in an unique way

From spectral decomposition we see that there is such quantity, namely, multivector amplitude, defined as $M_+ M_- \in \mathbb{C}$ using just natural conjugation $a + b\boldsymbol{f} \to a - b\boldsymbol{f}$, where $\boldsymbol{f}$ is our hypercomplex unit. But this is just Clifford conjugation and we see now new meaning for it: just “hypercomplex conjugation”. This is strong argument for regarding Lorentz transformations to be group of bilinear transformations that preserve multivector amplitude. It is verified on paravectors giving known special relativity, but now we should extend it on whole multivector including in this way all multivector symmetries of Euclidean 3D space. For multivector X (transformation) now we have $T(X) = X_+ X_- = 1$ and

$$X = e^M \Rightarrow M = \log X = \log|X| + \varphi\hat{\boldsymbol{F}} = \log 1 + \varphi\hat{\boldsymbol{F}} = \varphi\hat{\boldsymbol{F}} = \boldsymbol{F},$$

giving thus general bilinear (twelve parameters) transformation $M' = e^{\vec{p}+j\vec{q}} M e^{\vec{r}+j\vec{s}}$.

A2. *Hyperbolic inner and outer products*

Given two multivectors $M_1 = z_1 + z_{1F}\boldsymbol{f}$ and $M_2 = z_2 + z_{2F}\boldsymbol{f}$ we define a square of *multivector distance* (*conjugate products*, [13]) as

$$M_1^- M_2 = (z_1 - z_{1F}\boldsymbol{f})(z_2 + z_{2F}\boldsymbol{f}) = z_1 z_2 - z_{1F} z_{2F} + (z_1 z_{2F} - z_2 z_{1F})\boldsymbol{f} = hi + ho\boldsymbol{f},$$

where *hi* and *ho* stands for hyperbolic inner and hyperbolic outer products. If $M_1 = M_2 = M$ we have $M^- M = z^2 - z_F^2 + (zz_F - zz_F)\boldsymbol{f} = z^2 - z_F^2$ is just square of multivector amplitude. This suggests that *hi* and *ho* have to do something about being "parallel" or "orthogonal" besides being "near" and "close". For complex and hypercomplex plane (with real coordinates) meaning is obvious (fig A2).

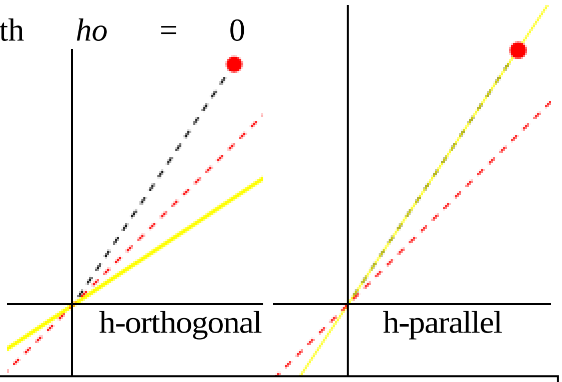


fig A2: *Yellow line represents h-numbers parallel or orthogonal to given h-number* (red dot).

With *ho* = 0 multivectors are said to be "h-parallel", while for *hi* = 0 they are "h-orthogonal". For *hi* = *ho* = 0 multivector distance is null and we said it to be "h-light-like", where h- stands for hyperbolic.
In "boost" formalism we have

$$hi = \rho_1\rho_2\gamma_1\gamma_2(1 - \vartheta_1\vartheta_2),$$
$$ho = \rho_1\rho_2\gamma_1\gamma_2(\vartheta_2 - \vartheta_1)/(1 - \vartheta_1\vartheta_2).$$

So, "h-parallel" multivectors have equal "velocities" and $hi = \rho_1\rho_2$, while for "h-orthogonal" multivectors "velocities" are reciprocal and *ho* becomes infinite (orthogonal multivectors belong to different hyper-quadrants delimited by light-like hyper-planes).

Lema: *Let* $M_1^- M_2 = 0$ *for* $M_1 \neq 0$ *and* $M_2 \neq 0$. Then $M_1 M_2 \neq 0$, *and vice versa*.

$$M_1 M_2 = (M_{1+}u_+ + M_{1-}u_-)(M_{2+}u_+ + M_{2-}u_-) = M_{1+}M_{2+}u_+ + M_{1-}M_{2-}u_-,$$
$$M_1^- M_2 = (M_{1+}u_- + M_{1-}u_+)(M_{2+}u_+ + M_{2-}u_-) = M_{1-}M_{2+}u_+ + M_{1+}M_{2-}u_-,$$

and we have $M_{1-}M_{2+} = 0$ or $M_{1+}M_{2-} = 0$, which means $M_{1-} = 0$ and $M_{2-} = 0$ or $M_{1+} = 0$ and $M_{2+} = 0$, but either case gives $M_1 M_2 \neq 0$. Converse proof is similar.

A3. *Polynomials*

Suppose we have simple equation $M^2 + 1 = 0$, then objects that squares to -1 are solutions. Using spectral decomposition we could explore it further, so,

$$M^2+1=\left(M_+u_+ + M_-u_-\right)^2+u_+ + u_- = \left(M_+^2+1\right)u_+ + \left(M_-^2+1\right)u_- = 0 \Rightarrow$$

$M_+^2+1=0,\ \ M_-^2+1=0$, so we have two equations over complex numbers. Obvious solutions are $\sqrt{-1},\ j,\ je_i,\ \hat{\boldsymbol{F}}$, but it is possible to investigate further. There is an infinite number of solutions, obviously, due to algebraically expanded paradigm of number.

Another simple equation is $M^2=M_+^2u_+ + M_-^2u_- = 0$. One obvious solution is $M=\boldsymbol{N}$ (nilpotent) which we cannot obtain using spectral decomposition (because there is no one for a multivector $\boldsymbol{N}$), so, some caution is necessary.

In physics we are using a lot of special polynomials and their roots and we probably should reconsider those in geometric algebra.

References:

[1] Josipović, *Some remarks on Cl3 and Lorentz transformations*, vixra: 1507.0045v1

[2] Chappell, Hartnett, Iannella, Abbott, *Deriving time from the geometry of space*,arXiv:1501.04857v2

[3] Chappell, Iqbal, Gunn, Abbott, *Functions of multivector variables*, arXiv:1409.6252v1

[4] Chappell, Iqbal, Iannella, Abbott, *Revisiting special relativity: A natural algebraic alternative to Minkowski spacetime*, PLoS ONE 7(12)

[5] Dorst, Fontijne, Mann, *Geometric Algebra for Computer Science* (Revised Edition), Morgan Kaufmann Publishers, 2007

[6] Hestenes, *New Foundations for Classical Mechanics*, Kluwer Academic Publishers, 1999

[7] Hestenes, Sobczyk, *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, Kluwer Academic Publishers, 1993

[8] Hitzer, Helmstetter, Ablamowicz, *Square Roots of -1 in Real Clifford Algebras*, arXiv:1204.4576v2

[9] Mornev, *Idempotents and nilpotents of Clifford algebra* (russian), Гиперкомплексные числавгеометрии и физике, 2(12), том 6, 2009

[10] Sobczyk, *New Foundations in Mathematics*: *The Geometric Concept of Number*, Birkhäuser, 2013

[11] Sobczyk, *Special relativity in complex vector algebra*, arXiv:0710.0084v1

[12] Sobczyk, *Geometric matrix algebra*, Linear Algebra and its Applications 429 (2008) 1163–1173

[13] Sobczyk, *The Hyperbolic Number Plane*, http://www.garretstar.com

[14] Sobczyk, *The Missing Spectral Basis in Algebra and Number Theory*, JSTOR

[15] Sobczyk, *A Complex Gibbs-Heaviside Vector Algebra for Space-time*, Acta Physica Polonica, B12, 1981