Demonstrating Lorenz Wealth Distribution and Increasing Gini Coefficient with the Iterating (Koch Snowflake) Fractal Attractor.

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Abstract

The Koch snowflake fractal attractor was analysed by Lorenz and Gini methods. It was found the fractal Lorenz curve fits the wealth distribution Lorenz curve. As the fractal grows and/or develops with iteration-time the Lorenz curve expands to the right, with a corresponding increasing Gini coefficient. All triangles (triangle sizes) grow with iteration-time from an arbitrary size, and iteration group size accelerate apart from each other with iteration time. It was concluded the Lorenz distribution is a property of the fractal and inextricably linked to (fractal) growth and development. This behaviour can be observed in any fractal structure.

Keywords:
fractals, Lorenz curve, Gini Coefficient, wealth distribution

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1 INTRODUCTION

The Lorenz curve – first developed by M. O. Lorenz (Lorenz 1905) – shows the distribution of size or income or wealth in a population, and been found useful in both natural and cultural economics. as shown below in Figure 1. This paper tested whether a Lorenz wealth distribution (a stock as opposed to the flow concept of income) is a natural geometric phenomenon, observed in every fractal structure.

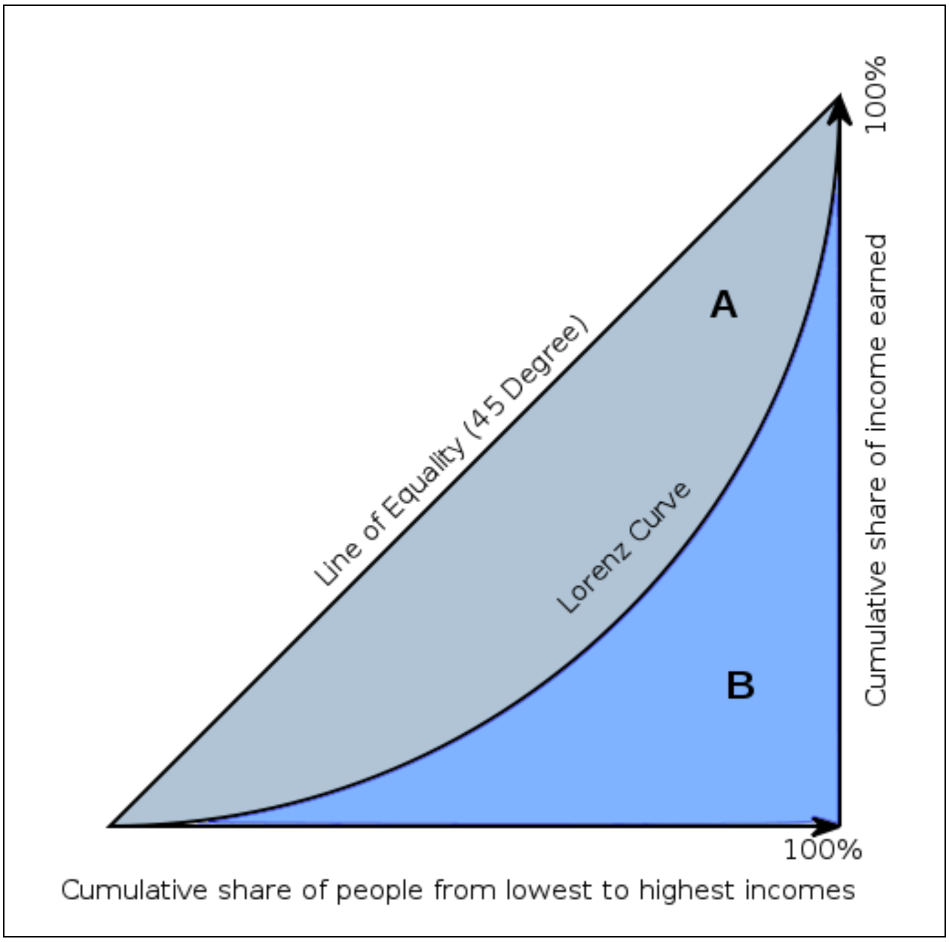

Figure 1. Lorenz Diagram. The graph shows that the Gini coefficient is equal to the area marked A divided by the sum of the areas marked A and B. that is, Gini = A / (A + B). It is also equal to 2*A due to the fact that A + B = 0.5 (since the axes scale from 0 to 1)("Gini Coefficient" 2015).

If the area of the triangle stands for the wealth of an individual, and the quantity of triangles for the population of individuals, does the fractal offer insight and explanation to Lorenz and Gini data through time? To test for this pattern, the distribution of triangle areas – in a Koch snowflake fractal attractor – were analysed using Lorenz methods; and Gini coefficients (the ratio of area A to area A+B above) were calculated for each iteration as the fractal grew (or developed).

1.1 The Classical Fractal

Fractals are described as emergent objects from iteration, possessing regular irregularity (same but different) at all scales, and is classically demonstrated by the original Mandelbrot Set and the Koch Snowflake (Figure 1 A and B below respectively).

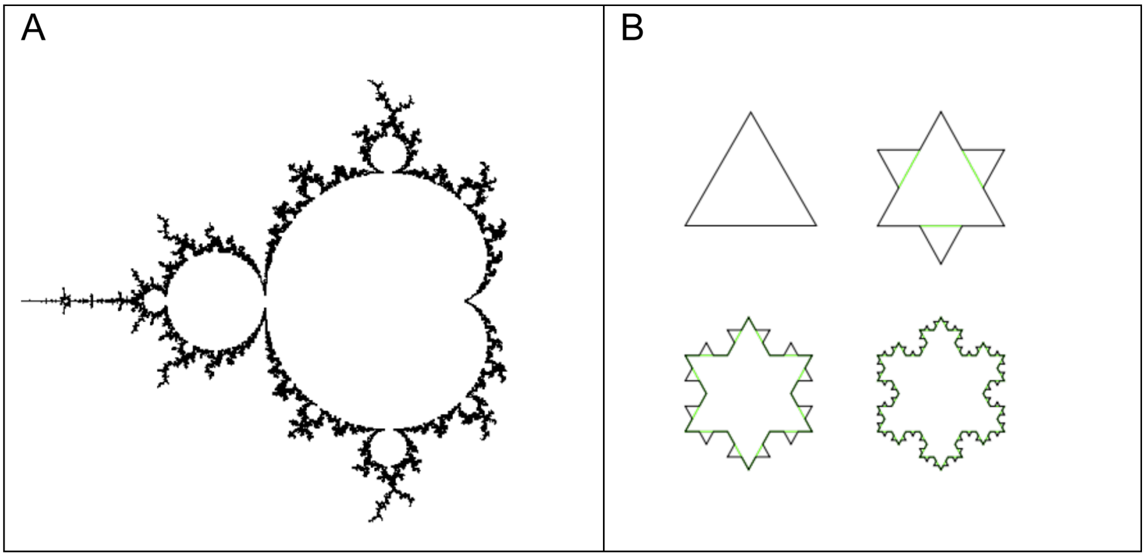

Figure 2. (Classical) Fractals. (A) boundary of the Mandelbrot set; (B) The Koch Snowflake fractal from iteration-time (*t*) 0 to 3. Reference: (A) (Prokofiev 2007); (B) ("Koch Snowflake" 2014).

The development or growth of the classical fractal shape emerges as a result of the iteration of a simple rule, the repeating the process of adding triangles – as demonstrated in the Koch Snowflake above. The complete emergent structure reaches a shape equilibrium (where no more detail can be observed – with additional iterations – to an observer of fixed position) at or around four to seven iteration-times.

1.2 Distribution of Triangles

If the area of the triangle stands for the wealth of an individual, and the quantity of triangles for the population of individuals, does the fractal offer insight and explanation to Lorenz and Gini data through time? To test for this pattern, the distribution of triangle areas – in a Koch snowflake fractal attractor – were analysed using Lorenz methods; and Gini coefficients (the ratio of area A to area A+B above) were calculated for each iteration as the fractal grew (or developed).

In this experiment the fractal was inverted so as to measure the growth in income from the bottom up. To do this the new triangle size was held constant (rather than diminish).

2 METHODS

To create a quantitative data series for analysis of the area distribution of the inverted fractal a spreadsheet model[1] (Macdonald 2015) was developed. The classical Koch Snowflake area equations were adapted to invert the fractal.
A data table was produced (Tab 'Table') to calculate the area growth at each, and every iteration of a single triangle. Area was calculated from the following formula (1) measured in standard (arbitrary) units (u)

$$A = \frac{l^2\sqrt{3}}{4} \quad (1)$$

where (A) is the area of a single triangle, and where l is the triangle's base length. l was placed in Table 1 and was set to 15.1967128766173 u so that the area of the first triangle (t=0) approximated an arbitrary area of 100 u^{-2}. To expand the triangle with iteration (and thus invert the fractal) the base length was multiplied by a factor of 3. The iteration-time (***t***) number was placed in a column, followed by the base length of the equilateral triangle, and in the final column the formula to calculate the area of the triangle. Calculations were made to the arbitrary 12th iteration, and the results graphed.

2.1 Lorenz Curve

For each iteration a table was created that ranked triangles by their size in ascending order. At each ranked quantity the following was calculated: a percentage quantity was created (Quantity/ Total Quantity) for the line of equality; a percentage area (Area/ Total Area) for the Lorenz curve; Cumulative percentage Area; and finally – for the calculation of the Gini Coefficient – the area under the Lorenz Curve was calculated by

$$\frac{Cum.\%\, Area\ of\ Iteration1\ +\ Cum.\%\, Area\ of\ Iteration2}{2} \times\%\, Quantity\ of\ Iteration1 \times 100$$

2.2 Gini Coefficient

The Gini Coefficient is a calculated by dividing the area between the line of equality and the Lorenz curve (area A figure 1) by the area under the line of equality.

$$\frac{A}{A+B}$$

A method of how the area A and B is calculated standard procedure and is not included in this work. Summing all the areas under the Lorenz Curve gives the area of B (figure Gini Coefficients were calculated for each iteration-time, and analysed.

3 RESULTS

Figures 3 below is a composite diagram of Lorenz curves from iteration-time 1 (the line of equality) to iteration time 5. All curves are derived from the spreadsheet model.

3.1 Lorenz Curves

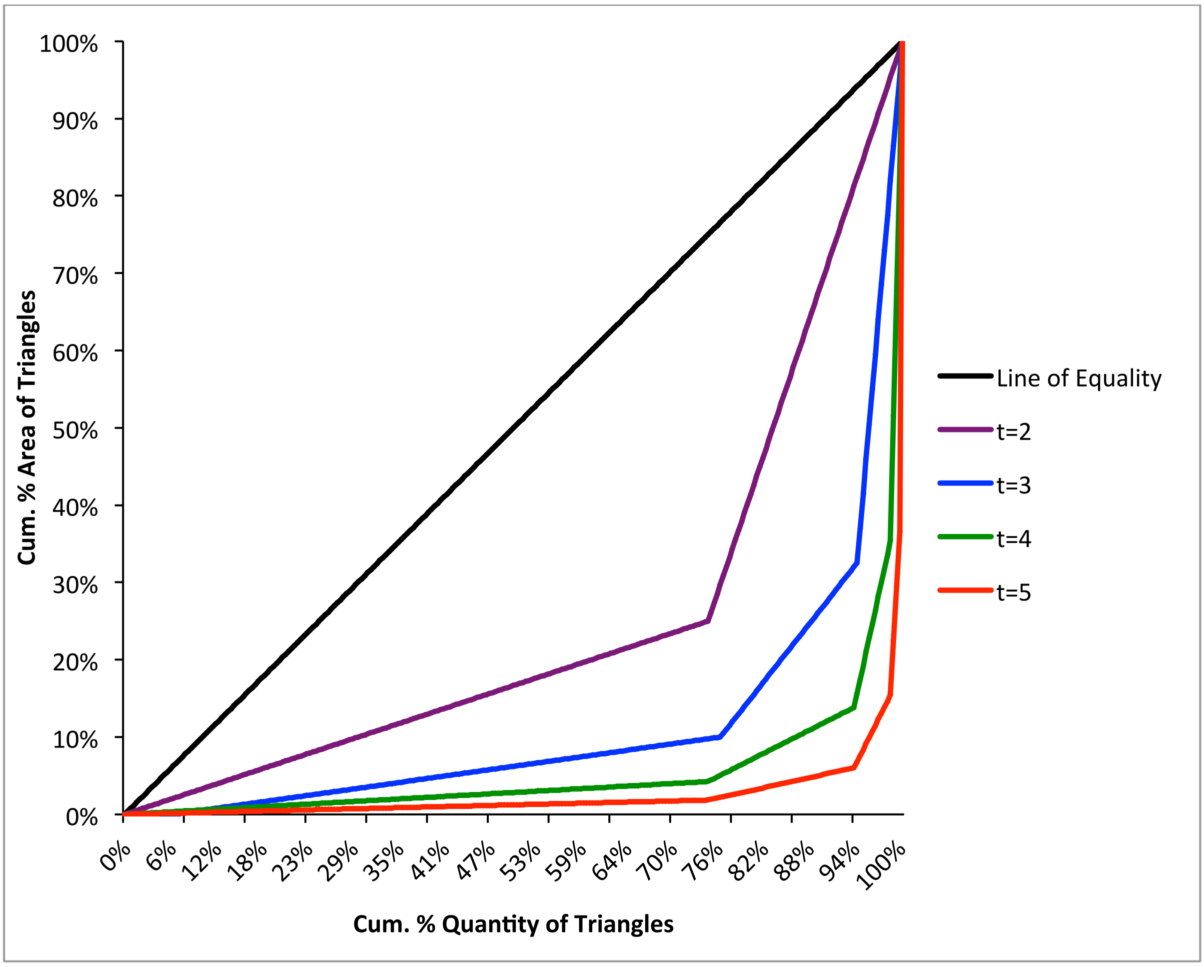

Figure 3. Fractal (Koch Snowflake) Lorenz Curves Expansion. As the (Koch Snowflake) fractal grows – and/or develops – with iteration time (*t*) the distribution of area – shown by the Lorenz Curves t=2 to t=5 – expands. At iteration-time 1 – the line of equality – distribution is homogenous. With growth the second iteration time (t=2) shows 25% of the area is found in 75% of the triangles; and conversely the remaining 75% of the area is found in the remaining 25% of the triangles. At iteration-time 5 (*t*=5) this distribution has expanded and shows only a small percentage of the triangles account for some 99% of the area.

3.2 Gini Coefficient

Gini coefficients at each iteration are listed below in Table 1. As iteration-time increases, so to did the Gini coefficient.

Table 1: Koch Snowflake Gini Coefficient by Iteration-time *t*. The Gini coefficient is a measure of the area between the line of equality and the Lorenz curve in relation to an area of perfect equality. As the (Koch Snowflake) fractal grows – and/or develops – with iteration time, the Gini Coefficient increases.

t	Gini Coefficient
2	0.75
3	0.7625
4	0.89
5	0.9498

4 DISCUSSIONS

Area distribution of the Koch Snowflake fractal clearly matches the Lorenz wealth distribution curve. As the fractal iterates the distribution of area increases or becomes more unequal.

4.1 Changing Lorenz Curve

A new Lorenz curve is drawn as iteration-time increases; each new is outside and to the right of the previous. Distribution becomes more unequal with iteration-time, or as the fractal grows and/or develops.

4.2 Increasing Gini coefficient

From Table 1 we see the Gini coefficients at each iteration-time increases. The greater the iteration-time or area (growth) of the fractal, the greater the inequality between area sizes. An increasing Gini coefficient with iteration time (at least for wealth) may suggest the Kuznet (reduction of Gini coefficient with time) maybe atypical – a cultural phenomenon. It would be interesting to test whether other biological systems redistribute wealth or income with (economic) growth.

4.3 Increasing Gini Coefficient with Iteration Number

It was shown in this model the Gini coefficient increases with iteration- time. Another interpretation maybe: different Gini coefficients for different economies maybe as a consequence of their inherent complexity, and not at all a function of iteraton-time. Greater iterations between smallest and largest (triangles) lead to greater Gini coefficient – the converse being true. A more homogenous group with have a low Gini coefficient.

4.4 Exponentially Accelerating Gap

It was found in and earlier study – by the author – the gap between the (area/size) groups (of triangles) expand apart exponentially, and at an accelerating rate with iteration time (Macdonald 2014). This acceleration between points is a property of the fractal, and may show itself in reality as the wealth (and income) gap between the poorest and richest expanding (exponentially) as the economy grows.

4.5 Inequality Natural Consequence of the Fractal

The inequality of income or wealth distribution is often seen as an undesirable trade-off of the free market, however, as revealed in this fractal experiment, it maybe more true that this inequality is a natural, fractal phenomenon.

4.6 Scale

The Lorenz distribution of the fractal is a scale invariant pattern that will be viewed at any iteration time – there will always be a Lorenz distribution.
This experiment can be demonstrated on all things fractal – tree plants for example.

“ One of the most pervasive and far-reaching realities in economic systems is inequality, the tendency for one party involved in an interaction over resources to gain more, or less, than its rival.” Geerat J. Vermeij. (Vermeij 2009)

5 CONCLUSIONS

This investigation found the Lorenz distribution is a property of all things fractal, revealed in fractal structures such as trees, clouds and economies. As the fractal develops (and grows) the income or wealth distribution increases, and the Gini coefficient also increases. The gap between area sizes (wealth) also expands apart exponentially (becomes more unequal) at an accelerating rate with iteration time.

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