# TWO RESULTS ON ZFC: (1) IF ZFC IS CONSISTENT THEN IT IS DEDUCTIVELY INCOMPLETE, (2) ZFC IS INCONSISTENT

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Abstract. The Zermelo-Fraenkel-Axiom-of-Choice (ZFC) system of axioms for set theory appears to be inconsistent. A step in developing this proof is the observation that ZFC would be deductively incomplete if it were consistent. Both points are proven by means of the singleton. The axioms are still too lax on the notion of a 'well-defined set'.

Contents



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## 1. Introduction

1.1. **The problem.** The Zermelo-Fraenkel-Axiom-of-Choice (ZFC) system of axioms for set theory is studied here with a focus on the singleton. Section 2 defines the case. Section 3 shows deductive incompleteness, i.e. that there is a truth that cannot be derived. Section 4 derives this truth and thus shows inconsistency. Section 5 discusses the results. Section 6 concludes.

This introduction proceeds with basic definitions and theorems.

1.2. **Definition of ZFC.** We take our definitions from a matricola course in set theory at Leiden and Delft.

**Definition 1.1.** (Coplakova et al. (2011:18), I.4.7): Let $A$ be a set. The power set of $A$ is the set of all subsets of $A$. Notation: $P[A]$. Another notation is $2^A$, whence its name.

(Coplakova et al. (2011:144-145)): ZFC.

*Remark.* This includes the axiom that each set has a power set. (POW)

**Definition 1.2.** The *Axiom of Separation* (Coplakova et al. (2011:145), inserting here a by-line on freedom) is: If $A$ is a set and $\gamma[x]$ is a formula with variable $x$, while $B$ is not free in $\gamma[x]$, then there exists a set $B$ that consists of the elements of $A$ that satisfy $\gamma[x]$:

$$(\forall A)(\exists B)(\forall x)\ ((x \in B)\ \Leftrightarrow\ ((x \in A)\ \wedge\ \gamma[x]))\ \ \text{(SEP)}$$

*Remark.* This is also called an axiom-schema since there is no quantifier on $\gamma$.

1.3. **Cantorian sets in ZFC.**

**Theorem 1.3.** *(Existence.) Let $A$ be a set, $P[A]$ its power set. For every function $f: A \to P[A]$ there is a set $\Psi = \{x \in A \mid x \notin f[x]\}$.*

*Proof.* (i) $P[A]$ exists because of the Axiom of the Power set. (ii) $f$ can be regarded as a subset of $A \times P[A]$, and $f$ exists because of Axiom of Pairing. (iii) $\Psi$ exists because of the Axiom of Separation. □

*Remark.* Find $\Psi \subseteq A$, and thus $\Psi \in P[A]$. Observe that $\Psi$ depends upon $f$, i.e. $\Psi = \Psi[f]$. When $\alpha \in A$ then we can use $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$.

**Definition 1.4.** Above $\Psi = \{x \in A \mid x \notin f[x]\}$ is called a strictly Cantorian set. A generalized Cantorian set has $x \notin f[x]$ as part of its definition. The meaning of 'Cantorian set' without qualification depends upon the context.

**Theorem 1.5.** *(Weakest Conjecture on strictly Cantorian sets.) Let $A$ be a set. For every $f: A \to P[A]$ there is a $\Psi \in P[A]$ such that for all $\alpha \in A$ it holds that $\Psi \neq f[\alpha]$.*

*Proof.* Define $\Psi = \{x \in A \mid x \notin f[x]\}$. Take $\alpha \in A$. Check the two cases:

Case 1: $\alpha \in \Psi$. In this case $\alpha \notin f[\alpha]$. Thus $\Psi \neq f[\alpha]$. (We have $\alpha \in \Psi \setminus f[\alpha]$.)

Case 2: $\alpha \notin \Psi$. In this case $\alpha \in f[\alpha]$. Thus $\Psi \neq f[\alpha]$. (We have $\alpha \in f[\alpha] \setminus \Psi$.) □

*Remark.* This theorem combines the definition of strictly Cantorian sets, the existence proof and an identification of their key property. It is essentially a rewrite of $\forall \alpha \in A : (\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$, deducible from $\forall \alpha : (\alpha \in \Psi) \Leftrightarrow (\alpha \in A \wedge \alpha \notin f[\alpha])$.

1.4. **About the appendices.** This paper leans more on logic than set theory. The author is no expert on ZFC but wrote a book on elementary logic, see Colignatus (1981, 2007, 2011), *A Logic of Exceptions* (ALOE), and see for background Colignatus (2013). This present paper is derived from a discussion of a condition inspired by Paul of Venice (1369-1429), see Colignatus (2014b, 2015) (PV-RP-CDA-ZFC). ALOE in 1981 applied the Paul of Venice consistency condition to the Russell set (p129), and applied it in 2007 (p239) also to Cantor's (diagonal) argument, in Russell's version, i.e. for the power set. ALOE's discussion may be seen as intermediate between naive set theory and this present paper. ALOE does not develop the ZFC system of axioms for set theory. *Appendix A* discusses the versions of ALOE, for proper reference. *Appendix B* has more on the genesis of this paper.

## 2. The singleton

2.1. **The singleton with a nutshell link between Russell and Cantor.** Let $A$ be a set with a single element, $A = \{\alpha\}$. Thus $P[A] = \{\emptyset, A\}$. Let $f : A \to P[A]$.

If $f[\alpha] = \emptyset$ then $\alpha \notin f[\alpha]$. If $f[\alpha] = A$ then $\alpha \in f[\alpha]$.

Thus $(f[\alpha] = \emptyset) \Leftrightarrow (\alpha \notin f[\alpha])$. Consider:

(1) In steps: define $\Psi = \{x \in A \mid x \notin f[x]\}$, then try $f[\alpha] = \Psi$.

(2) Directly: $f[\alpha] = \{x \in A \mid x \notin f[x]\}$.

(3) Either directly or indirectly via (1) or (2): $\Psi = \{x \in A \mid x \notin \Psi\}$.

The latter is a variant of Russell's paradox: $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin \Psi)$. Thus (1) - (3) are only consistent when $\Psi \neq f[\alpha]$. This is an instance of Theorem 1.5.

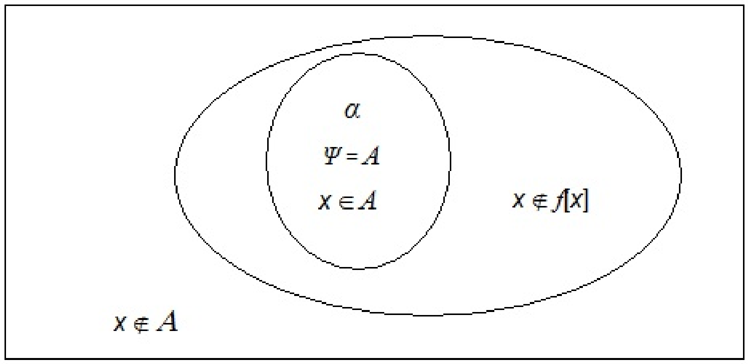
Choosing $f[\alpha] = \Psi$ in (1) assumes freedom that conflicts with the other properties. We have liberty to choose $f[\alpha] = \emptyset$ or $f[\alpha] = A$. This choice defines $f$ and we should write $\Psi = \Psi[f]$ indeed. This shows why (2) with $f[\alpha] = \Psi[f]$ is tricky. If (2) is an implicit definition of $f$ then it doesn't exist. If it exists then this$f[\alpha]$ will not be in its definition.

2.2. **Possibilities for the singleton.** Checking all possibilities in the former subsection gives Table 1. The cells are labeled with $\Delta$-*case-numbers*. The $\Delta$ refers to a *difference analysis* when a set is extended with single element. Because of $\alpha \notin f[\alpha]$, the row $f[\alpha] = \emptyset$ is important for us. The case of $\Delta 2$ is depicted in a Venn-diagram in Figure 2.1.

Table 1. Possibilities for $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$, given that $\alpha \in A$

| For all cases: $\alpha \in A$ | $\Psi = \emptyset,\ \alpha \notin \Psi$ | $\Psi = A,\ \alpha \in \Psi$ |
|---|---|---|
| $f[\alpha] = \emptyset$ | $\Delta 1 : \alpha \in \emptyset \Leftrightarrow \alpha \notin \emptyset$ | $\Delta 2 :\ \alpha \in A \Leftrightarrow \alpha \notin \emptyset$ |
| $\alpha \notin f[\alpha]$ | $f[\alpha] = \Psi$, *impossible* | $f[\alpha] \neq \Psi$, possible |
| $f[\alpha] = A$ | $\Delta 3 : \alpha \in \emptyset \Leftrightarrow \alpha \notin A$ | $\Delta 4 : \alpha \in A \Leftrightarrow \alpha \notin A$ |
| $\alpha \in f[\alpha]$ | $f[\alpha] \neq \Psi$, possible | $f[\alpha] = \Psi$, *impossible* |

FIGURE 2.1. Cantorian set for the singleton, case $\Delta 2$: $f[\alpha] = \emptyset \neq \Psi$




## 3. Deductive incompleteness

3.1. **Existence of $\Delta 1$.** An idea is that $\Psi$ in Theorems 1.3 and 1.5 or Table 1 covers all $\alpha \notin f[\alpha]$. This appears to be false: it doesn't cover $\Delta 1$. The cell is declared *impossible*. Let us first verify that it exists as a truth (outside of ZFC), and then accept deductive incompleteness.

**Theorem 3.1.** *Case $\Delta 1$ exists as a possibility with $\alpha \notin f[\alpha]$.*

*Proof.* We consider the case $f[\alpha] = \emptyset$ so that $\alpha \notin f[\alpha]$. Take $q = (\alpha \notin f[\alpha]) = (\alpha \notin \emptyset)$ and use tautology T1: $\forall p, q: \ q \Rightarrow (p \Leftrightarrow (q \wedge p))$, see Table 2. We are free to take $p = (\alpha \in A)$ which would give $\Delta 2$, or $p = (\alpha \in \emptyset)$ which would give $\Delta 1$. Take the latter, apply modus ponens on $q$ and tautology T1, and find $(\alpha \in \emptyset) \Leftrightarrow (\alpha \notin \emptyset \wedge \alpha \in \emptyset)$. The equivalence reduces into $\alpha \notin \emptyset$ or $\alpha \in A$. The equivalence is by itself consistent, so that it is possible for $\alpha \in \{\alpha\}$. Case $\Delta 1$ with both $f[\alpha] = \emptyset$ and $\Psi = \emptyset$ fits this equivalence: $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha] \wedge \alpha \in \Psi)$. We merely establish possibility, and thus the deduction stops here. □

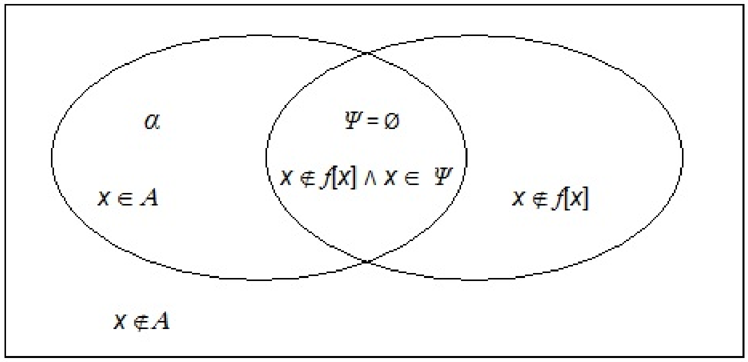
*Remark.* We are tempted to derive inconsistency now, but for understanding of the situation it is better to formally establish deductive incompleteness. See Section 5.3 for existence of $\Delta 1$ based upon other axioms than ZFC.

TABLE 2. Truthtable for T1: $q \Rightarrow (p \Leftrightarrow (q \wedge p))$ with $q = (\alpha \notin f[\alpha])$

| Case | $\alpha \notin f[\alpha]$ | $\Rightarrow$ | $(p$ | $\Leftrightarrow$ | $(\alpha \notin f[\alpha]$ | $\wedge$ | $p))$ |
|---|---|---|---|---|---|---|---|
| $\Delta 2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\Delta 4$ | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| $\Delta 1$ | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| $\Delta 3$ | 0 | 1 | 0 | 1 | 0 | 0 | 0 |

The case shows up by requiring that *all* properties of the case hold, thus jointly $(\alpha \notin \emptyset \wedge \alpha \in \emptyset)$ and not only $(\alpha \notin \emptyset)$. See Figure 3.1. When we test $\alpha \in \Psi$, in this approach for $\Delta 1$ or in the figure, then we test $\alpha \notin f[\alpha] \wedge \alpha \in \Psi$ jointly. From this joint test the decision follows that $\alpha \notin \Psi$, or $\alpha \in A$.

FIGURE 3.1. Cantorian set for the singleton, case $\Delta 1$: $f[\alpha] = \emptyset = \Psi$




### 3.2. **Definition, theorem and proof.**

**Definition 3.2.** (DeLong (1971:132)): "A formal system is deductively complete if under the intended interpretation there is no truth which is not also a theorem."

**Theorem 3.3.** *If ZFC is consistent then it is deductively incomplete.*

*Proof.* Let $A = \{\alpha\}$, with a single element. Thus $P[A] = \{\emptyset, A\}$. Let $f : A \to P[A]$ with $f[\alpha] = \emptyset$. Then $\alpha \notin \emptyset$ and $\alpha \notin f[\alpha]$. Under the intended interpretation, there is the case $\Delta 1$ that has $\alpha \notin f[\alpha]$. $\Psi$ is formulated such that it should contain *all* cases with $\alpha \notin f[\alpha]$. However, trying to prove that $\Delta 1$ fits $\Psi$, causes a shift to $\Delta 2$ or $\Psi = A$ (Theorem 1.5). If ZFC is consistent then there is no path to $\Delta 1$. □

*Remark.* If there is such a path then ZFC becomes inconsistent.

## 4. INCONSISTENCY

4.1. **An implication for the singleton Cantorian set.** For the singleton we have $\alpha \in A$, and thus we have $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$. It is possible to weaken this by means of another tautology T2: $\forall p, q : \ (p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \wedge p))$. The truthtable for the singleton Cantorian set is in Table 3. The truthtable holds for every $f$ while $\Psi = \Psi[f]$.

TABLE 3. Truthtable for T2: $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \wedge p))$, applied to the singleton Cantorian set

| Case | $(\alpha \in \Psi$ | $\Leftrightarrow$ | $\alpha \notin f[\alpha])$ | $\Rightarrow$ | $(\alpha \in \Psi$ | $\Leftrightarrow$ | $(\alpha \notin f[\alpha]$ | $\wedge$ | $\alpha \in \Psi))$ |
|---|---|---|---|---|---|---|---|---|---|
| $\Delta 2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\Delta 4$ | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| $\Delta 1$ | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 |
| $\Delta 3$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |

Consider again $f[\alpha] = \emptyset$. The equivalence on the LHS only allows $\Psi = A$. Look at row $\Delta 1$ in Table 3. On the LHS we have $\Delta 1$ with $(\alpha \notin \Psi) \wedge (\alpha \notin f[\alpha])$, and the equivalence would declare this combination impossible. However, there is also the relaxed condition on the RHS, that we already encountered in Theorem 3.1.

The crucial step is to distill the RHS from the table. Theorems 1.3 and 1.5 establish the LHS. Modus ponens with T2 gives the RHS as a separate expression FT2 ('from T2') - provided that we maintain $\Psi = \Psi[f]$:

$$(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha] \wedge \alpha \in \Psi) \quad \text{(FT2)}$$

For FT2 we get Table 4. The same $\Delta$-case-numbers apply. Now $\Delta 1$ is allowed too: a possible $f[\alpha] = \Psi$.

TABLE 4. Possibilities for $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha] \wedge \alpha \in \Psi)$, given $\alpha \in A$

| Always: $\alpha \in A$ | $\Psi = \emptyset,\ \alpha \notin \Psi$ | $\Psi = A,\ \alpha \in \Psi$ |
|---|---|---|
| $f[\alpha] = \emptyset$ | $\Delta 1 : \alpha \in \emptyset \Leftrightarrow (\alpha \notin \emptyset \wedge \alpha \in \emptyset)$ | $\Delta 2 :\ \alpha \in A \Leftrightarrow (\alpha \notin \emptyset \wedge \alpha \in A)$ |
| $\alpha \notin f[\alpha]$ | $f[\alpha] = \Psi$, $\alpha \notin \emptyset$, *possible* | $f[\alpha] \neq \Psi$, possible |
| $f[\alpha] = A$ | $\Delta 3 : \alpha \in \emptyset \Leftrightarrow (\alpha \notin A \wedge \alpha \in \emptyset)$ | $\Delta 4 : \alpha \in A \Leftrightarrow (\alpha \notin A \wedge \alpha \in A)$ |
| $\alpha \in f[\alpha]$ | $f[\alpha] \neq \Psi$, possible | $f[\alpha] = \Psi$, *impossible* |

Note that $f[\alpha] = \emptyset$ doesn't give a unique $\Psi$ now. Both $\Psi = \emptyset$ (Figure 3.1) and $\Psi = A$ (Figure 2.1) are possible. Note that $f$ is still a function and no correspondence.

PM. We can also gain access to $\Delta 4$ by another relaxing condition but we are interested in the $\alpha \notin f[\alpha]$ case.

4.2. **A counterexample for Theorem 1.5.** Let us make the latter observations formal. The discovery of $\Delta 1$ and tautology T2 gives a contradiction to Theorem 1.5. While Theorem 3.3 did not see a path towards $\Delta 1$, we now found that path, namely tautology T2, which gives Theorem 4.1. When $\Delta 1$ not merely exists as a truth outside of ZFC (using tautology T1) but also can been proven from $\Psi$ (using tautology T2), it becomes a counterexample for Theorem 1.5.

**Theorem 4.1.** *For the singleton Cantorian case there are a $f$ and $\Psi$ with $f[\alpha] = \Psi$.*

*Proof.* Let $A = \{\alpha\}$ have a single element. Thus $P[A] = \{\emptyset, A\}$. Let $f : A \to P[A]$ with $f[\alpha] = \emptyset$. Then $\alpha \notin \emptyset$ and $\alpha \notin f[\alpha]$.

Consider $\Psi = \{x \in A \mid x \notin f[x]\}$. Since $\alpha \in A$ we can use $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$. Look at Table 3. Use tautology T2 and modus ponens, and find FT2: $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha] \wedge \alpha \in \Psi)$. In this deduction we have maintained the definition of $\Psi$. The modus ponens is independent of the possibility that also $\Psi = A$ might be derived via another route. Thus FT2 *stands as a separate relation* for $\Psi$.

A substitution of $f[\alpha] = \emptyset$ and $\Psi = \emptyset$ into FT2 gives case $\Delta 1$: $\alpha \in \emptyset \Leftrightarrow (\alpha \notin \emptyset \wedge \alpha \in \emptyset)$, that we saw above in Theorem 3.1 and subsequently in Table 4, which reduces to $\alpha \notin \emptyset$, or $\alpha \in A$. The case is consistent by itself, and thus we have established a path to it. The case has $f[\alpha] = \emptyset = \Psi$. □

**Theorem 4.2.** *ZFC is inconsistent.*

*Proof.* For the singleton, Theorems 1.3 and 1.5 generate that $\Psi = A$. Theorem 4.1 generates the possibility that $\Psi = \emptyset$. Thus it is possible that $A = \emptyset$. This is a clear contradiction. □

## 5. Discussion

5.1. **Nominalism versus realism.** This paper deals with self-reference and derives a contradiction. It may thus be difficult to follow. The reader can maintain clarity by holding on to the key notion of freedom of definition. When a restriction on that freedom generates a consistent framework, while release of that restriction generates confusion, then the restriction is to be preferred above too much freedom. Amendment of ZFC thus will tend to reduce the freedom of definition, unless one allows for a three-valued logic that is strong enough to recognise nonsense.

We can look at Table 1 and Table 4 in horizontal or vertical direction. This reflects the schism in philosophy between *nominalism* and *realism*. (See William of Ockham.)

(1) The horizontal view gives the *realists* who take predicates as 'real': $\alpha \notin f[\alpha]$ versus $\alpha \in f[\alpha]$. They are also sequentialist: $\Delta 1$ & $\Delta 2$ versus $\Delta 3$ & $\Delta 4$.

(2) The vertical view gives the nominalists (Occam) who regard the horizontal properties as mere stickers, and who more realistically look at $\Psi = \emptyset$ versus $\Psi = A$. They see the table in *even* versus *uneven* fashion: $\Delta 2$ & $\Delta 4$ versus $\Delta 1$ & $\Delta 3$.

The nominalist reasoning is: The sets $\emptyset$ and $A$ exist. We are merely discussing how they are referred to. The expression for $\Psi$ is not a *defining* statement but a *derivative* observation. Once the functions have been mapped out, the criteria can be used to see whether the underlying sets may get also another sticker $\Psi$. We are discussing *'consistent referring'* and not existence.

At issue is now whether ZFC has sufficient logical strength to block nonsensical situations. ZFC has a realist bend. It translates predicates into sets (their extensions). Instead we better employ an axiomatic system to only *test* whether a predicate is useful. Merely cataloguing differently what already exists should not be confused with existence itself. The freedom of definition can be a mere illusion and then should not be abused to create nonsense.

A relation for $\Psi$ on the LHS of Table 3 results via this tautology into a weaker relation on the RHS that contradicts this relation.

The problem with Theorems 1.3 and 1.5 is that they *impose* the equivalence. This assumes a freedom of definition, whence this assumes that the truthtable on the LHS is true, whence $\Delta 1$ is forbidden. But that freedom of definition does not exist. Something exists, that is infringed upon by the definition. When $\Psi$ is the empty set, as in the singleton possibility of Figure 3.1, then one no longer has the freedom to switch from $\emptyset$ to $A$, as in Figure 2.1.

The discussion is not without consequence, see PV-RP-CDA-ZFC:

> The logical construction $x \notin f[x]$ and only a single problematic element, in badly understood self-reference, should not be abused to draw conclusions on the infinite. There are ample reasons to look for ways how this can be avoided.

5.2. **Diagnosis, and an axiom for a solution set.** ZFC blocks Russell's paradox by the Axiom of Separation (SEP). When the paradoxical $\gamma[x] = (x \notin x)$ is separated from $A$ to create some set $R$ then the conclusion follows that $R \notin A$, so that the separation cannot be achieved. For the Cantorian set we use $\alpha \in A$, or for the singleton $\alpha \in \{\alpha\}$, so that we can use $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$, and there is no separation escape anymore. Separation is an irrelevant solution concept here, and what is at issue is self-reference that requires a fundamental solution.

The diagnosis is that $\Psi$ is rather a variable (name) than a constant. There is a solution set $\Psi^*=\{\emptyset, A\}$, and $\Psi$ is a variable that runs over $\Psi^*$. Compare to algebra, when one uses a variable $x$ with value $x = 2$ in one case and $x = 4$ in another case: then one might derive $2 = x = 4$, but this goes against the notion of a variable. The inconsistency in ZFC is caused by that it does not allow for that $\Psi$ is such a variable.

The following is not in ZFC but will help to understand ZFC.

**Definition.** An *Axiom of a Solution Set* (this paper) might be:

$$(\forall A)(\exists Z)(\exists B)((B \in Z) \Leftrightarrow (\forall x)((x \in B) \Leftrightarrow ((x \in A) \wedge \gamma[x]))) \quad \text{(SOL)}$$

This SOL could reduce to the Axiom of Separation (SEP). A way is to eliminate $B \in Z$ as superfluous, with $Z = \{B\}$, or self-evident (which it apparently isn't). Another way is to replace $B \in Z$ by $B = Z$. This imposes uniqueness. When $\gamma[x]$ has more solutions then a contradiction arises when SEP requires that a single solution $B$ is also the whole set $Z$. ZFC has the latter effect.

For the singleton $A = \{\alpha\}$ with $f[\alpha] = \emptyset$, Theorem 1.5 finds $B = \Psi = A$ but we find $Z = \{\emptyset, A\} = P[A]$. In itself it is true that $\Psi \in P[A]$, but when $Z = P[A]$ then it is erroneous to require $Z \in P[A]$.

It is not just an issue of notation. It is not sufficient to suggest to read Theorem 1.5 now as generating a value for the variable, rather than restricting the solution set to that value. For, then one reads something into SEP which it does not do: for it really restricts that solution set.

In ZFC $\Psi$ creates the illusion of a unique set, and thus we need amendment of ZFC to correct that. One might hold that Theorems 1.3 and 1.5 are not necessarily wrong, since one can find for any $f$ a $\Psi[f]$ such that for all $\alpha$: $f[\alpha] \neq \Psi$. (For the singleton $f[\alpha] = \emptyset$ gives $\Psi = A$.) But the formula of $\Psi$ allows Theorem 4.1 to also find another case with $f[\alpha] = \Psi$. (For the singleton $f[\alpha] = \emptyset$ also gives $\Psi = \emptyset$.) One can conceive that the two options co-exist, but Theorem 1.5 does not allow for $\Psi = \emptyset$. Thus Theorem 4.1 is a real counterexample for Theoreom 1.5. The freedom of definition used in Theorem 1.5 depends upon the existence Theorem 1.3. Then something is wrong with Theorem 1.3, that proved the existence of what 1.5 uses. The theorems were derived in ZFC. Thus ZFC has a counterexample and thus is inconsistent.

Again, consider $f[\alpha] = \emptyset = \Psi$ ($\Delta 1$). This is consistent, but cannot be seen directly by $\Psi$, even though it is covered in Table 3 by the falsehood of $\alpha \in \Psi$. In a realist mode of thought, we deduce from $f[\alpha] = \emptyset$ that $\Psi = A$, which is the only possibility on the LHS for $\alpha \notin f[\alpha]$ that $\Psi$ recognises (row $\Delta 2$). This is not necessarily the proper response. The problem with ZFC is that it focuses on the LHS and neglects the RHS. We can derive a relaxed condition FT2 and then Theorem 4.1 allows to recover $\Delta 1$. The latter deductions are actually within ZFC and thus there is scope to argue that Theorem 1.5 presents only part of the picture. However, that part is formulated in such manner that it causes the contradiction in Theorem 4.2. We must switch to a better axiomatic system that covers the *intended interpretation* and that blocks the paradoxical $\Psi$. The better system blocks the LHS and allows only the RHS.

While this paper has a destructive flavour on ZFC, it is actually constructive since it indicates what the improvement will be. See PV-RP-CDA-ZFC.

5.3. **Logical structure of this paper.** The inconsistency shows itself in Table 3 with two cases on the LHS for Theorem 1.5 and three cases on the RHS for Theorem 4.1. In itself it might be possible to use only this table and forget about deductive incompleteness. However, it is useful to build up understanding by first explaining such existence by the use of tautology T1.

The idea that $\Psi$ in Theorems 1.3 and 1.5 or Table 1 covers all $\alpha \notin f[\alpha]$ appears to be false: it doesn't cover $\Delta 1$. Thus when $\alpha \notin \Psi$ then there still exists a case of $\alpha \notin f[\alpha]$. Now, isn't $\Psi$ *supposed* to cover *all* such cases ? The conclusion is: Theorems 1.3 and 1.5 do not cover the *intended interpretation* (DeLong (1971)). However: since Theorem 4.1 deducts this neglected truth, and still is in ZFC, ZFC becomes inconsistent, can prove everything, and the notion of deductive incompleteness loses meaning. Looking at only consistency would cause us to lose sight of Theorems 3.1 and 3.2.

Since Theorems 1.3 and 1.5 are well accepted in the literature and Theorem 4.1 is new, there is great inducement to find error in Theorem 4.1. Indeed, Theorem 4.1 allows the deduction of a contradiction in Theorem 4.2, and thus one might hold that it should go. However, its steps are correct. It is more productive for the reader to accept inconsistency of ZFC.

A discussion about self-reference that identifies a contradiction is always difficult to follow. The problem lies not in the identification of the logical framework of the situation but in the inconsistency of ZFC. Potentially the distinction between constant and variable has most effect for clarity. But it also helps to see the distinct role of the two tautologies T1 and T2.

These tautologies were found to be useful following an analysis that was inspired by what Paul of Venice (1369-1429) wrote on the Liar paradox. The tautologies generate an amendment to SEP that gives SEP-PV. If SEP in ZFC is replaced by SEP-PV then we get ZFC-PV. In this new system of axioms, Table 1 can no longer be derived, but Table 4 can. This is another way to see that $\Delta 1$ is a truth that would exist outside of ZFC if it would not be inconsistent.

5.4. **Further reading.** This paper is a rewrite of sections of PV-RP-CDA-ZFC version June 17 2015. See there for a longer discussion and a link to Cantor's Theorem and the transfinites, and suggestions for new axioms for set theory. CCPO-PCWA deproves Cantor's original proofs of 1874 and 1890/91, and presents the notion of *bijection by abstraction*. Though ALOE presents a course in elementary logic, it contains various innovations that are relevant but little-known. The common self-referential logical paradoxes have solution methods: (1) the theory of types (levels), (2) proof theory, (3) three-valued logic with *true, false* and *nonsense*. ALOE shows that the latter is consistent and superior, with a way to deal with the three-valued-Liar as well.

It would not be a solution to repair ZFC in such a way that the transfinites would be saved, since they are a figment of $x \notin f[x]$ confusions, and there is no intended interpretation for them outside of those Cantorian confusions.

If one holds that ZFC is consistent, against all logic, then there still is the issue of the transfinites in terms of modeling. Those make one wonder what ZFC is a model for. We can agree with Cantor that the essence of mathematics lies in its freedom, but the freedom to create nonsense would no longer be mathematics. CCPO-PCWA defines the notion of *bijection by abstraction,* and this definition conflicts with transfinites as generated by ZFC.

## 6. Conclusion

(1) If ZFC is consistent then it is deductively incomplete. (Via tautology T1.)
(2) ZFC is inconsistent. (Via T2. See PV-RP-CDA-ZFC for alternatives.)

*Acknowledgements.* Let me repeat my gratitude stated in the other papers CCPO-PCWA and PV-RP-CDA-ZFC. For this paper, I thank Richard Gill (Leiden) for various discussions, and Klaas Pieter Hart (Delft) over 2011-2015 and Bas Edixhoven (Leiden) in 2014 for some comments and for causing me to look closer at ZFC. Jan Bergstra (Amsterdam) gave the final inducement to select only ZFC from PV-RP-CDA-ZFC. Hart and Edixhoven apparently have missed the full analysis and take a 'Cantorian position'. I am sorry to have to report a breach in scientific integrity, see Colignatus (2015e). All errors remain mine.

## 7. Appendix A: Versions of ALOE

The following comments are relevant for accurate reference.

(1) Colignatus (1981, 2007, 2011) (ALOE) existed first unpublished in 1981 as *In memoriam Philetas of Cos,* then in 2007 it was rebaptised and self-published. It was both retyped and programmed in the computer-algebra environment of *Mathematica* to allow ease of use of three-valued logic. In 2011 it was marginally adapted with a new version of *Mathematica.*

(2) Gill (2008) reviewed the 1st edition of ALOE of 2007. This edition refers to Cantor's standard set-theoretic argument on the power set and rejects it.

(3) Gill (2008) did not review the 2nd edition of ALOE of 2011. This edition also refers to Cantor's original argument on the natural and real numbers in particular. This edition mentions the suggestion that $\mathbb{N} \sim \mathbb{R}$. The discussion itself is not in ALOE but is now in Colignatus (2012, 2013) (CCPO-PCWA). The latter also presents the notion of *bijection by abstraction.* See Colignatus (2015af) on abstraction.

(4) ALOE is a book on logic and not a book on set theory. It presents the standard notions of naive set theory (membership, intersection, union) and the standard axioms for first order predicate logic that of course are relevant for set theory. But I have always felt that discussing *axiomatic* set theory (with ZFC) was beyond the scope of the book and my actual interest and developed expertise. This present paper is in my sentiment rather exploratory.

## 8. Appendix B: On the genesis of this paper

Colignatus (2013) explains my background and *Appendix A* explains about ALOE. It is joy to see that basic propositional logic still is so useful to resolve the issue of this paper. It is quite conceivable that ZFC theorists simply don't have this affinity with both logic and empirical science that I can advise to every student.

This paper uses the literature reference style of *Econometrica* (author (year)), which is more informative than plain numbers.

Theorem 1.3 is a reformulation of the addendum provided by B. Edixhoven, statement in Colignatus (2014a), its appendix D.

Theorem 1.5 was given by K.P. Hart (TU Delft), 2012, in Colignatus (2015b).

A visit to a restaurant in October 27 2014 and subsequent e-mail exchange with Edixhoven (Leiden), co-author of Coplakova et al. (2011), led to the memos Colignatus (2014ab), and the inspiration to write about ZFC. Originally I asked

Edixhoven the question on the relation between Cantorian $\Psi$ and Pauline $\Phi$ (see PV-RP-CDA-ZFC). Edixhoven agreed that the Pauline consistency condition should have no effect, and I asked him to explain that it could have an effect. Since November 2014, see Colignatus (2014ab), I have not received a response even though the question was clear and articulate. Hart (Delft), who has invested deeply into the transfinites, apparently rejects the usefulness of these questions. Having seen ZFC more often in the course of these exchanges, I decided on the morning of Wednesday May 27 2015 to provide for the answers myself, and established the singleton case before noon. The rest is didactics. Advised reading is Colignatus (2015e).

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