Quaternionic versus Maxwell based differential calculus

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Abstract

Two quite different forms of differential calculus exist that both have physical significance. The most simple version is quaternionic differential calculus. Maxwell based differential calculus is based on the equations that Maxwell and others have developed in order to describe electromagnetic phenomena. Both approaches can be represented by four-component “fields” and four-component differential operators. Both approaches result in a dedicated non-homogeneous second order partial differential equation. These equations differ and offer solutions that differ in details.

Maxwell based differential calculus uses coordinate time *t*, where quaternionic differential calculus uses proper time τ. The consequence is that also the interpretation of speed differs between the two approaches. A more intriguing fact is that these differences involve a different space-progression model and different charges and currents. The impacts of these differences are not treated in this paper.

By adding an extra Maxwell based differential equation the conformance between the two approaches increases significantly.

The formulation of physics in Maxwell based differential calculus differs significantly from the formulation of physics in quaternionic differential calculus. It results in a different space-progression model. The choice between the two approaches influences the description of physical reality. However, the selected formulation does not affect physical reality. The description does not affect the described field.

The conclusion of the paper is that depending on the type of investigated phenomena either the Maxwell based approach or the quaternionic approach fits better as a descriptor. The Maxwell based approach fits better for describing wave behavior. The quaternionic approach fits better for the description of the embedding process.

Quaternionic differential calculus also fits better with the application of Hilbert spaces in quantum physics than Maxwell based differential calculus does. However, Maxwell based differential calculus is the general trend in current physical theories.

Contents

1 Introduction

In this paper the quaternionic differential equations are compared to Maxwell based differential equations [1][2].

In order to ease the comparison of the two approaches, we apply four-component “fields” and four-component operators. The parameter space is represented in a similar way by a similar but flat four-component “field”.

We start with a four-component differentiable “field” φ and we also define the corresponding four-component differential operator ∇. This nabla operator is applicable in situations in which the continuity of the field is not too violently disrupted. We tolerate point-like artefacts that manifest as sources, drains, charges or transient embedding locations.

The four-component approach is sometimes implemented with the help of spinors and corresponding matrices. Here we could, but will not apply that methodology. The method confuses more than that it elucidates the situation. Instead, we consider the scalar part as a separate part and we apply base vectors $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ rather than the corresponding Pauli matrices [3]. For the same reason we do not apply Clifford algebra, Jordan algebra or Grassmann algebra.

The investigated approaches both start with a basic “field” φ. Gravitation concerns applications where this “field” φ is always and everywhere present. This kind of field is suited as continuum for embedding discrete objects. It is also suited as long range transport medium for carriers of information and energy. Electromagnetic theory concerns applications where the existence of the “field” φ is determined by a set of charges in the form of nearby point-like artifacts. These two kinds of basic fields are related, but that is subject of another paper [4].

Double differentiation results in a non-homogeneous second order partial differential equation that reveals how the basic “field” φ can be deformed or vibrated and how the artifacts control the behavior and the existence of the field. This second order partial differential equation differs between the two approaches.

The investigated field φ exists independent of the fact which kind of functions and parameter spaces are used to describe it. The investigated subject is the continuum eigenspace of an operator that resides in a non-separable quaternionic Hilbert space.

2 Notation

Italic font face without subscript indicates four-component “fields” or four-component operators. Bold italic font face indicates 3D vectors and vector functions or 3D operators.

The four-component “fields” consist of a combination of a scalar field and a field of 3D vectors.

$$\varphi = \{\varphi_0,\varphi_1,\varphi_2,\varphi_3\} = \{\varphi_0,\boldsymbol{\varphi}\} = \{\varphi_0, \boldsymbol{i}\,\varphi_1 + \boldsymbol{j}\,\varphi_2 + \boldsymbol{k}\,\varphi_3\} \tag{1}$$

Both approaches start with a basic “field” φ. A set of related “fields” is derived from this basic “field”.

Both approaches use the 3D nabla operator $\boldsymbol{\nabla}$.

$$\boldsymbol{\nabla} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{2}$$

This vector operator can be applied when the deformations of the subjected fields are not too violent.

$$\nabla = \{\nabla_0, \nabla_1, \nabla_2, \nabla_3\} = \{\nabla_0, \boldsymbol{\nabla}\} \tag{3}$$

For example, the four-component "field" ϕ is defined as:

$$\phi = \{\varphi_0, \boldsymbol{\varphi}\} = \nabla\varphi = \{\nabla\varphi_0, \nabla\boldsymbol{\varphi}\} \tag{4}$$

The four-component differential operator differs between the two approaches. Quaternionic differential calculus uses proper time τ and partial derivative $\nabla_0 = \nabla_\tau = \frac{\partial}{\partial\tau}$ and Maxwell based differential calculus uses coordinate time t and partial derivative $\nabla_0 = \nabla_t = \frac{\partial}{\partial t}$.

We suppose that ∇_0 commutes with $\boldsymbol{\nabla}$.

3 Parameter spaces

The parameter space is represented by a four-component flat "field":

$$\{x_0, \boldsymbol{i}\, x_1 + \boldsymbol{j}\, x_2 + \boldsymbol{k}\, x_3\} = \{x_0, \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}\};\ x_0 = \tau \text{ or } x_0 = t \tag{1}$$

Infinitesimal coordinate time steps Δt and infinitesimal proper time $\Delta\tau$ steps are related by:

$$\text{Coordinate time step vector} = \text{proper time step vector} + \text{spatial step vector} \tag{2}$$

Or in Pythagoras format:

$$(\Delta t)^2 = (\Delta\tau)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{3}$$

In the quaternionic model, the formula indicates that the coordinate time step corresponds to the step of a full quaternion, which is a superposition of a proper time step and a perpendicular pure spatial step.

An infinitesimal spacetime step Δs is usually presented as an infinitesimal proper time step $\Delta\tau$.

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \quad (4)$$

The signs on the right side form the Minkowski signature $(+,-,-,-)$.

The quaternionic model offers a Euclidean signature $(+,+,+,+)$ as is shown in formula (3).

4 Definition of the differential

Locally, the deformation of the "field" φ is supposed to be sufficiently moderate, such that the nabla operator ∇ can be applied.

In the two approaches, the differentiations $\{\phi_0, \boldsymbol{\phi}\} = \{\nabla_0, \boldsymbol{\nabla}\}\{\varphi_0, \boldsymbol{\varphi}\}$ have different definitions.

We do not go further than double differentiation. This double differentiation results in a non-homogeneous second order partial differential equation. In the two approaches, the non-homogeneous second order partial differential equations have a different format. In the Maxwell based approach this equation it is known as wave equation.

5 Mathematical facts

5.1 Quaternions

Quaternions are a combination of a real scalar a_0 and a 3D vector $\boldsymbol{a}$, which forms the imaginary part. Quaternionic number systems are division rings. This means that every non-zero element has an inverse. Hilbert spaces can only cope with number systems that are division rings. Quaternionic number systems form the most elaborate division rings.

Continuous quaternionic functions represent skew fields. Quaternionic differential calculus uses proper time τ as progression parameter. For that reason all quaternionic differential equations are inherently Lorentz invariant.

Due to their four dimensions quaternionic number systems exist in 16 symmetry flavors that only differ in their discrete symmetry sets [1][4].

$$a \equiv a_0 + \boldsymbol{a} \quad (1)$$

The quaternionic conjugate is defined as:

$$a^* \equiv a_0 - \boldsymbol{a} \quad (2)$$

The norm is defined as:

$$|a| \equiv \sqrt{a^* a} \quad (3)$$

The norm of a quaternionic function φ is defined as

$$\|\varphi\| \equiv \sqrt{\int_V \varphi^* \varphi \, dV} \quad (4)$$

The quaternionic product is defined as:

$$c = ab = a_0 b_0 + a_0 \boldsymbol{b} + b_0 \boldsymbol{a} - \langle \boldsymbol{a}, \boldsymbol{b} \rangle \pm \boldsymbol{a} \times \boldsymbol{b} \tag{5}$$

The $\pm$ sign indicates the freedom of choice between a left handed and a right handed external vector product. This indicates that quaternionic number systems exist in several versions.

5.2 Special differential equations

In the next equations, α is a real or complex valued scalar function. $\boldsymbol{a}$ is a vector function. Both equation sets use the nabla operator:

$$\boldsymbol{\nabla} \equiv \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} \equiv + \boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z} \tag{1}$$

The following formulas are just mathematical facts that generally hold for vector differential calculus:

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla a} \rangle \equiv \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{a} \tag{2}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \alpha \rangle \equiv \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \alpha \tag{3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \alpha = \boldsymbol{0} \tag{4}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \times \boldsymbol{a} \rangle = 0 \tag{5}$$

$$\langle \boldsymbol{\nabla} \times \boldsymbol{\nabla}, \boldsymbol{a} \rangle = \boldsymbol{0} \tag{6}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) = \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{a} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{a} \tag{7}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \alpha) = \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \alpha \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\nabla} \alpha \tag{8}$$

6 Quaternionic differential calculus

The quaternionic nabla is defined by:

$$\nabla \equiv \left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} \equiv \frac{\partial}{\partial \tau} + \boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z} = \nabla_0 + \boldsymbol{\nabla} \tag{1}$$

$$\nabla^* = \nabla_0 - \boldsymbol{\nabla} \tag{2}$$

In quaternionic differential calculus ***the differential can be defined as a product***.

$$\phi = \nabla\varphi \equiv (\nabla_0 + \boldsymbol{\nabla})(\varphi_0 + \boldsymbol{\varphi}) = \nabla_0\varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle + \boldsymbol{\nabla}\varphi_0 + \nabla_0\boldsymbol{\varphi} \pm \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{3}$$

$$\phi_0 = \nabla_0\varphi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{4}$$

$$\boldsymbol{\phi} = \boldsymbol{\nabla}\varphi_0 + \nabla_0\boldsymbol{\varphi} \pm \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{5}$$

The second derivative delivers a non-homogeneous equation:

$$\zeta = \nabla^*\phi = \nabla^*\nabla\varphi = (\nabla_0 - \boldsymbol{\nabla})(\nabla_0 + \boldsymbol{\nabla})(\varphi_0 + \boldsymbol{\varphi}) \tag{6}$$

$$= \{\nabla_0\nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\}\varphi = \frac{\partial^2\varphi}{\partial\tau^2} + \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

$$\zeta_0 = \nabla_0\phi_0 - \langle \boldsymbol{\nabla}, \boldsymbol{\phi}\rangle \tag{7}$$

$$\boldsymbol{\zeta} = \boldsymbol{\nabla}\phi_0 + \nabla_0\boldsymbol{\phi} \pm \boldsymbol{\nabla} \times \boldsymbol{\phi} \tag{8}$$

Notice that:

$$\nabla_0\phi = \nabla_0\nabla_0\varphi_0 - \nabla_0\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle + \nabla_0\boldsymbol{\nabla}\varphi_0 + \nabla_0\nabla_0\boldsymbol{\varphi} \pm \nabla_0\boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{9}$$

$$-\boldsymbol{\nabla}\phi = -\boldsymbol{\nabla}\nabla_0\varphi_0 + \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\varphi_0 + \nabla_0\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \mp \nabla_0\boldsymbol{\nabla} \times \boldsymbol{\varphi} - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) \tag{10}$$

$$= -\boldsymbol{\nabla}\nabla_0\varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\varphi_0 + \nabla_0\langle \boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \mp \nabla_0\boldsymbol{\nabla} \times \boldsymbol{\varphi} + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\boldsymbol{\varphi}$$

$$\rho_0 = \langle \boldsymbol{\nabla}, \boldsymbol{\phi}\rangle = \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\varphi_0 \tag{11}$$

$$\boldsymbol{\rho} = \boldsymbol{\nabla}\phi_0 \pm \boldsymbol{\nabla} \times \boldsymbol{\phi} = \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\boldsymbol{\varphi} \tag{12}$$

The gauge transformation $\varphi \to \varphi + \chi$, where $\nabla^*\chi = 0$, does not change ϕ in $\phi = \nabla\varphi$.

6.1 Solutions of the quaternionic second order partial differential equation

Solutions of the second order partial differential equation depend on start and boundary conditions. This second order partial differential equation only holds for moderate discontinuity conditions.

$$\nabla^*\nabla\varphi \equiv \{\nabla_0\nabla_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\}\varphi = \frac{\partial^2\varphi}{\partial\tau^2} + \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = \zeta \tag{1}$$

Apart from more detailed conditions the equation can be reduced to special forms. Examples are the Poisson equation, the screened Poisson equation and the homogeneous second order partial differential equation.

6.1.1 Poisson Equations

In the screened Poisson equation the first term is reduced to multiplication with a real constant λ:

$$\nabla_0 \nabla_0 \varphi = -\lambda^2 \, \varphi \tag{1}$$

$$\{-\lambda^2 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\} \varphi = \zeta \tag{2}$$

The corresponding solution is superposition of screened Green's functions.

Green functions represent solutions for point sources.

$$\{-\lambda^2 + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle\} G(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{3}$$

$$\varphi = \iiint G(\boldsymbol{r} - \boldsymbol{r}') \, \zeta(\boldsymbol{r}') \, d^3 \boldsymbol{r}' \tag{4}$$

In this case the Green's function for spherical symmetric conditions ($\boldsymbol{r} = \boldsymbol{0}$) is:

$$G(r) = \frac{\exp(-\lambda \, r)}{r} \tag{5}$$

A zero value of λ offers the normal Poisson equation.

If $\lambda \neq 0$ then equation (1) has a solution

$$\varphi = a(\boldsymbol{x}) \exp(\pm i \, \omega \, \tau); \; \lambda = \pm i \, \omega \tag{6}$$

ω represents a parameter space wide clock frequency.

6.1.2 Coherent swarm of charges

A coherent swarm of charges that can be described by a continuous quaternionic location density function represents a blurred Green's function. For example, in case of an isotropic Gaussian distribution ζ_0 the N contributions of the swarm elements to the integral φ will on average equal $\mathfrak{G}(r) = \mathrm{ERF}(r)/r$. $N \, \mathfrak{G}(r)$ represents the local potential.

6.1.3 The homogeneous quaternionic second order partial differential equation

Despite the fact that the equation is quite similar to a wave equation it does not support waves. Locally, this quaternionic second order partial differential equation is considered to act in a rather flat continuum φ.

$$\nabla^*\nabla\varphi = \nabla_0\nabla_0\varphi + \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\varphi = 0 \tag{1}$$

First we look at:

$$\nabla^*\nabla\varphi_0 = 0 \tag{2}$$

φ_0 is a scalar function. For isotropic conditions in three participating dimensions equation (2) has three dimensional spherical wave fronts as one group of its solutions.

By changing to polar coordinates it can be deduced that a general solution is given by:

$$\varphi_0(r,\tau) = \frac{f_0(\boldsymbol{i}r - c\tau)}{r} = \frac{f_0(\boldsymbol{r} - c\tau)}{|\boldsymbol{r}|} \tag{3}$$

where $c = \pm 1$ and $\boldsymbol{i}$ represents a base vector in radial direction. This solution describes a shape keeping front.

In fact the parameter $\boldsymbol{i}r - c\tau$ of f_0 can be considered as a complex number valued function.

We use

$$\langle\nabla,\nabla\rangle\varphi_0 \equiv \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi_0}{\partial r}\right)\right) = -\frac{f''(\boldsymbol{i}r - c\tau)}{r} = \frac{1}{c^2}\frac{\partial^2\varphi_0}{\partial\tau^2} \tag{4}$$

Next we consider the vector function $\boldsymbol{\varphi}$

$$\nabla^*\nabla\boldsymbol{\varphi} = 0 \tag{5}$$

Equation (5) has one dimensional wave fronts as one group of its solutions:

$$\boldsymbol{\varphi}(z,\tau) = \boldsymbol{f}(\boldsymbol{i}z - c\tau) \tag{6}$$

Again the parameter $\boldsymbol{i}z - c\tau$ of $\boldsymbol{f}$ can be interpreted as a complex number based function. Again the solution describes a shape keeping front, but this time the traveling front also keeps its amplitude.

The imaginary $\boldsymbol{i}$ represents a normalized base vector in the x, y plane. Its orientation θ may be a function of z.

That orientation determines the polarization of the one dimensional wave front.

These solutions do not represent waves. Instead they represent moving fronts that keep their shape.

6.1.4 No dynamic waves

A solution based on

$$\varphi = a(\boldsymbol{x}) \exp(\pm i\, \omega\, \tau) \tag{1}$$

$$\nabla_0 \nabla_0 \varphi = -\omega^2 \varphi \tag{2}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi = \omega^2 \varphi \tag{3}$$

does not lead to dynamic spatial waves. Thus separation of variables does not work well for the quaternionic homogeneous second order partial differential equation.

In order to show waves, a change to another parameter space is required. Instead of parameters τ and x we might select parameter t which is a function of τ and $\boldsymbol{x}$. Taking $t = |x| = |\tau + \boldsymbol{x}|$ will do the job. This delivers a homogeneous (spherical) wave equation of the form:

$$\frac{\partial^2 f}{\partial t^2} - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle f = 0 \tag{4}$$

In contrast to equation (4) the equation

$$\frac{\partial^2 f}{\partial t^2} + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle f = 0 \tag{5}$$

does not describe dynamic waves, but like equation (4) it can describe shape keeping fronts.

7 Maxwell based differential calculus

We know that Maxwell based differential calculus supports a wave equation. As has been indicated above, the quaternionic second order partial differential equation is not suitable as a wave equation. For that reason, we introduce a new variable t, which replaces parameter τ.

We will take $t = |x| = |\tau + \boldsymbol{x}|$.

In Maxwell based differential calculus, the partial differential $\frac{\partial}{\partial t}$ replaces $\frac{\partial}{\partial \tau}$.

The majority of the Maxwell based differential equations are quite similar to the quaternionic differential equations. The differences are very subtle. This fact can be very confusing.

7.1 Maxwell-like equations

We start from the quaternionic differential and use control switch $\alpha = -1$.

$$\phi = \nabla\varphi = (\nabla_t + \boldsymbol{\nabla})(\varphi_0 + \boldsymbol{\varphi}) = \nabla_\tau\varphi_0 - \langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle + \boldsymbol{\nabla}\varphi_0 + \nabla_\tau\boldsymbol{\varphi} \pm \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{1}$$

$$\phi_0 = -\alpha\, \nabla_\tau\varphi_0 - \langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{2}$$

$$\boldsymbol{\phi} = \boldsymbol{\nabla}\varphi_0 + \nabla_\tau\boldsymbol{\varphi} - \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{3}$$

We define new symbols:

$$\mathfrak{E} \equiv -\boldsymbol{\nabla}\varphi_0 - \nabla_\tau\boldsymbol{\varphi} \tag{4}$$

$$\boldsymbol{\mathcal{B}} \equiv \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{5}$$

$$\nabla_\tau\boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times \nabla_\tau\boldsymbol{\varphi} = -\boldsymbol{\nabla} \times \mathfrak{E} \tag{6}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) = \boldsymbol{\nabla}\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle - \langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\boldsymbol{\varphi} \tag{7}$$

$$\nabla_\tau\mathfrak{E} \equiv -\nabla_\tau\boldsymbol{\nabla}\varphi_0 - \nabla_\tau\nabla_\tau\boldsymbol{\varphi} \tag{8}$$

$$\langle\boldsymbol{\nabla}, \mathfrak{E}\rangle = -\langle\boldsymbol{\nabla}, \boldsymbol{\nabla}\rangle\varphi_0 - \nabla_\tau\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{9}$$

$$\nabla_\tau\phi_0 = -\alpha\, \nabla_\tau\nabla_\tau\varphi_0 - \nabla_\tau\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{10}$$

$$\boldsymbol{\nabla}\phi_0 \equiv -\alpha\, \nabla_\tau\boldsymbol{\nabla}\varphi_0 - \boldsymbol{\nabla}\langle\boldsymbol{\nabla}, \boldsymbol{\varphi}\rangle \tag{11}$$

$$\nabla_\tau \phi_0 - \langle \boldsymbol{\nabla}, \mathfrak{E} \rangle = -\alpha\, \nabla_\tau \nabla_\tau \varphi_0 - \nabla_\tau \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 + \nabla_\tau \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{12}$$

$$= (-\alpha\, \nabla_\tau \nabla_\tau + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle) \varphi_0 = \xi_0$$

$$-\boldsymbol{\nabla} \phi_0 + \alpha\, \nabla_\tau \mathfrak{E} - \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} \tag{13}$$

$$= +\alpha\, \nabla_\tau \boldsymbol{\nabla} \varphi_0 + \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \alpha\, \nabla_\tau \boldsymbol{\nabla} \varphi_0 - \alpha\, \nabla_\tau \nabla_\tau \boldsymbol{\varphi} - \boldsymbol{\nabla} \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi}$$

$$= (-\alpha\, \nabla_\tau \nabla_\tau + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle) \boldsymbol{\varphi} = \boldsymbol{\xi}$$

$$\boldsymbol{\phi} = -\mathfrak{E} \mp \boldsymbol{\mathcal{B}} \tag{14}$$

$$(-\alpha\, \nabla_\tau \nabla_\tau + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle) \varphi_0 = -\alpha \frac{\partial^2 \varphi_0}{\partial \tau^2} + \frac{\partial^2 \varphi_0}{\partial x^2} + \frac{\partial^2 \varphi_0}{\partial y^2} + \frac{\partial^2 \varphi_0}{\partial z^2} = \zeta_0 \tag{15}$$

$$= \nabla_\tau \phi_0 + \langle \boldsymbol{\nabla}, \mathfrak{E} \rangle$$

$$(-\alpha\, \nabla_\tau \nabla_\tau + \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle) \boldsymbol{\varphi} = -\alpha \frac{\partial^2 \boldsymbol{\varphi}}{\partial \tau^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial x^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial y^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial z^2} = \boldsymbol{\zeta} \tag{16}$$

$$= -\boldsymbol{\nabla} \phi_0 + \alpha\, \nabla_\tau \mathfrak{E} - \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}}$$

$$-\alpha \frac{\partial^2 \varphi}{\partial \tau^2} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \zeta = \nabla^* \nabla \varphi \tag{17}$$

With $\alpha = -1$, this corresponds to a Euclidean signature.

7.2 Maxwell equations

The Maxwell based formulas that are used here are taken from Bo Thidé; “Electromagnetic field theory”; second edition.

We use these formulas without units. Thus $c = 1;\ \varepsilon_0 = 1; \mu = 1$.

The Maxwell equations use coordinate time t. Just changing parameter τ into variable t, which is a function of τ and $\boldsymbol{x}$, does not affect field φ. It only changes the parameter space and the formulas that describe φ. This means that φ still obeys all the quaternionic partial differential equations, including the second order partial differential equation! With other words, Maxwell equations just offer a different view on field φ. We will use a selector α that will distinguish pure quaternionic differential formulas ($\alpha = -1$) from nearly equivalent Maxwell based differential formulas ($\alpha = +1$).

∇_t stands for $\frac{\partial}{\partial t}$. In quaternionic parameter space, function t plays the role of quaternionic distance $|x|$, where:

$$t = |x| = |\tau + \boldsymbol{x}| \tag{1}$$

In Maxwell equations the symbol $\boldsymbol{E}$ is usually used for the electrical field and symbol $\boldsymbol{B}$ is usually used for the magnetic field. Here we use the special symbols $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ in order to indicate the more general usage.

$$\boldsymbol{\mathcal{E}} \equiv -\boldsymbol{\nabla}\varphi_0 - \nabla_t \boldsymbol{\varphi} \tag{2}$$

$$\boldsymbol{\mathcal{B}} \equiv \boldsymbol{\nabla} \times \boldsymbol{\varphi} \tag{3}$$

In order to support the comparison, we introduce $\varkappa$ as a new scalar field. *This field is not subject of a regular Maxwell equation.*

$$\varkappa \equiv \alpha\, \nabla_t \varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \Leftrightarrow -\phi_0 = \alpha\, \nabla_\tau \varphi_0 + \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{4}$$

In EMFT the scalar field $\varkappa$ is taken as a gauge with

$\alpha = 1$; Lorentz gauge

$\alpha = 0$; Coulomb gauge

$\alpha = -1$; Kirchhoff gauge.

In Maxwell based differential calculus the scalar field $\varkappa$ is ignored or it is taken equal to zero. As will be shown, zeroing $\varkappa$ is not necessary for the derivation of the Maxwell based wave equation.

$$\nabla_t \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times \nabla_t \boldsymbol{\varphi} = -\boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} \tag{5}$$

$$\boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varphi}) = \boldsymbol{\nabla}\langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle - \langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \boldsymbol{\varphi} \tag{6}$$

$$\nabla_t \boldsymbol{\mathcal{E}} \equiv -\nabla_t \boldsymbol{\nabla} \varphi_0 - \nabla_t \nabla_t \boldsymbol{\varphi} \tag{7}$$

$$\langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}} \rangle = -\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \varphi_0 - \nabla_t \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{8}$$

$$\nabla_t \varkappa = \alpha \nabla_t \nabla_t \varphi_0 + \nabla_t \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \Leftrightarrow \nabla_\tau \phi_0 = -\alpha\, \nabla_\tau \nabla_\tau \varphi_0 - \nabla_\tau \langle \boldsymbol{\nabla}, \boldsymbol{\varphi} \rangle \tag{9}$$

$$\boldsymbol{\nabla}\varkappa \equiv \alpha\nabla_t\boldsymbol{\nabla}\varphi_0 + \boldsymbol{\nabla}\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle \Leftrightarrow \boldsymbol{\nabla}\phi_0 = -\alpha\,\nabla_\tau\boldsymbol{\nabla}\varphi_0 - \boldsymbol{\nabla}\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle \tag{10}$$

$$\nabla_t\varkappa + \langle\boldsymbol{\nabla},\boldsymbol{\mathcal{E}}\rangle = \alpha\nabla_t\nabla_t\varphi_0 + \nabla_t\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\varphi_0 - \nabla_t\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle \tag{11}$$

$$= \alpha\nabla_t\nabla_t\varphi_0 - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\varphi_0 = (\alpha\nabla_t\nabla_t - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\varphi_0$$

$$-\boldsymbol{\nabla}\varkappa - \alpha\,\nabla_t\boldsymbol{\mathcal{E}} + \boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}} \tag{12}$$

$$= -\alpha\nabla_t\boldsymbol{\nabla}\varphi_0 - \boldsymbol{\nabla}\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle + \alpha\nabla_t\boldsymbol{\nabla}\varphi_0 + \alpha\nabla_t\nabla_t\boldsymbol{\varphi} + \boldsymbol{\nabla}\langle\boldsymbol{\nabla},\boldsymbol{\varphi}\rangle - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\boldsymbol{\varphi}$$

$$= \alpha\nabla_t\nabla_t\boldsymbol{\varphi} - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle\boldsymbol{\varphi}$$

In quaternionic differential calculus $\alpha = -1$ and $\phi_0 = -\varkappa$. If in Maxwell based differential calculus the Lorentz gauge $\alpha = 1$ is applied, then the Maxwell based wave functions result:

$$(\alpha\,\nabla_t\nabla_t - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\varphi_0 = \rho_0 = \nabla_t\varkappa + \langle\boldsymbol{\nabla},\boldsymbol{\mathcal{E}}\rangle \Leftrightarrow \nabla_\tau\phi_0 + \langle\boldsymbol{\nabla},\mathfrak{E}\rangle \tag{13}$$

$$\alpha\,\frac{\partial^2\varphi_0}{\partial t^2} - \frac{\partial^2\varphi_0}{\partial x^2} - \frac{\partial^2\varphi_0}{\partial y^2} - \frac{\partial^2\varphi_0}{\partial z^2} = \rho_0 \tag{14}$$

$$(\alpha\,\nabla_t\nabla_t - \langle\boldsymbol{\nabla},\boldsymbol{\nabla}\rangle)\boldsymbol{\varphi} = \boldsymbol{J} = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}} - \alpha\,\nabla_t\boldsymbol{\mathcal{E}} - \boldsymbol{\nabla}\varkappa \Leftrightarrow \boldsymbol{\nabla}\phi_0 - \alpha\,\nabla_\tau\mathfrak{E} + \boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}} \tag{15}$$

$$\alpha\,\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2} - \frac{\partial^2\boldsymbol{\varphi}}{\partial x^2} - \frac{\partial^2\boldsymbol{\varphi}}{\partial y^2} - \frac{\partial^2\boldsymbol{\varphi}}{\partial z^2} = \boldsymbol{J} \tag{16}$$

This corresponds to the Minkowski signature.

$$\{\rho_0,\boldsymbol{J}\} \Leftrightarrow \{\nabla_t\varkappa - \langle\boldsymbol{\nabla},\boldsymbol{\mathcal{E}}\rangle, -\boldsymbol{\nabla}\varkappa + \boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}} - \alpha\,\nabla_\tau\boldsymbol{\mathcal{E}}\} \tag{17}$$

$$= \{\nabla_t\varkappa, -\boldsymbol{\nabla}\varkappa\} + \{\langle\boldsymbol{\nabla},\boldsymbol{\mathcal{E}}\rangle, \boldsymbol{\nabla}\times\boldsymbol{\mathcal{B}} - \alpha\,\nabla_\tau\boldsymbol{\mathcal{E}}\} \tag{18}$$

Notice that we did not need to take $\varkappa = 0$, which is used in the gauge. Adding equation (3) as an extra Maxwell equation would bring Maxwell equations more in conformance with the equations of quaternionic differential calculus.

Notice the difference of the Minkowski signature of these equations with the Euclidean signature of the wave function of quaternionic differential calculus. This difference is enforced by the selection of the value of α.

7.2.1 Poisson equations

The Poisson equations for the Maxwell based differential calculus are similar to the Poisson equations for the quaternionic differential calculus.

$$\frac{\partial^2 \varphi_0}{\partial x^2} + \frac{\partial^2 \varphi_0}{\partial y^2} + \frac{\partial^2 \varphi_0}{\partial z^2} = -\rho_0 = -\nabla_t \varkappa - \langle \boldsymbol{\nabla}, \boldsymbol{\mathcal{E}} \rangle \tag{1}$$

$$\frac{\partial^2 \boldsymbol{\varphi}}{\partial x^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial y^2} + \frac{\partial^2 \boldsymbol{\varphi}}{\partial z^2} = -\boldsymbol{J} = \boldsymbol{\nabla} \varkappa + \alpha \nabla_t \boldsymbol{\mathcal{E}} - \boldsymbol{\nabla} \times \boldsymbol{\mathcal{B}} \tag{2}$$

7.2.2 The screened Poisson equation

The screened Poisson equation runs:

$$\langle \boldsymbol{\nabla}, \boldsymbol{\nabla} \rangle \chi - \lambda^2 \chi = \rho \tag{3}$$

In Maxwell based differential calculus this corresponds to:

$$\nabla_t \nabla_t \chi = \lambda^2 \chi \tag{4}$$

A solution of this equation is

$$\chi = a(\boldsymbol{x}) \exp(\pm \lambda t) \tag{5}$$

This differs significantly from the quaternionic differential calculus version.

7.2.3 The Maxwell-Huygens homogeneous wave equation

In Maxell format the homogeneous wave equation uses coordinate time t. It runs as:

$$\frac{\partial^2 \varphi_0}{\partial t^2} - \frac{\partial^2 \varphi_0}{\partial x^2} - \frac{\partial^2 \varphi_0}{\partial y^2} - \frac{\partial^2 \varphi_0}{\partial z^2} = 0 \tag{1}$$

Papers on Huygens principle work with the homogeneous version of this formula or it uses the version with polar coordinates.

For isotropic conditions in three participating dimensions the general solution runs:

$$\varphi_0 = f(r - ct)/r, \text{ where } c = \pm 1; f \text{ is real} \tag{2}$$

This follows from

$$\langle \nabla, \nabla \rangle \varphi_0 \equiv \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi_0}{\partial r}\right)\right) = \frac{f''(r - ct)}{r} = \frac{1}{c^2}\partial^2 \varphi_0 / \partial t^2 \tag{3}$$

In a single participating dimension the general solution runs:

$$\varphi_0 = f(x - ct), \text{ where } c = \pm 1; f \text{ is real} \tag{4}$$

7.3 Asymmetric tensor

The Maxwell-based equation

$$\phi \Leftrightarrow \{\phi_0, \boldsymbol{\phi}\} \Leftrightarrow \{\nabla_0, \boldsymbol{\nabla}\}\{\varphi_0, \boldsymbol{\varphi}\} = \{\nabla_0, -\boldsymbol{\nabla}\}\{A_0, \boldsymbol{A}\} \tag{1}$$

$$\boldsymbol{\phi} = -\mathfrak{E} \pm \mathfrak{B} \tag{2}$$

$$\mathfrak{E}_\nu \equiv -\left(\frac{\partial \varphi_0}{\partial x_\nu} + \frac{\partial \varphi_\nu}{\partial t}\right) = -F_{0\nu} = \partial_0 A_\nu - \partial_\nu A_0; \nu = 1..3 \tag{3}$$

$$\mathfrak{B}_{\mu\nu} = (\boldsymbol{\nabla} \times \boldsymbol{\varphi})_{\mu\nu} = \left(\frac{\partial \varphi_\mu}{\partial x_\nu} - \frac{\partial \varphi_\nu}{\partial x_\mu}\right) = \partial_\mu A_\nu - \partial_\nu A_\mu; \mu = 1..3; \nu = 1..3; \tag{4}$$

corresponds with the asymmetric tensor $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{bmatrix} 0 & -\mathfrak{E}_x & -\mathfrak{E}_y & -\mathfrak{E}_z \\ \mathfrak{E}_x & 0 & \mp\mathfrak{B}_z & \pm\mathfrak{B}_y \\ \mathfrak{E}_y & \mp\mathfrak{B}_z & 0 & \mp\mathfrak{B}_x \\ \mathfrak{E}_y & \pm\mathfrak{B}_y & \mp\mathfrak{B}_x & 0 \end{bmatrix} \tag{5}$$

For the quaternionic differential calculus the same tensor can be generated. This tensor does not show the nature of the partial derivatives that are contained in the $\mathfrak{E}_\nu$ terms. The tensor hides the real parts of the differential.

7.4 The space-progression model

This paper supports two space progression models. Quaternions, quaternionic functions and quaternionic differential equations support parameter spaces that have an Euclidean signature and correspond to a metric tensor:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

Elements of this model can directly be stored as eigenvalues of operators that reside in quaternionic Hilbert spaces.

The Maxwell based equations and the parameter space of these equations support a space-time model with Minkowski signature and correspond to a metric tensor:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{2}$$

Elements of this model must first be dismantled into their real components before they can be stored as eigenvalues of Hermitian operators that reside in real, complex or quaternionic Hilbert spaces

The fact that the quaternionic field can be stored in the eigenspace of an operator that resides in a non-separable quaternionic Hilbert space and that after dismantling into real components the same can be done for a Maxwell based field means that the stored fields can represent one and the same object. It also means that both differential equation sets can investigate the same field and offer different views onto that field that reveal different aspects of the behavior of that field.

It also means that both space-progression models represent different views of the same reality.

8 Phenomena

The two approaches are two different views of the same investigated field. Each view corresponds to a set of equations. These sets differ in the format of some of the equations and the equations differ in the selected scalar parameter. The actual behavior of the field and its features are not affected by the selected view. The two sets of equations might describe different aspects of the reaction of the field on discontinuities. Two fields might show different behavior if the type of artifacts that cause the discontinuities of the field differ.

The used nabla operator can only be applied for simple discontinuities that can be represented by Dirac delta functions. The field response on such discontinuity is represented by a Green's function. Three different kinds of these discontinuities can occur.

- Oscillatory point-like discontinuities. This requires an oscillating trigger.
- Persistent point-like discontinuities. These can still move around in the field.
- Transient point-like discontinuities. These can still be grouped. These groups can move as a group.
    - The grouping can result in a coherent swarm
    - The grouping can result in a linear string

The kind of discontinuities will influence the characteristics of the field.

The definition of the quaternionic differential as

$$\phi = \nabla\varphi \tag{1}$$

defines this formula as a differential continuity equation. In fact the quaternionic second order partial differential equation represents the combination of two continuity equations

$$\zeta = \nabla^* \phi \tag{2}$$

$$\phi = \nabla \varphi \tag{3}$$

$$\zeta = \nabla^* \nabla \varphi \tag{4}$$

The phenomena are all solutions of the second order partial differential equation.

Thus the discontinuities can be interpreted as sources, as drains, as oscillatory triggers, as charges or as transient embedding events.

Examples are [5]:

- Electric charges. These can be interpreted as persistent sources or drains. These objects may move around.
- Elementary particles. Stochastic processes control the recurrent transient embedding of point-like artifacts that together form a coherent swarm and a hopping path. The swarm is characterized by a continuous location density distribution that conforms to the squared modulus of the wave function of the particle.
- Linear strings of moving artifacts. The fronts that represent the Green's functions of the artifacts move with constant speed along the path of the string and may rotate around the axis of the string. These strings may represent photons.

8.1 Coupling equation

The coupling equation represents a peculiar property of the quaternionic differential equation.

We start with two normalized functions ψ and φ and a normalizable function $\Phi = m\,\varphi$.

Here m is a fixed quaternion. Function φ can be adapted such that m becomes a real number.

$$\|\psi\| = \|\varphi\| = 1 \tag{1}$$

These normalized functions are supposed to be related by:

$$\Phi = \nabla\psi = m\,\varphi \tag{2}$$

$\Phi = \nabla\psi$ defines the ***differential equation***. (3)

$\nabla\psi = \Phi$ formulates a differential ***continuity equation***. (4)

All quaternionic functions ψ and ψ that obey $\|\psi\| = \|\varphi\| = 1$, will also obey the ***coupling equation***.

$$\nabla\psi = m\,\varphi \tag{5}$$

9 Quaternionic Hilbert spaces

Separable Hilbert spaces are linear vector spaces in which an inner product is defined. This inner product relates each pair of Hilbert vectors. The value of that inner product must be a member of a division ring. Suitable division rings are real numbers, complex numbers and quaternions. This paper uses quaternionic Hilbert spaces [1].

Paul Dirac introduced the bra-ket notation that eases the formulation of Hilbert space habits [6].

$$\langle x|y\rangle = \langle y|x\rangle^* \tag{1}$$

$$\langle x+y|z\rangle = \langle x|z\rangle + \langle y|z\rangle \tag{2}$$

$$\langle \alpha x|y\rangle = \alpha\, \langle x|y\rangle \tag{3}$$

$$\langle x|\alpha y\rangle = \langle x|y\rangle\, \alpha^* \tag{4}$$

$\langle x|$ is a bra vector. $|y\rangle$ is a ket vector. α is a quaternion.

This paper considers Hilbert spaces as no more and no less than structured storage media for dynamic geometrical data that have an Euclidean signature. Quaternions are ideally suited for the storage of such data.

The operators of separable Hilbert spaces have countable eigenspaces. Each infinite dimensional separable Hilbert space owns a Gelfand triple. The Gelfand triple embeds this separable Hilbert space and offers as an extra service operators that feature continuums as eigenspaces. In the corresponding subspaces the definition of dimension loses its sense.

9.1 Representing continuums and continuous functions

Operators map Hilbert vectors onto other Hilbert vectors. Via the inner product the operator T may be linked to an adjoint operator $T^\dagger$.

$$\langle Tx|y\rangle \equiv \langle x|T^\dagger y\rangle \tag{1}$$

$$\langle Tx|y\rangle = \langle y|Tx\rangle^* = \langle T^\dagger y|x\rangle^* \tag{2}$$

A linear quaternionic operator T, which owns an adjoint operator $T^\dagger$ is normal when

$$T^\dagger T \;=\; T\, T^\dagger \tag{3}$$

$T_0 = \left(T + T^\dagger\right)/2$ is a self adjoint operator and $\boldsymbol{T} = \left(T - T^\dagger\right)/2$ is an imaginary normal operator. Self adjoint operators are also Hermitian operators. Imaginary normal operators are also anti-Hermitian operators.

By using what we will call ***reverse bra-ket notation***, operators that reside in the Hilbert space and correspond to continuous functions, can easily be defined by starting from an orthonormal base of vectors. In this base the vectors are normalized and are mutually orthogonal. The vectors span a subspace of the Hilbert space. We will attach eigenvalues to these base vectors via the reverse bra-ket notation. This works both in separable Hilbert spaces as well as in non-separable Hilbert spaces.

Let $\{q_i\}$ be the set of rational quaternions in a selected quaternionic number system and let $\{|q_i\rangle\}$ be the set of corresponding base vectors. They are eigenvectors of a normal operator $\mathcal{R} = |q_i\rangle q_i \langle q_i|$. Here we enumerate the base vectors with index i.

$$\mathcal{R} = |q_i\rangle q_i \langle q_i| \tag{4}$$

$\mathcal{R}$ is the configuration parameter space operator.

$\mathcal{R}_0 = (\mathcal{R} + \mathcal{R}^\dagger)/2$ is a self-adjoint operator. Its eigenvalues can be used to arrange the order of the eigenvectors by enumerating them with the eigenvalues. The ordered eigenvalues can be interpreted as ***progression values***.

$\boldsymbol{\mathcal{R}} = (\mathcal{R} - \mathcal{R}^\dagger)/2$ is an imaginary operator. Its eigenvalues can also be used to order the eigenvectors. The eigenvalues can be interpreted as ***spatial values*** and can be ordered in several ways.

Let $f(q)$ be a mostly continuous quaternionic function. Now the reverse bra-ket notation defines operator f as:

$$f = |q_i\rangle f(q_i)\langle q_i| \tag{5}$$

f defines a new operator that is based on function $f(q)$. Here we suppose that the target values of f belong to the same version of the quaternionic number system as its parameter space does.

Operator f has a countable set of discrete quaternionic eigenvalues.

For this operator the reverse bra-ket notation is a shorthand for

$$\langle x|f\ y\rangle = \sum_i \langle x|q_i\rangle f(q_i)\langle q_i|y\rangle \tag{6}$$

In a non-separable Hilbert space, such as the Gelfand triple, the continuous function $\mathcal{F}(q)$ can be used to define an operator, which features a continuum eigenspace.

$$\mathcal{F} = |q\rangle \mathcal{F}(q)\langle q| \tag{7}$$

Via the continuous quaternionic function $\mathcal{F}(q)$, the operator $\mathcal{F}$ defines a curved continuum $\mathcal{F}$. This operator and the continuum reside in the Gelfand triple, which is a non-separable Hilbert space.

$$\mathfrak{R} = |q\rangle q\langle q| \tag{8}$$

The function $\mathcal{F}(q)$ uses the eigenspace of the reference operator $\mathfrak{R}$ as a flat parameter space that is spanned by a quaternionic number system $\{q\}$. The continuum $\mathcal{F}$ represents the target space of function $\mathcal{F}(q)$.

Here we no longer enumerate the base vectors with index i. We just use the name of the parameter. If no conflict arises, then we will use the same symbol for the defining function, the defined operator and the continuum that is represented by the eigenspace.

For the shorthand of the reverse bra-ket notation of operator $\mathcal{F}$ the integral over q replaces the summation over q_i.

$$\langle x|\mathcal{F}\ y\rangle = \int_q \langle x|q\rangle \mathcal{F}(q)\langle q|y\rangle\ dq \tag{9}$$

Remember that quaternionic number systems exist in several versions, thus also the operators f and $\mathcal{F}$ exist in these versions. The same holds for the parameter space operators. When relevant, we will use superscripts in order to differentiate between these versions.

Thus, operator $f^x = |q_i^x\rangle f^x(q_i^x)\langle q_i^x|$ is a specific version of operator f. Function $f^x(q_i^x)$ uses parameter space $\mathcal{R}^x$.

Similarly, $\mathcal{F}^x = |q^x\rangle \mathcal{F}^x(q^x)\langle q^x|$ is a specific version of operator $\mathcal{F}$. Function $\mathcal{F}^x(q^x)$ and continuum $\mathcal{F}^x$ use parameter space $\mathfrak{R}^x$. If the operator $\mathcal{F}^x$ that resides in the Gelfand triple $\mathcal{H}$ uses the same defining function as the operator $\mathcal{F}^x$ that resides in the separable Hilbert space, then both operators belong to the same quaternionic ordering version.

In general the dimension of a subspace loses its significance in the non-separable Hilbert space.

The continuums that appear as eigenspaces in the non-separable Hilbert space $\mathcal{H}$ can be considered as quaternionic functions that also have a representation in the corresponding infinite dimensional separable Hilbert space $\mathfrak{H}$. Both representations use a flat parameter space $\mathfrak{R}^x$ or $\mathcal{R}^x$ that is spanned by quaternions. $\mathcal{R}^x$ is spanned by rational quaternions.

The parameter space operators will be treated as reference operators. The rational quaternionic eigenvalues $\{q_i^x\}$ that occur as eigenvalues of the reference operator $\mathcal{R}^x$ in the separable Hilbert space map onto the rational quaternionic eigenvalues $\{q_i^x\}$ that occur as subset of the quaternionic eigenvalues $\{q^x\}$ of the reference operator $\mathfrak{R}^x$ in the Gelfand triple. In this way the reference operator $\mathcal{R}^x$ in the infinite dimensional separable Hilbert space $\mathfrak{H}$ relates directly to the reference operator $\mathfrak{R}^x$, which resides in the Gelfand triple $\mathcal{H}$.

All operators that reside in the Gelfand triple and are defined via a mostly continuous quaternionic function have a representation in the separable Hilbert space.

9.2 Stochastic operators

Stochastic operators do not get their data from a continuous quaternionic function. Instead a stochastic process delivers the eigenvalues. Again these eigenvalues are quaternions and the real parts of these quaternions can be interpreted as progression values. The generated eigenvalues are picked from a selected parameter space.

Stochastic operators only act in a step-wise fashion. Their eigenspace is countable. Stochastic operators may act in a cyclic fashion.

The mechanisms that control the stochastic operator can synchronize the progression values with the model wide progression that is set by a selected reference operator. These mechanisms and the stochastic processes are not part of the functionality of the Hilbert space.

Characteristic for stochastic operators is that the imaginary parts of the eigenvalues are not smooth functions of the real values of those eigenvalues.

9.2.1 Density operators

The eigenspace of a stochastic operator may be characterized by a continuous spatial density distribution. In that case the corresponding stochastic process must ensure that this continuous density distribution fits. The density distribution can be constructed afterwards or after each regeneration cycle. Constructing the density distribution involves a reordering of the imaginary parts of the produced eigenvalues. This act will usually randomize the real parts of those eigenvalues. A different operator can then use the continuous density distribution in order to generate its functionality. The old real parts of the eigenvalues may then reflect the reordering. The construction of the density distribution is a pure administrative action that is performed as an aftermath. The constructed density operator represents a continuous function and may reside both in the separable Hilbert space and in the Gelfand triple. The construction of the density function involves a selected parameter space. That parameter space need not be the same as the parameter space from which the stochastic process picked its eigenvalues.

9.3 Storing Maxwell based field components in Hilbert space

As shown above it is easy to store quaternionic functions and their parameter spaces in a quaternionic non-separable Hilbert space. If the components of the Maxwell based function must be stored in the non-separable Hilbert space, then the function must first be dismantled into its real components. After that action these components can be stored in the eigenspaces of Hermitian operators. Physical theories usually use complex number based Hilbert spaces for this purpose. In that way at least the coordinate time part can stay coupled with one spatial dimension. This approach may generate problems when multiple spatial dimensions interact. That happens in the realm of elementary particles [5].

10 Conclusion

Great similarities, but also essential differences exist between quaternionic differential equations and Maxwell based differential equations. In the quaternionic differential calculus the differential can be seen as a product between the four-component differential operator ∇ and the four-component field φ. That simple interpretation is not possible in Maxwell based differential calculus. It is not possible to interpret the Maxwell field as a function of a parameter space that directly corresponds to a number system. In the Maxwell approach, the parameter space has a Minkowski signature and does not form a division ring. In quaternionic function theory the parameter space has a Euclidean signature. This shows in the structure of the second order partial differential equations of the two approaches. These equations have solutions that differ between the two approaches. However, the Poisson part of the two second order partial differential equations is similar. This does not hold for the screened Poisson equation. The corresponding Green's functions are similar. Both homogeneous second order partial differential equations have solutions in the form of one dimensional and three dimensional fronts that keep their shape. The one dimensional fronts also keep their amplitude. Between the two approaches, these fronts have different mathematical representations. In applications, the fronts can act as carriers of information or energy. Only the Maxwell based version supports harmonic oscillations in the form of waves.

This paper replaces in the Maxwell based differential calculus the usage of gauges by the introduction of an extra scalar field $\varkappa$. This results in the same form of the Maxwell based wave equation as the Lorentz gauge delivers, but the non-homogeneous equation applies different charge and current distributions. The impact of the difference in charges and currents is not treated here.

The two approaches offer different views on the same basic field. These views reveal different phenomena of that basic field. They might also split basic fields in categories. One category reveals their properties with Maxwell based differential calculus and another category reveals their properties with quaternionic differential calculus. EM fields fit better in the first category. The field that describes our living space fits better in the second category.

10.1 Extra

Maxwell based differential calculus can be implemented with complex numbers. In that way it does not have to cope so intensively with non-commutative operators. As a consequence, gauges can be implemented easily.

A disadvantage of Maxwell based differential calculus is that spacetime based dynamic geometric data must first be dismantled into real numbers or complex numbers before Hilbert spaces can handle them.

Quaternionic differential calculus must circumvent most gauges. On the other hand, the quaternionic approach offers compensating advantages.

Hilbert spaces can directly cope with quaternions as eigenvalues of operators. This holds for separate quaternions and for quaternionic continuums [1][4].

Since proper time is Lorentz invariant, all quaternionic differential equations are inherently Lorentz invariant.

Due to the fact that quaternions form a number system with a non-commutative product, they can implement rotations:

$$c = a\ b/a$$

In this way they can implement the functionality of gluons [4]. This is not possible with parameters of the Maxwell based “field”.

11 References

[1] “Quaternions and quaternionic Hilbert spaces”; http://vixra.org/abs/1411.0178

[2] The vector differential calculus formulas and the Maxwell based equations that are used here are taken from the online textbook Bo Thidé; “Electromagnetic field theory”; second edition; http://www.plasma.uu.se/CED/Book/, or https://www.calvin.edu/~pribeiro/courses/engr315/EMFT_Book.pdf .

[3] The relation between spinor notation and quaternionic formulation is shown in “The Dirac Equation in Quaternionic Format”; http://vixra.org/abs/1505.0149

[4] “Foundation of a Mathematical Model of Physical Reality”; http://vixra.org/abs/1511.0074

[5] “On the Origins of Physical Fields”; http://vixra.org/abs/1511.0007 .

[6] Paul Dirac introduced the bra-ket notation, which popularized the usage of Hilbert spaces. Dirac also introduced its delta function, which is a generalized function. Spaces of generalized functions offered continuums before the Gelfand triple arrived.

See: P.A.M. Dirac.(1958), The Principles of Quantum Mechanics, Fourth edition, Oxford Univerity Press, ISBN 978 0 19 852011 5.