Demonstrating Lorenz Wealth Distribution and Increasing Gini Coefficient with the Iterating (Koch Snowflake) Fractal Attractor.

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Abstract

The Koch snowflake fractal attractor was analysed by Lorenz and Gini methods. It was found the fractal Lorenz curve fits the wealth (stock) distribution Lorenz curve. Gini coefficient analysis showed an increasing coefficient by iteration (time). It was concluded the Lorenz distribution is a property of the fractal and inextricably linked to (fractal) growth and development.

Keywords:

fractals, Lorenz curve, Gini Coefficient, wealth distribution

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1 INTRODUCTION

The Lorenz curve – first developed by M. O. Lorenz in 1905 [1] – shows the distribution of income in a population as show below in Figure 1. This paper tested whether a Lorenz wealth distribution (a stock as opposed to the flow concept of income) is a fractal phenomenon.

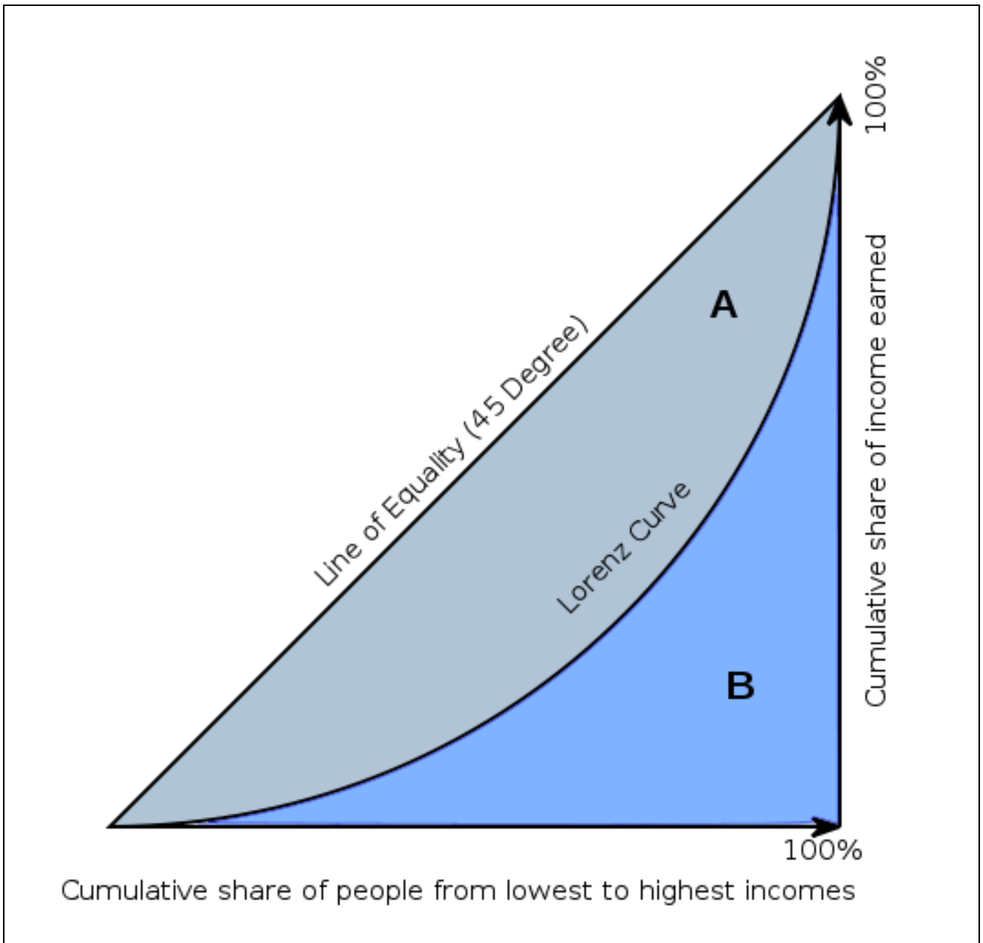

Figure 1. Lorenz Diagram. The graph shows that the Gini coefficient is equal to the area marked A divided by the sum of the areas marked A and B. that is, Gini = A / (A + B). It is also equal to 2*A due to the fact that A + B = 0.5 (since the axes scale from 0 to 1)[2].

Does the fractal offer insight and explanation to Lorenz and Gini data through time? To test for this pattern, the distribution of triangle areas – in a Koch snowflake fractal attractor – were analysed using Lorenz methods; and Gini coefficients (the ratio of area

A to area A+B above) were calculated for each iteration as the fractal grew (or developed).

1.1 The Classical Fractal

Fractals are described as emergent objects from iteration, possessing regular irregularity (same but different) at all scales, and is classically demonstrated by the original Mandelbrot Set (Figure 1 A below).

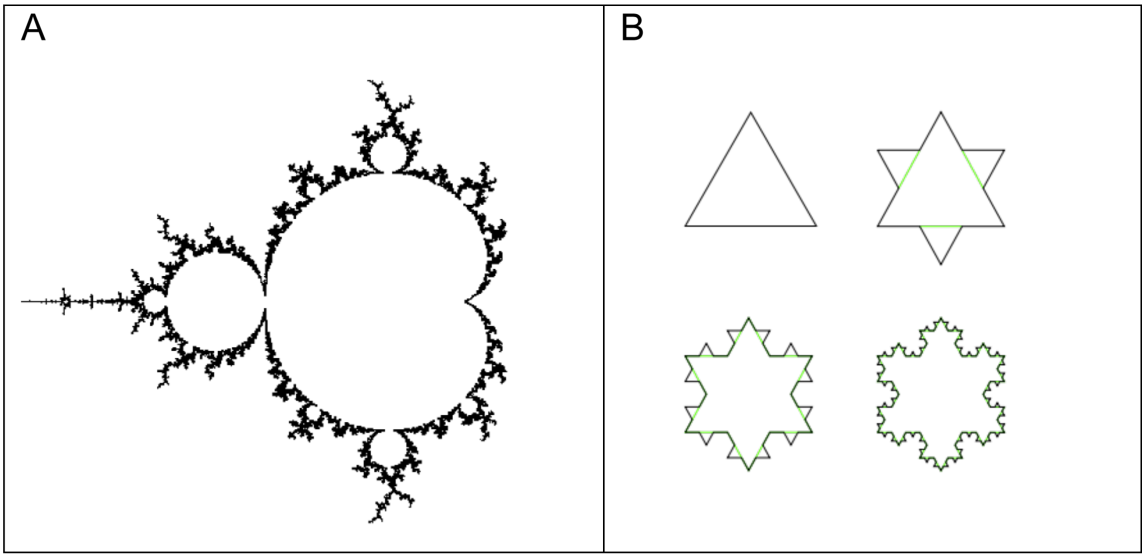

Figure 2. (Classical) Fractals: (A) boundary of the Mandelbrot set; (B) The Koch Snowflake fractal from iteration 0 to 3. Reference: (A) [3]; (B) [4].

The classical fractal shape – as demonstrated in the Koch Snowflake – emerges as a result of the iteration of a simple rule: the repeating the process of adding triangles in the case of the Koch Snowflake. The complete emergent structure is at shape equilibrium (where no more detail can be observed – with additional iterations – to an observer of fixed position) at or around four to seven iterations. This equilibrium iteration count is the observable fractal distance, relative to the observer. This distance is constant irrespective of magnification. For the purposes of this experiment, the equilibrium iteration count is iteration 4.

1.2 Fractspansion – The Fractal Viewed From Within

By inverting the fractal the new triangle size remains constant; while the previous – older – triangles in the series are allowed to expand – rather than diminish as with the classic fractal. The inverted fractal reveals this fractal expansion – termed fractspansion as demonstrated in Figure 2 (B). Colours (red, blue, black followed by purple) and numbers are used to demonstrate the expansion.

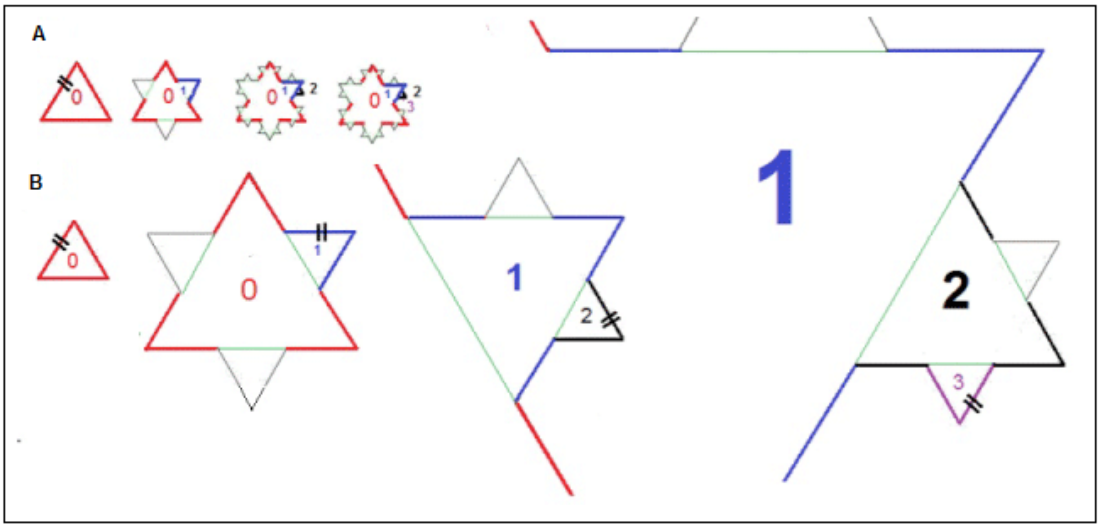

Figure 3. Expansion of the inverted Koch Snowflake fractal (fractspansion): The schematics above demonstrate fractal development by (A) the classical snowflake perspective, where the standard sized thatched (iteration '0') is the focus, and the following triangles diminish in size from colour red iteration 0 to colour purple iteration 3; and (B) the inverted, fractspanding perspective where the new (thatched) triangle is the focus and held at standard size while the original red iteration 0 triangle expands in area – as the fractal iterates.

The size of the initial red iteration 0 triangle, with fractspansion, expands relative to the new. A practical example of this fractspansion principle is to think of the growth of a tree. Follow the first (new growth) stem size – keeping this stem/branch size at a constant size – as the rest of the tree grows. To grow more branches, the volume of the earlier/older branches must expand. Now think of sitting on one the branches of a tree that is infinitely large, infinitely growing.

2 METHODS

To create a quantitative data series for analysis of the area distribution of the inverted fractal, the classical Koch Snowflake area equations were adapted to account for this perspective, and a spreadsheet model [5] was developed.

A data table was produced (Tab 'Table') to calculate the area growth at each, and every iteration of a single triangle. Area was calculated from the following formula (1) measured in standard (arbitrary) units ($\boldsymbol{u}$)

$$A = \frac{l^2\sqrt{3}}{4} \quad (1)$$

where ($\boldsymbol{A}$) is the area of a single triangle, and where l is the triangle's base length. l was placed in Table 1 and was set to 1.51967128766173 $\boldsymbol{u}$ so that the area of the first triangle ($\boldsymbol{t_0}$) approximated an arbitrary area of 1 $\boldsymbol{u}^2$. To expand the triangle with iteration the base length was multiplied by a factor of 3. The iteration number was placed in a column, followed by the base length of the equilateral triangle, and in the final column the formula to calculate the area of the triangle. Calculations were made to the arbitrary 12th iteration, and the results graphed.

2.1 Lorenz Curve

For each iteration a table was created that ranked triangles by their size in ascending order. At each ranked quantity the following was calculated: a percentage quantity was created (Quantity/ Total Quantity) for the line of equality; a percentage area (Area/ Total Area) for the Lorenz curve; Cumulative percentage Area; and finally – for the calculation of the Gini Coefficient – the area under the Lorenz Curve was calculated by

$$\frac{Cum.\%\ Area\ of\ Iteration1\ +\ Cum.\%\ Area\ of\ Iteration2}{2} \times\%\ Quantity\ of\ Iteration1 \times 100$$

2.2 Gini Coefficient

Summing all the areas under the Lorenz Curve gives the area of B. The Area of A is calculated by subtracting B from the area under the line of equality.
The Gini Coefficient is a calculated by

$$\frac{A}{A+B}.$$

Gini Coefficients were calculated for each iteration time, and analysed.

3 RESULTS

Figures 4 to 7 show graphically the results of the experiment.

3.1 Lorenz Curves

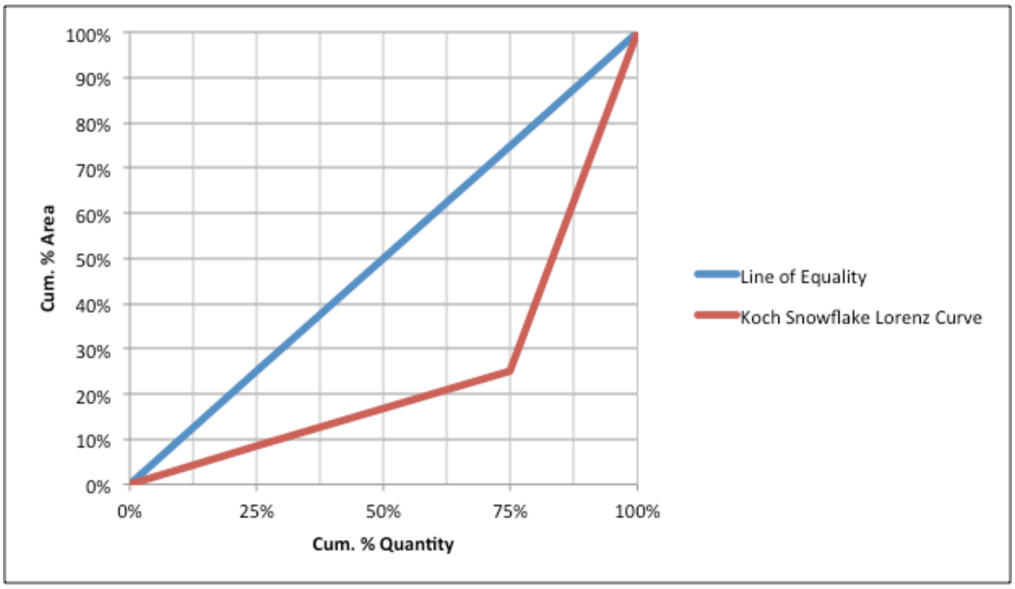

Figure 4. Koch Snowflake Fractal Lorenz Curve at Iteration 2.

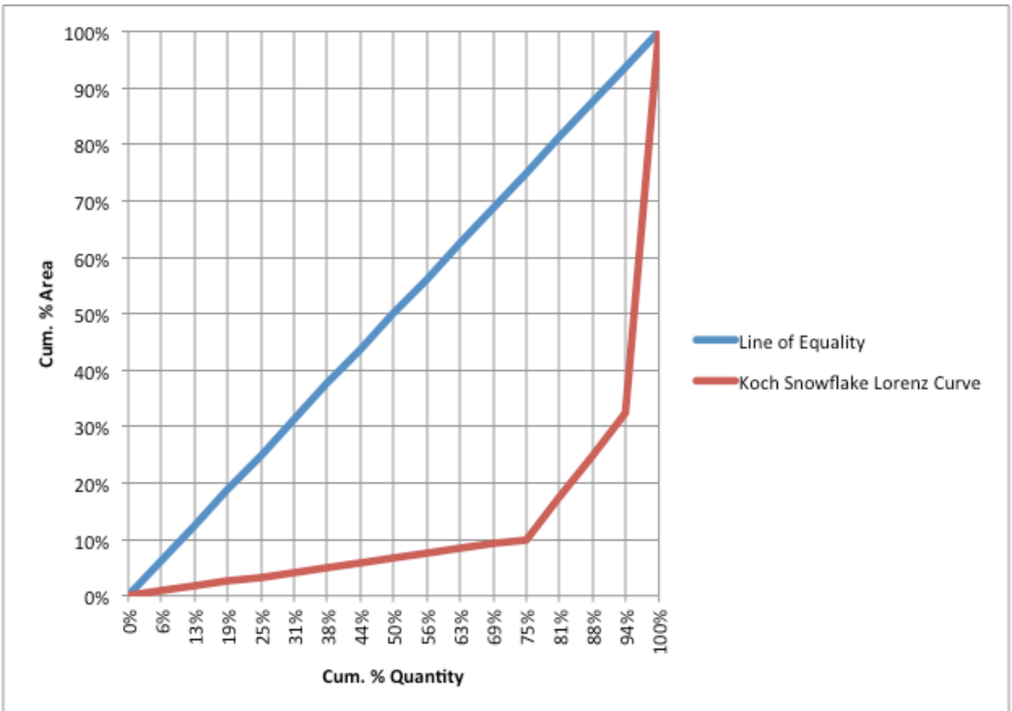

Figure 5. Koch Snowflake Fractal Lorenz Curve at Iteration 3.

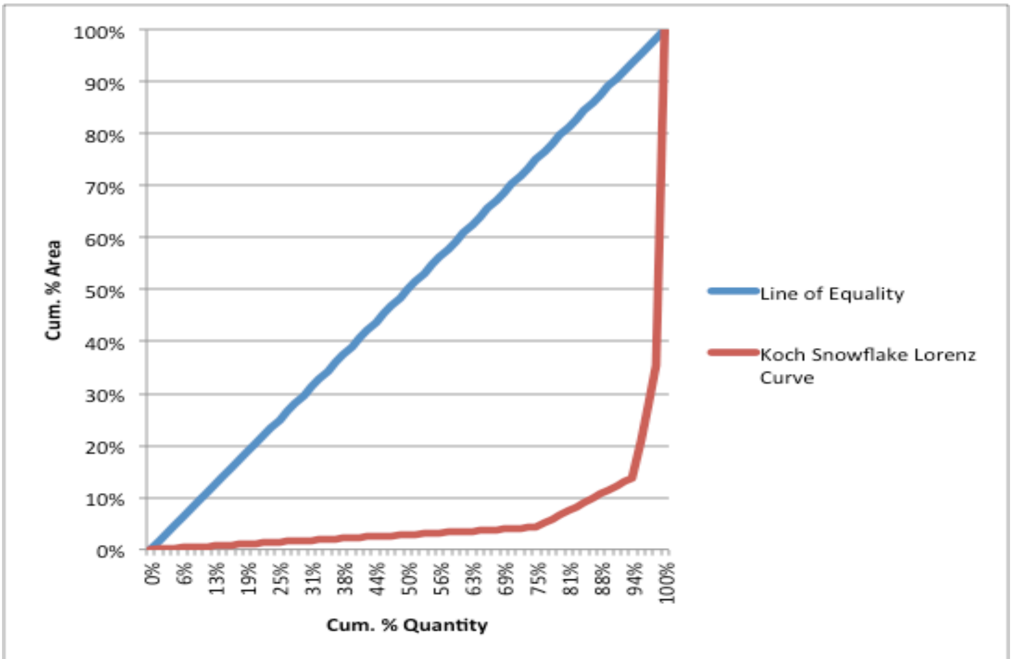

Figure 6. Koch Snowflake Fractal Lorenz Curve at Iteration 4.

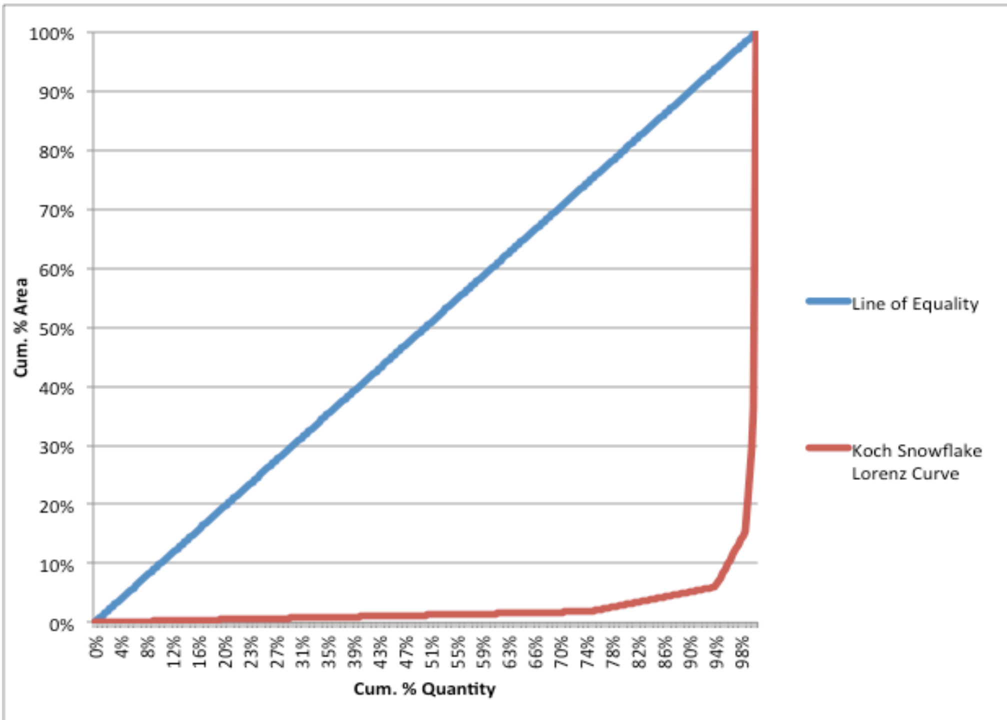

Figure 7. Koch Snowflake Fractal Lorenz Curve at Iteration 5

3.2 Gini Coefficient

Gini coefficients at each iteration are listed below. As iteration time increases, so to does the Gini coefficient.

Table 1: Koch Snowflake Gini Coefficient by Iteration.

Iteration	Gini Coefficient
2	0.75
3	0.7625
4	0.8900
5	0.9498

4 DISCUSSIONS

Area distribution of the Koch Snowflake fractal clearly resembles of wealth distribution. Gini coefficients increase as iteration time increases. The greater the area (growth) of the fractal, the greater the Gini coefficient.

4.1 Exponential Area Increase

In an earlier study from the author on fractal expansion [6], the area of the initial triangle of the inverted Koch Snowflake fractal (and thus the area of any triangle) increased exponentially – as shown below in Figure 8.

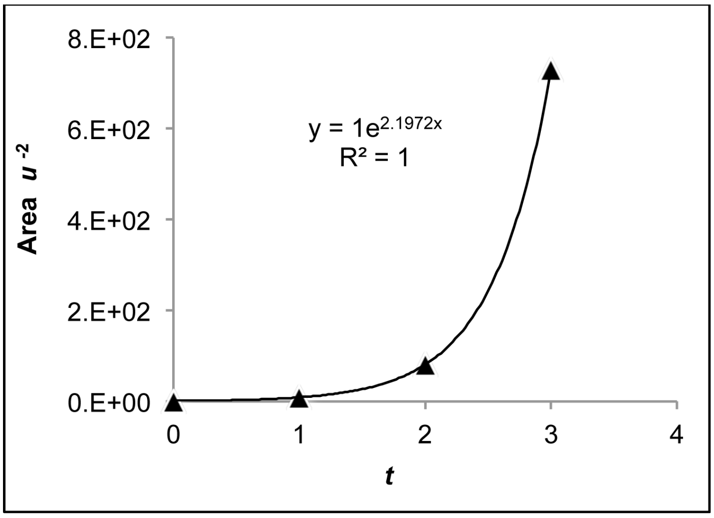

Figure 8. Area Expansion of a single triangle in the inverted Koch Snowflake fractal by iteration time (t) [6]. u = arbitrary length, *t* = iteration.

This expansion with respect to iteration time is written as

$$A = 1e^{2.1972t}. \tag{6}$$

4.2 Accelerated Expansion of Gap

In the same early study[6], the area of the total fractal (Figure 9A) and the distance between points (Figure 9B) on the inverted fractal also expanded exponentially.

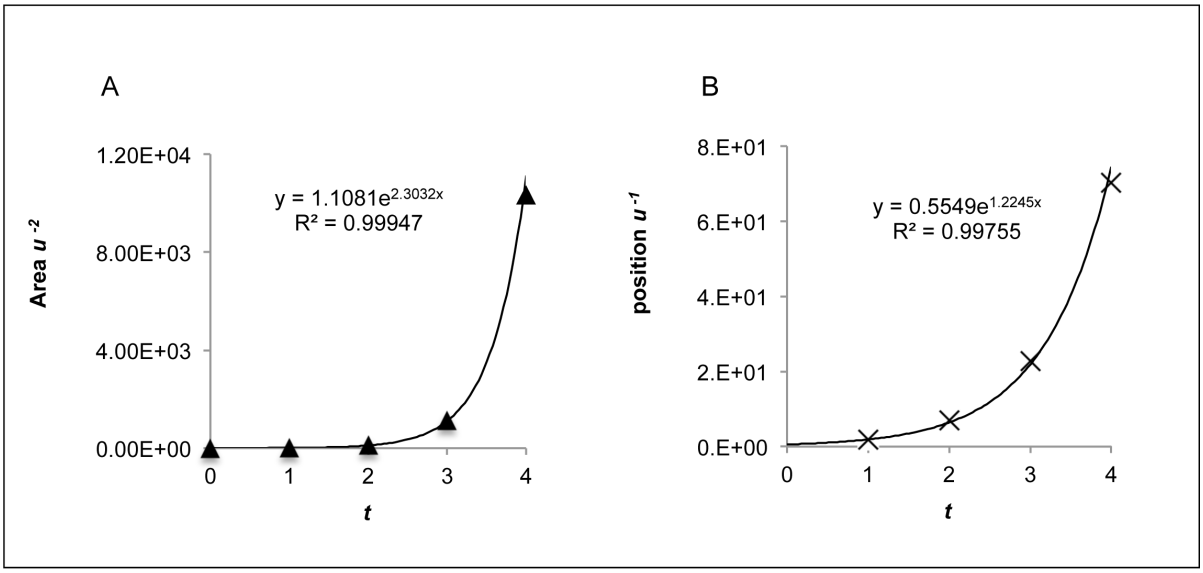

Figure 9. Inverted Koch Snowflake fractal expansion per iteration time (t). (A) total area expansion and (B) distance between points. u = arbitrary length *t* = iteration.

The expansion of area is described as

$$A^{T} = 1.1081e^{2.3032t} \tag{7}$$

where A^{T} is the total area.

The expansion of distance between points is described by the equation

$$D = 0.5549e^{1.2245t} \tag{8}$$

where D is the distance between points.

This acceleration between points is a property of the fractal, and may show itself in reality as the wealth (and income) gap between the poorest and richest expanding (exponentially) as the economy grows.

The disparity of income or wealth distribution is often seen as a trade off with market efficiency; if this is so, the disparity – as revealed in this experiment to be fractal –may also suggest the market is a fractal phenomenon.

An increasing Gini coefficient with iteration time (at least for wealth) may suggest the Kuznet (reduction of Gini coefficient with time) maybe atypical – a cultural phenomenon. It would be interesting to test whether other biological systems redistribute wealth or income with (economic) growth.

5 CONCLUSIONS

This investigation it was found Lorenz distribution is a property of all things fractal, revealed in fractal structures such as trees, clouds and societies.

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