Demonstrating Lorenz Wealth Distribution and Increasing Gini Coefficient with the Iterating (Koch Snowflake) Fractal Attractor.

First published: May 17th, 2015.
Updated: October, 13th, 2015.
Blair D. Macdonald

Abstract

The Koch snowflake fractal attractor was analysed by Lorenz and Gini methods. It was found the fractal Lorenz curve fits the wealth (stock) distribution Lorenz curve. Gini coefficient analysis showed an increasing coefficient by iteration (time). It was concluded the Lorenz distribution is a property of the fractal and inextricably linked to (fractal) growth and development.
Keywords:
fractals, Lorenz curve, Gini Coefficient, wealth distribution

1 INTRODUCTION

The Lorenz curve – first developed by M. O. Lorenz in 1905 [1] – shows the distribution of income in a population as show below in Figure 1. This paper tested whether a Lorenz wealth distribution (a stock as opposed to the flow concept of income) is a fractal phenomenon.

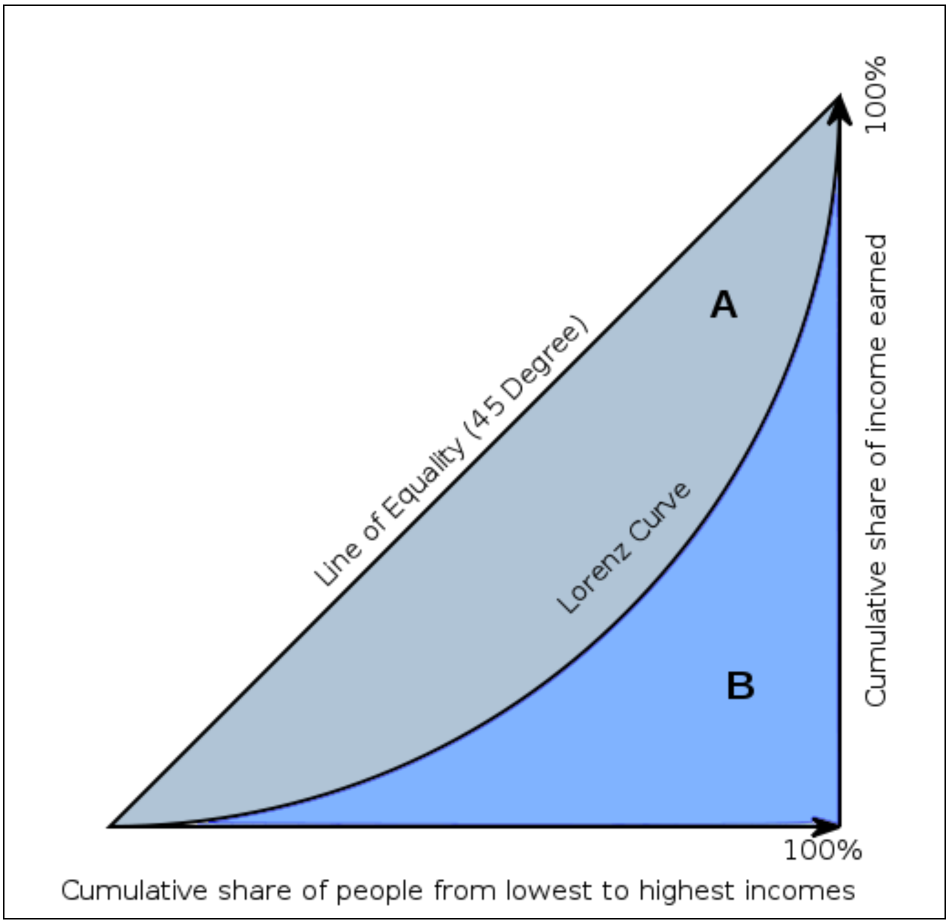

Figure 1. Lorenz Diagram. The graph shows that the Gini coefficient is equal to the area marked A divided by the sum of the areas marked A and B. that is, Gini = A / (A + B). It is also equal to 2*A due to the fact that A + B = 0.5 (since the axes scale from 0 to 1)[2].

If the area of the triangle stands for the wealth of an individual, and the quantity of triangles for the population of individuals, does the fractal offer insight and explanation to Lorenz and Gini data through time? To test for this pattern, the distribution of triangle areas – in a Koch snowflake fractal attractor – were analysed using Lorenz methods; and Gini coefficients (the ratio of area A to area A+B above) were calculated for each iteration as the fractal grew (or developed).

1.1 The Classical Fractal

Fractals are described as emergent objects from iteration, possessing regular irregularity (same but different) at all scales, and is classically demonstrated by the original Mandelbrot Set (Figure 1 A below).

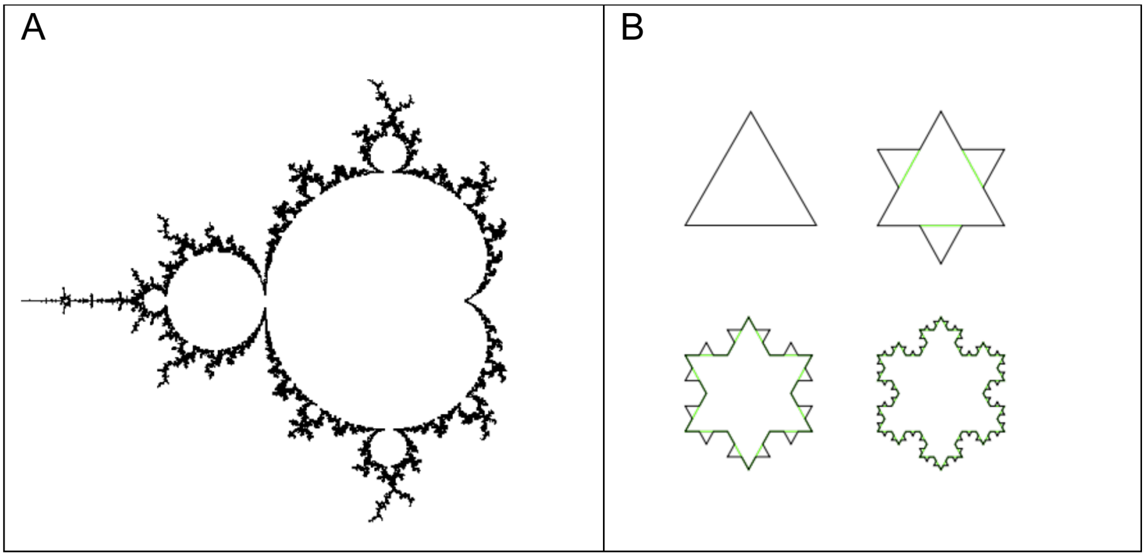

Figure 2. (Classical) Fractals. (A) boundary of the Mandelbrot set; (B) The Koch Snowflake fractal from iteration-time (***t***) 0 to 3. Reference: (A) [3]; (B) [4].

The classical fractal shape – as demonstrated in the Koch Snowflake – emerges as a result of the iteration of a simple rule: the repeating the process of adding triangles in the case of the Koch Snowflake. The complete emergent structure is at shape equilibrium (where no more detail can be observed – with additional iterations – to an observer of fixed position) at or around four to seven iteration-times. This equilibrium iteration count is the observable fractal distance, relative to the observer. This distance is constant irrespective of magnification. For the purposes of this experiment, the equilibrium iteration count is iteration 4.

2 METHODS

To create a quantitative data series for analysis of the area distribution of the inverted fractal, the classical Koch Snowflake area equations were adapted to account for this perspective, and a spreadsheet model [5] was developed.

A data table was produced (Tab 'Table') to calculate the area growth at each, and every iteration of a single triangle. Area was calculated from the following formula (1) measured in standard (arbitrary) units (u)

$$A = \frac{l^2\sqrt{3}}{4} \quad (1)$$

where (A) is the area of a single triangle, and where l is the triangle's base length. l was placed in Table 1 and was set to 1.51967128766173 u so that the area of the first triangle (t0) approximated an arbitrary area of 1 u2. To expand the triangle with iteration the base length was multiplied by a factor of 3. The iteration number was placed in a column, followed by the base length of the equilateral triangle, and in the final column the formula to calculate the area of the triangle. Calculations were made to the arbitrary 12th iteration, and the results graphed.

Lorenz Curve
For each iteration a table was created that ranked triangles by their size in ascending order. At each ranked quantity the following was calculated: a percentage quantity was created (Quantity/ Total Quantity) for the line of equality; a percentage area (Area/ Total Area) for the Lorenz curve; Cumulative percentage Area; and finally – for the calculation of the Gini Coefficient – the area under the Lorenz Curve was calculated by

$$\frac{Cum.\%\,Area\ of\ Iteration1\ +\ Cum.\%\,Area\ of\ Iteration2}{2} \times\%\,Quantity\ of\ Iteration1\times 100$$

2.1 Gini Coefficient

Summing all the areas under the Lorenz Curve gives the area of B. The Area of A is calculated by subtracting B from the area under the line of equality.
The Gini Coefficient is a calculated by

$$\frac{A}{A+B}$$

Gini Coefficients were calculated for each iteration-time, and analysed.

3 RESULTS

Figures 4 to 7 show graphically the results of the experiment. All data is derived directly from the spreadsheet model.

3.1 Lorenz Curves

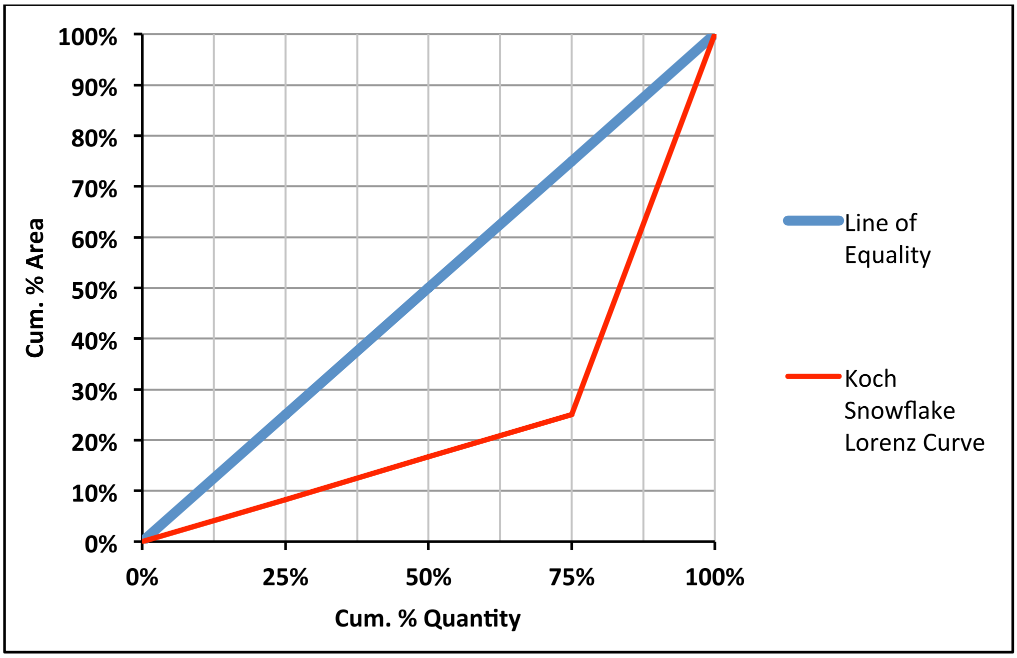

Figure 3. Koch Snowflake Fractal Lorenz Curve at Iteration-Time 2.

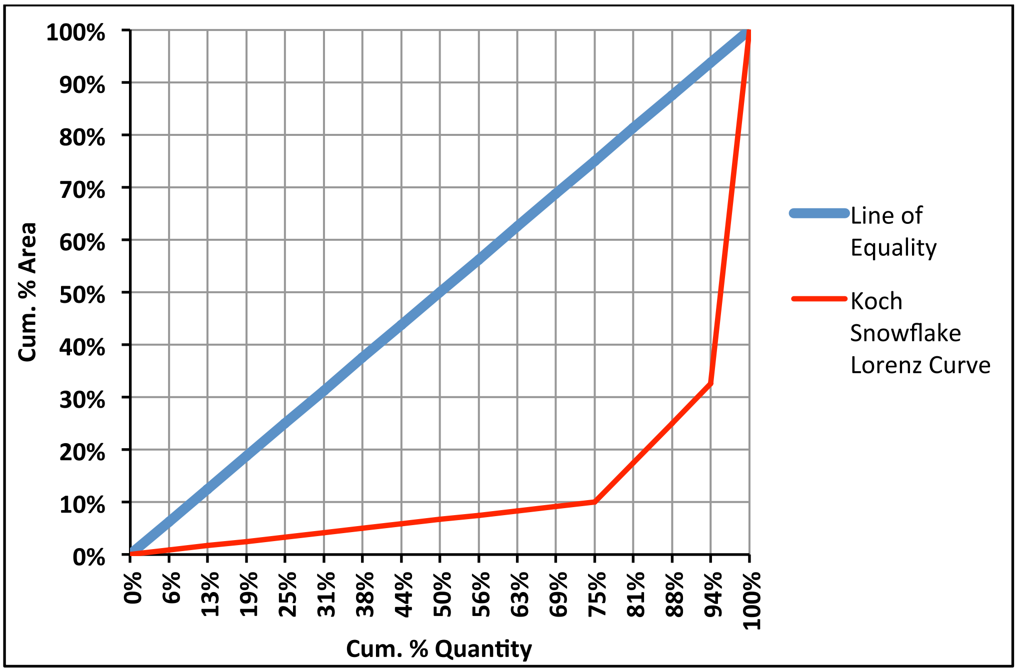

Figure 4. Koch Snowflake Fractal Lorenz Curve at Iteration-Time 3.

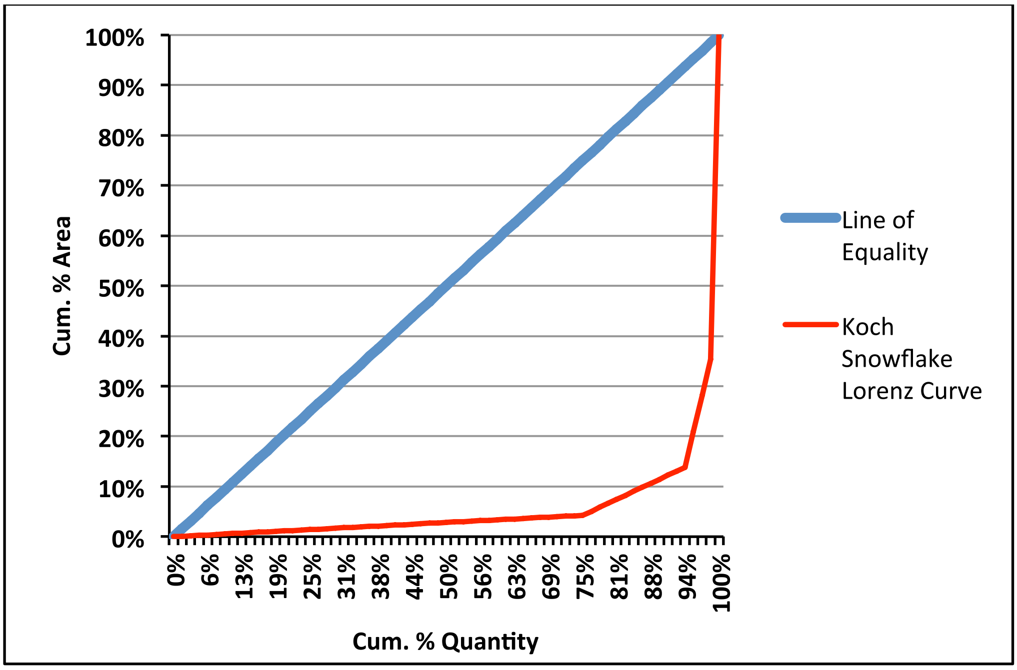

Figure 5. Koch Snowflake Fractal Lorenz Curve at Iteration Time 4.

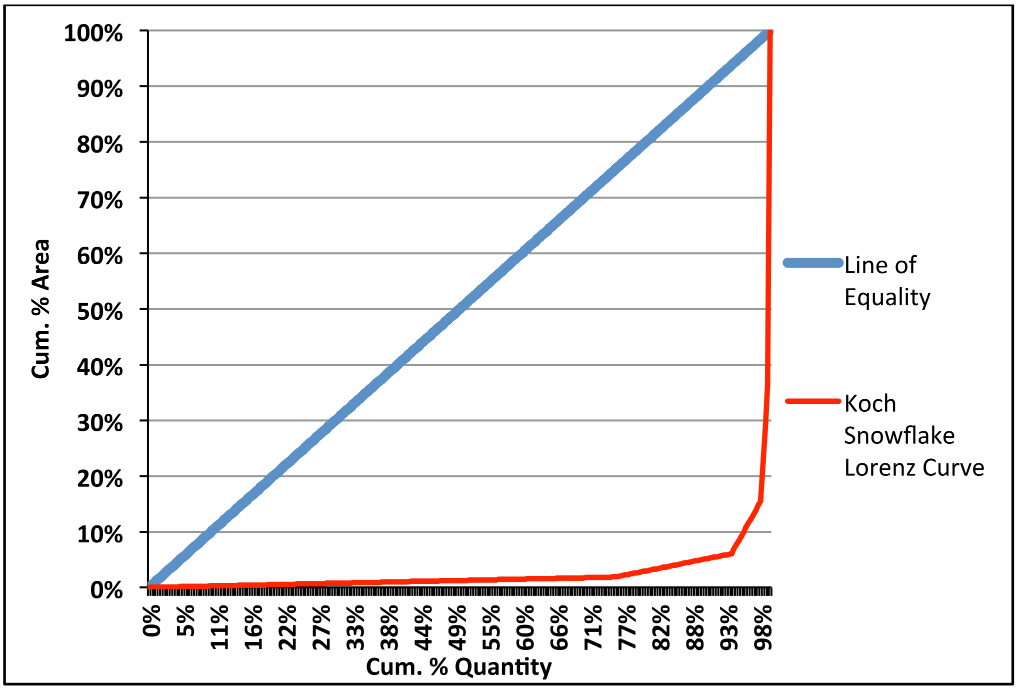

Figure 6. Koch Snowflake Fractal Lorenz Curve at Iteration Time 5.

3.2 Gini Coefficient

Gini coefficients at each iteration are listed below. As iteration time increases, so to does the Gini coefficient.

Table 1: Koch Snowflake Gini Coefficient by Iteration time *t*.

t	Gini Coefficient
2	0.75
3	0.7625
4	0.89
5	0.9498

4 DISCUSSIONS

Area distribution of the Koch Snowflake fractal clearly matches the Lorenz wealth distribution curve. As the fractal iterates the distribution of area increases or becomes more equal.

Gini coefficients increase as iteration-time increases. The greater the area (growth) of the fractal, the greater the Gini coefficient.

As the fractal is an exponential model, it maybe that the inequality gap will accelerate apart. This acceleration between points is a property of the fractal, and may show itself in reality as the wealth (and income) gap between the poorest and richest expanding (exponentially) as the economy grows.

The inequality of income or wealth distribution is often seen as an undesirable trade-off of the free market, however, as revealed in this fractal experiment, it maybe more true that this inequality is a natural, fractal phenomenon.

The Lorenz distribution of the fractal is a scale invariant pattern that will be viewed at any iteration time – there will always be a Lorenz distribution.

An increasing Gini coefficient with iteration time (at least for wealth) may suggest the Kuznet (reduction of Gini coefficient with time) maybe atypical – a cultural phenomenon. It would be interesting to test whether other biological systems redistribute wealth or income with (economic) growth.

This experiment should be able to be demonstrated with all thing fractal – plants for example.

5 CONCLUSIONS

This investigation found the Lorenz distribution is a property of all things fractal, revealed in fractal structures such as trees, clouds and economies. As the fractal develops (and grows) the income or wealth distribution increases, and the Gini coefficient also increases.

References

1. Lorenz MO. Methods of Measuring the Concentration of Wealth. Publ Am Stat Assoc. 1905;9: 209–219. doi:10.1080/15225437.1905.10503443

2. Gini coefficient [Internet]. Wikipedia, the free encyclopedia. 2015. Available: http://en.wikipedia.org/w/index.php?title=Gini_coefficient&oldid=660810809

3. Prokofiev. Boundary of the Mandelbrot set. [Internet]. 2007. Available: http://commons.wikimedia.org/wiki/File:Boundary_mandelbrot_set.png

4. Koch snowflake [Internet]. Wikipedia, the free encyclopedia. 2014. Available: http://en.wikipedia.org/w/index.php?title=Koch_snowflake&oldid=624336883

5. Macdonald B. Koch Snowflake Fractal Lorenz Model [Internet]. 2015. Available: http://figshare.com/articles/Koch_Snowflake_Fractal_Lorenz_Model/1409389