Gauge invariance of sedeonic Klein-Gordon equation

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We discuss the gauge invariance of generalized Klein-Gordon wave equation based on sedeonic space-time operators and sedeonic wave function.

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The operator Klein-Gordon (KG) wave equation corresponds to the fundamental Einstein relation between energy and momentum [1]. Substantially it is the basis for the description of relativistic quantum particles and fields. However the scalar KG equation describes only scalar bosons but does not specify the spin properties of particles. Therefore the generalization of KG equation to a wider class of operators and multicomponent wave functions is the subject of intensive investigations.

First, to describe the quantum particles with spin 1/2 P.A.M.Dirac proposed the matrix equation for the spinor wave function [2], which leads to non-scalar KG equation in a case of electromagnetic interaction. This idea was further generalized for the integer spin 0 and 1 in the frames of Duffin-Kemmer-Petiau (DKP) formalism [3-5].

On the other hand, in recent years many attempts have been made to generalize KG equation using different algebras of hypercomplex numbers, such as four-component quaternions and eight-component octonion [6-8]. The authors discuss the reformulation of the Proca equation [9] as the system of first-order equations similar to the equations of electromagnetic field but with a massive "photon". However, compared with electrodynamics the resulting Proca-Maxwell (PM) equations for massive field are not gauge invariant [10,11].

Recently we proposed an alternative approach based on sixteen-component sedeons generating noncommutative associative space-time algebra [12]. The sedeons take into account the space-time properties of physical values and realize the scalar-vector representation of Poincare group. In particular, we proposed the symmetric second-order and first-order wave equations describing the massive and massless fields [13]. In present paper we focus main attention on the gauge invariance of sedeonic KG wave equation based on sedeonic space-time operators and sedeonic wave function.

The sedeonic KG equation [13] can be written in the following compact form:

$$\overset{\frown}{\nabla}\overset{\frown}{\nabla}\ \tilde{\mathbf{W}}\ = 0. \tag{1}$$

Here the wave function $\tilde{\mathbf{W}}$ is the sixteen-component sedeon and we use the complex operator

$$\overset{\frown}{\nabla} = \left(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m\right), \tag{2}$$

with

$$\begin{aligned} &\partial = \frac{1}{c}\frac{\partial}{\partial t}, \\ &\vec{\nabla} = \frac{\partial}{\partial x}\ \mathbf{a_1} + \frac{\partial}{\partial y}\ \mathbf{a_2} + \frac{\partial}{\partial z}\ \mathbf{a_3}, \\ &m = \frac{m_0 c}{\hbar}, \end{aligned} \tag{3}$$

where $\mathbf{e_t}, \mathbf{e_r}, \mathbf{e_{tr}}$ are the sedeonic space-time units; $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ are the sedeonic unit vectors; c is the speed of light; m_0 is the mass of quantum; $\hbar$ is the Planck constant. On the other hand, if we introduce the filed strength

$$\tilde{\mathbf{E}} = \overset{\frown}{\nabla}\ \tilde{\mathbf{W}}, \tag{4}$$

then the wave equation (1) takes the following form:

$$\overset{\frown}{\nabla}\ \tilde{\mathbf{E}}\ = 0. \tag{5}$$

The sedeonic formalism of $\overset{\frown}{\nabla}$ operator enables the direct conclusion that the equation (1) and field strength definition (4) are invariant with respect to the replacement of wave function

$$\tilde{\mathbf{W}} \Rightarrow \tilde{\mathbf{W}} + \tilde{\mathbf{F}} + \overset{\frown}{\nabla}\tilde{\mathbf{G}}, \tag{6}$$

where $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ are arbitrary sedeons satisfy the following conditions:

$$\overset{\frown}{\nabla}\ \tilde{\mathbf{F}} = 0, \tag{7}$$

$$\overset{\frown}{\nabla}\overset{\frown}{\nabla}\ \tilde{\mathbf{G}} = 0. \tag{8}$$

Besides, the equation (5) is invariant with respect to the replacement

$$\tilde{\mathbf{E}} \Rightarrow \tilde{\mathbf{E}} + \tilde{\mathbf{F}} + \overset{\frown}{\nabla}\tilde{\mathbf{G}}. \tag{9}$$

The gauge relations (6) - (9) are the direct consequence of the operator $\overset{\frown}{\nabla}$ formalism.

Let us consider the field equations and gauge conditions in detail. We choose the wave function as

$$\begin{aligned} \tilde{\mathbf{W}} = \ & i a_1 \mathbf{e_t} - i a_2 \mathbf{e_r} + a_3 - i a_4 \mathbf{e_{tr}} \\ & + \vec{A}_1 \mathbf{e_r} + \vec{A}_2 \mathbf{e_t} - \vec{A}_3 \mathbf{e_{tr}} \ + i \vec{A}_4, \end{aligned} \tag{10}$$

where components a_{s} and $\vec{A}_{\mathrm{s}}$ are real functions of coordinates and time. Here and further the index s = 1, 2, 3, 4. Multiplying the operators in the left part of equation (1) and separating the values with different space-time properties, we obtain the following KG equations for the components of wave function:

$$\begin{aligned} \left(\partial^2 - \Delta + m^2\right) a_{\mathrm{s}} = 0, \\ \left(\partial^2 - \Delta + m^2\right) \vec{A}_{\mathrm{s}} = 0, \end{aligned} \tag{11}$$

where $\triangle$ is the Laplace operator. On the other hand, if we take the sedeon $\tilde{\mathbf{E}}$ as

$$\begin{aligned} \tilde{\mathbf{E}} = & -\varepsilon_1 + i\varepsilon_2 \mathbf{e_{tr}} + i\varepsilon_3 \mathbf{e_t} - i\varepsilon_4 \mathbf{e_r} \\ & + \vec{E}_1 \mathbf{e_{tr}} - i\vec{E}_2 + \vec{E}_3 \mathbf{e_r} + \vec{E}_4 \mathbf{e_t}, \end{aligned} \tag{12}$$

we have the following definitions for scalar ε_{s} and vector $\vec{E}_{\mathrm{s}}$ components:

$$\begin{aligned} \varepsilon_1 &= \partial a_1 + \left(\vec{\nabla} \cdot \vec{A}_1\right) + m a_4, \\ \varepsilon_2 &= \partial a_2 + \left(\vec{\nabla} \cdot \vec{A}_2\right) - m a_3, \\ \varepsilon_3 &= \partial a_3 + \left(\vec{\nabla} \cdot \vec{A}_3\right) + m a_2, \\ \varepsilon_4 &= \partial a_4 + \left(\vec{\nabla} \cdot \vec{A}_4\right) - m a_1, \\ \vec{E}_1 &= -\partial \vec{A}_1 - \vec{\nabla} a_1 + i\left[\vec{\nabla} \times \vec{A}_2\right] + m\vec{A}_4, \\ \vec{E}_2 &= -\partial \vec{A}_2 - \vec{\nabla} a_2 - i\left[\vec{\nabla} \times \vec{A}_1\right] - m\vec{A}_3, \\ \vec{E}_3 &= -\partial \vec{A}_3 - \vec{\nabla} a_3 - i\left[\vec{\nabla} \times \vec{A}_4\right] + m\vec{A}_2, \\ \vec{E}_4 &= -\partial \vec{A}_4 - \vec{\nabla} a_4 + i\left[\vec{\nabla} \times \vec{A}_3\right] - m\vec{A}_1, \end{aligned} \tag{13}$$

and the sedeonic equation (5) is equivalent to the following system of equations for the field strengths:

$$\begin{aligned}
&\partial\varepsilon_1 + \left(\vec{\nabla}\cdot\vec{E}_1\right) - m\varepsilon_4 = 0,\\
&\partial\varepsilon_2 + \left(\vec{\nabla}\cdot\vec{E}_2\right) + m\varepsilon_3 = 0,\\
&\partial\varepsilon_3 + \left(\vec{\nabla}\cdot\vec{E}_3\right) - m\varepsilon_2 = 0,\\
&\partial\varepsilon_4 + \left(\vec{\nabla}\cdot\vec{E}_4\right) + m\varepsilon_1 \ = 0,\\
&\partial\vec{E}_1 + \vec{\nabla}\varepsilon_1 + i\left[\vec{\nabla}\times\vec{E}_2\right] + m\vec{E}_4 = 0,\\
&\partial\vec{E}_2 + \vec{\nabla}\varepsilon_2 - i\left[\vec{\nabla}\times\vec{E}_1\right] - m\vec{E}_3 = 0,\\
&\partial\vec{E}_3 + \vec{\nabla}\varepsilon_3 - i\left[\vec{\nabla}\times\vec{E}_4\right] + m\vec{E}_2 = 0,\\
&\partial\vec{E}_4 + \vec{\nabla}\varepsilon_4 + i\left[\vec{\nabla}\times\vec{E}_3\right] - m\vec{E}_1 = 0.
\end{aligned} \tag{14}$$

Let us consider the gauge invariance for the components of wave function and field strength. We take the arbitrary sedeons $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ as

$$\begin{aligned}
\tilde{\mathbf{F}} = \ & if_1\mathbf{e_t} - if_2\mathbf{e_r} + f_3 - if_4\mathbf{e_{tr}}\\
& + \vec{F}_1\mathbf{e_r} + \vec{F}_2\mathbf{e_t} - \vec{F}_3\mathbf{e_{tr}} \ + i\vec{F}_4,
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\tilde{\mathbf{G}} = \ & -g_1 + ig_2\mathbf{e_{tr}} + ig_3\mathbf{e_t} - ig_4\mathbf{e_r}\\
& - \vec{G}_1\mathbf{e_{tr}} + i\vec{G}_2 - \vec{G}_3\mathbf{e_r} - \vec{G}_4\mathbf{e_t}.
\end{aligned} \tag{16}$$

Then the replacement (6) leads us to the following substitutions:

$$\begin{aligned}
&a_1 \Rightarrow a_1 + f_1 - \partial g_1 + \left(\vec{\nabla}\cdot\vec{G}_1\right) + mg_4,\\
&a_2 \Rightarrow a_2 + f_2 - \partial g_2 + \left(\vec{\nabla}\cdot\vec{G}_2\right) - mg_3,\\
&a_3 \Rightarrow a_3 + f_3 - \partial g_3 + \left(\vec{\nabla}\cdot\vec{G}_3\right) + mg_2,\\
&a_4 \Rightarrow a_4 + f_4 - \partial g_4 + \left(\vec{\nabla}\cdot\vec{G}_4\right) - mg_1,\\
&\vec{A}_1 \Rightarrow \vec{A}_1 + \vec{F}_1 - \partial\vec{G}_1 + \vec{\nabla}g_1 - i\left[\vec{\nabla}\times\vec{G}_2\right] - m\vec{G}_4,\\
&\vec{A}_2 \Rightarrow \vec{A}_2 + \vec{F}_2 - \partial\vec{G}_2 + \vec{\nabla}g_2 + i\left[\vec{\nabla}\times\vec{G}_1\right] + m\vec{G}_3,\\
&\vec{A}_3 \Rightarrow \vec{A}_3 + \vec{F}_3 - \partial\vec{G}_3 + \vec{\nabla}g_3 + i\left[\vec{\nabla}\times\vec{G}_4\right] - m\vec{G}_2,\\
&\vec{A}_4 \Rightarrow \vec{A}_4 + \vec{F}_4 - \partial\vec{G}_4 + \vec{\nabla}g_4 - i\left[\vec{\nabla}\times\vec{G}_3\right] + m\vec{G}_1,
\end{aligned} \tag{17}$$

which do not change the field strengths definitions (13) and equations (14). The substitutions (17) include the gradient invariance of electromagnetic potentials [14] as a particular case. Indeed, if we take $m = 0$, $f_s = 0$, $\vec{F}_s = 0$, $\vec{G}_s = 0$ then the substitutions (17) are rewritten as

$$\begin{aligned}
&a_s \Rightarrow a_s + \partial g_s,\\
&\vec{A}_s \Rightarrow \vec{A}_s - \vec{\nabla}g_s.
\end{aligned} \tag{18}$$

Similarly, the replacement (9) is equivalent to the following substitutions for the field strengths

$$\begin{aligned}
&\varepsilon_1 \Rightarrow \varepsilon_1 - f_3 + \partial g_3 - \left(\vec{\nabla}\cdot\vec{G}_3\right) - m g_2,\\
&\varepsilon_2 \Rightarrow \varepsilon_2 - f_4 + \partial g_4 - \left(\vec{\nabla}\cdot\vec{G}_4\right) + m g_1,\\
&\varepsilon_3 \Rightarrow \varepsilon_3 + f_1 - \partial g_1 + \left(\vec{\nabla}\cdot\vec{G}_1\right) + m g_4,\\
&\varepsilon_4 \Rightarrow \varepsilon_4 + f_2 - \partial g_2 + \left(\vec{\nabla}\cdot\vec{G}_2\right) - m g_3,\\
&\vec{E}_1 \Rightarrow \vec{E}_1 + \vec{F}_3 - \partial\vec{G}_3 + \vec{\nabla} g_3 + i\left[\vec{\nabla}\times\vec{G}_4\right] - m\vec{G}_2,\\
&\vec{E}_2 \Rightarrow \vec{E}_2 + \vec{F}_4 - \partial\vec{G}_4 + \vec{\nabla} g_4 - i\left[\vec{\nabla}\times\vec{G}_3\right] + m\vec{G}_1,\\
&\vec{E}_3 \Rightarrow \vec{E}_3 - \vec{F}_1 + \partial\vec{G}_1 - \vec{\nabla} g_1 + i\left[\vec{\nabla}\times\vec{G}_2\right] + m\vec{G}_4,\\
&\vec{E}_4 \Rightarrow \vec{E}_4 - \vec{F}_2 + \partial\vec{G}_2 - \vec{\nabla} g_2 - i\left[\vec{\nabla}\times\vec{G}_1\right] - m\vec{G}_3,
\end{aligned} \tag{19}$$

which do not change the equations (14).

Note that the equations (14) include the PM equations as the partial case. Indeed, if we suppose that the field is described only by a_1 and $\vec{A}_1$ components:

$$\tilde{\mathbf{W}} = i a_1 \mathbf{e_t} + \vec{A}_1 \mathbf{e_r}, \tag{20}$$

with Lorentz gauge

$$\partial a_1 + \left(\vec{\nabla}\cdot\vec{A}_1\right) = 0, \tag{21}$$

then we have only the following nonzero field's strengths (see the definitions (13)):

$$\begin{aligned}
&\varepsilon_4 = -m a_1,\\
&\vec{E}_1 = -\partial\vec{A}_1 - \vec{\nabla} a_1,\\
&\vec{E}_2 = -i\left[\vec{\nabla}\times\vec{A}_1\right],\\
&\vec{E}_4 = -m\vec{A}_1,
\end{aligned} \tag{22}$$

and the system (14) is rewritten as

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{E}_1\right) - m\varepsilon_4 = 0,\\
&\left(\vec{\nabla}\cdot\vec{E}_2\right) = 0,\\
&\partial\varepsilon_4 + \left(\vec{\nabla}\cdot\vec{E}_4\right) = 0,\\
&\partial\vec{E}_1 + i\left[\vec{\nabla}\times\vec{E}_2\right] + m\vec{E}_4 = 0,\\
&\partial\vec{E}_2 - i\left[\vec{\nabla}\times\vec{E}_1\right] = 0,\\
&i\left[\vec{\nabla}\times\vec{E}_4\right] - m\vec{E}_2 = 0,\\
&\partial\vec{E}_4 + \vec{\nabla}\varepsilon_4 - m\vec{E}_1 = 0.
\end{aligned} \tag{23}$$

The system (23) is the sedeonic analog of PM equations with considerably reduced symmetry and broken gauge invariance.

Note that all relations were derived in the sedeonic algebra. For the transition to the common used Gibbs-Heaviside vector algebra the change $i[\vec{\nabla}\times\vec{A}] \Rightarrow -[\vec{\nabla}\times\vec{A}]$ should be made in all equations.

Thus it was shown that the approach based on sedeonic wave functions and space-time operators allows us to formulate the symmetric KG equation and scalar-vector Maxwell-like equations describing a massive field. We have demonstrated that the gauge invariance of the sedeonic equations is the specific property of the operator $(i\mathbf{e_t}\partial - \mathbf{e_r}\vec{\nabla} - i\mathbf{e_{tr}}m)$ and can be reformulated for a wider class of scalar-vector substitutions. On the example of PM equations we have shown that the decrease in the number of components of wave function leads to the breaking of symmetry and gauge invariance violation. The proposed idea of gauge substitutions can be generalized for the Dirac and DKP matrix operators.

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