Generalized Sophomore's Dream Identity

Danil Krotkov

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The identity of Johann Bernoulli, nowadays known as "Sophomore's Dream", states that

$$\sum_{n=1}^{\infty} \frac{1}{n^n} = \int_0^1 \frac{1}{x^x} dx.$$

This paper represents the proof of generalized fact that

$$\sum_{n=k+1}^{\infty} \frac{n^k}{n^n} = \int_0^1 \frac{x^k}{x^x} dx$$

for every natural k, and moreover,

$$\sum_{n=1}^{\infty} \frac{a^n}{(n+c)^n} = \int_{-c}^{a-c} \frac{a^t}{(t+c)^t} dt \quad \left(= a \int_0^1 t^{c-at} dt \right) \qquad (*)$$

for every appropriate a and c.

Proof:

$$\sum_{n=1}^{\infty} \frac{a^n}{(n+c)^n} = \sum_{n=1}^{\infty} a^n \int_0^{\infty} \frac{t^{n-1}}{(n-1)!} e^{-t(n+c)} dt = a \int_0^{\infty} e^{-t(c+1)+ate^{-t}} dt = a \int_0^1 t^{c-at} dt.$$

Even more,

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{\Gamma(n+b)}{(n+c)^{n+b}} = \sum_{n=0}^{\infty} \frac{a^n}{n!} \int_0^{\infty} t^{n+b-1} e^{-t(n+c)} dt = \int_0^{\infty} t^{b-1} e^{-ct+ate^{-t}} dt.$$

You can use the fact $(*)$ in more general way:

$$\sum_{n=1}^{\infty} \frac{a^n}{(n+c)^n} \int_p^q t^{n-1} f(t) dt = a \int_p^q f(t) \int_0^1 x^{c-atx} dx dt = a \int_0^1 x^c \int_p^q f(t) e^{-atx \ln x} dt dx.$$

So, if you have closed-form expressions for every natural n and every real c of integrals $\int\limits_p^q t^{n-1} f(t) dt$ and $\int\limits_p^q f(t) e^{-ct} dt$, you have the closed-form integral expression of the sum

$$\sum_{n=1}^{\infty} \frac{a^n}{(n+c)^n} \int\limits_p^q t^{n-1} f(t) dt.$$

One can use it, for example, to show that

$$\sum_{n=1}^{\infty} \frac{a^n \Gamma(n+b)}{(n+c)^n} = a\Gamma(1+b) \int\limits_0^{\infty} \frac{e^{-x(c-b)}}{(e^x - ax)^{b+1}} dx.$$

That's why, for example,

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} = \int_0^1 \frac{dx}{(1 + x \ln x)^2}$$