TROUBLES WITH SINGLE-VALUE ALGEBRAIC STRUCTURES' DEFINITION IN SET THEORY AND SOME WAYS TO SOLVE THEM

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ABSTRACT. It seems that statements determining features of some algebraic structures behavior are based on just intuitive assumptions or empiric observations and for sake of convenience (simplest example is the phrase: "let's consider 0! =1"… perhaps, just because sir I. Newton entrusted, so, why not 2, 5, 7.65 – choose any). So, without logical explanation these are looking a little mysterious or sometimes even magic. This article is a humble attempt to get it straight rather formally. Some troubles may appear on the way – e.g. as it was shown earlier (in the ref. [2], for example), there are at least two binary relations having properties of idempotent equivalences – algebra's elements that may aspire to be an identity. Apparently, probable obtaining of some well-known results in the text is not an attempt of their re-discovering, but it is rather "check-points" that confirm theory validity, more by token that it was made by using of the only exceptionally formal way, while usually they are obtained rather intuitively. Usually the notion of tensor product is determined for each kind of algebraic structure – especially for modulus (in group theory it is often called direct product – but this is a matter of semantics, so, it's rather negligible). Here it is shown that tensor product may be introduced without defining of concrete algebraic structure. Without such introduction defining of algebraic operation is strongly complicated.

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1. Terms and notations

So, it becomes clear that transitivity does be necessary condition for single-value set ordering[1] by the relation. But it is also clear that it rather cannot be possible in case of any correspondence just due to the reversibility indeterminacy in the right part of inclusion (1.3.17[2]) that may be expressed by inequality

$$\boldsymbol{A}_I \cap \boldsymbol{B}_I^* \neq \boldsymbol{B}_I \cap \boldsymbol{A}_I^*.$$

Thus, correspondence "transitivity" rather doesn't exist. But in this case there's reversibility existence for both of intersections $\boldsymbol{A}_I \cap \boldsymbol{A}_I^*$ and $\boldsymbol{B}_I \cap \boldsymbol{B}_I^*$, but separately. Now it is prematurely to talk about correspondence one-one defining criteria, but if such correspondences would exist, then they called *functions* mapping set A into set B. As a rule, to do accent correspondence's *functionality* there may be used special symbols

$$\left.\begin{array}{c} \varphi: A \to B \\ \varphi(x) \Rightarrow y \\ \varphi(a) \Rightarrow b \\ \varphi(A) = A\varphi \subseteq B \\ A \xrightarrow{\varphi} B \end{array}\right\}. \tag{1.1}$$

In accordance with accepted denotations for set elements logical symbols (implications) are used. They are supposed previously to be one directed – just because it's not so clear whether or not they'll still be functional under their inversion. But there may be said for sure – such kind of inclusion direction occurs in the fourth of these expressions, because it is harmonized to implication direction inherent to conjunction. Set $A\varphi$ is named *image* of set A in function's range (co-domain). It might be interpret as a right composition with set. It's clear that there's no obligatory commutativity for such kind of activity, but this is associative[3] – this permits to suspect it as some multiplicative action. For binary relation arrow-function and mapping sets put together a *triple* that is labeled *morphism*.

Formula (1.3.2) describes correspondence multiplication – for functions, in new terms of formulae (1.1) it looks like

$$\left.\begin{array}{c} A \xrightarrow{\varphi} B \wedge C \xrightarrow{\pi} D \Rightarrow A \xrightarrow{\psi} D \\ A \xrightarrow{\varphi} (B \cap C) \xrightarrow{\pi} D \subseteq A \xrightarrow{\psi} D \\ (\varphi \circ \chi)(x) = \pi[\varphi(x)] \subseteq \psi(x) \\ (B \cap C = \emptyset) \Rightarrow (\varphi \circ \chi = \emptyset) \end{array}\right\}. \tag{1.2}$$

As it was before, the last of these conditions admits an existence of empty compositions of non-empty participants. Actually, it does not contradict with axiom of choice because non-empty Cartesian product contains empty subset too, but the last one boils it down to its ambiguous definition. To avoid it the notion of *mapping* is introduced. Its definition

[1] More specifically – it's done to lead in one-one correspondence between two sets equal to each other.
[2] As it was earlier – the first cipher labels external reference number – e.g. (1.3.17) means formula (3.17) in ref. [1].
[3] This is used in the fourth formula.

does almost not differ from the function's one but with only exception – not any of their composition is mapping's one but just as a result of their *sequential* multiplication

$$\left.\begin{array}{c}\pi\colon A\to C\wedge\varphi\colon C\to B\Rightarrow\psi\colon A\to B\\ A\overset{\pi}{\to}(C\cap C)\overset{\varphi}{\to}B\subseteq A\overset{\psi}{\to}B\\ \pi\circ\varphi\subseteq\psi\end{array}\right\}. \tag{1.3}$$

In case of equality as kind of such inclusion one may talk about mapping's *factorization* – its representation by *factors'* compositions. It may be seen that if all sets here are equal to each other, such expression coincides to idempotent transitivity condition.

Therefore, correspondence functionality is something akin to relation's transitivity; however it's still not clear how to propagate symmetry activities on the correspondences. Apprehensions are based on the fact that among initial correspondences multiplication an "identity" is left diagonal but among reversed ones – right one – and there's no genuine identity anymore that may be pointed out. But anyway it has been shown [2] there is a possibility to construct symmetric compositions by using antisymmetric relations. Such kind of them may be written for correspondences too

$$\left.\begin{array}{c}\ker\varphi=\varphi\circ\varphi^{-1}\subseteq A\\ \operatorname{im}\varphi=\varphi^{-1}\circ\varphi\subseteq B\end{array}\right\}. \tag{1.4}$$

In these terms the first one defines function *kernel*; the second one – its *image*. The last one is not obliged to coincide with set image but in any case it is a subset of co-domain

$$\operatorname{im}\varphi\subseteq A\varphi\subseteq B.$$

So, for non-commutative functions' compositions such order does reside

$$\left.\begin{array}{c}\varphi=\varphi\circ\varphi^{-1}=\varphi^{-1}[\varphi(x)]\\ \varphi=\varphi^{-1}\circ\varphi=\varphi[\varphi^{-1}(y)]\\ \varphi\circ\varphi^{-1}\neq\varphi[\varphi^{-1}(y)]\\ \varphi^{-1}\circ\varphi\neq\varphi^{-1}[\varphi(x)]\end{array}\right\}. \tag{1.5}$$

In contrary, for commutative compositions, it's not important. Two last expressions describe the order that is not inherent to non-commutative functional multiplication[4].

As it has been shown for relations [1], such compositions are not a result of their mutual inversion; more by token, here it is aggravated by the presence of two different "identities". But among inclusions there may be extracted equality. Function is called an *injection* of set A *into* set B (or their *monomorphism*) if it is performed by the expression

$$\left.\begin{array}{c}\ker\varphi=\mathrm{id}_A\\ B(x)=\varphi^{-1}B=A\end{array}\right\}. \tag{1.6}$$

The fact, that diagonal of any set cannot be empty, permits to state of injection existence without domain

[4] It may be possible but such product is not functional.

$$\left.\begin{array}{c}\ker\varphi = \mathrm{id}_{\emptyset}\\ \varphi\colon\emptyset\to B\end{array}\right\}. \tag{1.7}$$

There may be shown that for injection such kind of correspondence is inherent

$$\left.\begin{array}{c}(a = c) \Rightarrow [\varphi(a) = \varphi(c)]\\ (a \neq c) \Rightarrow [\varphi(a) \neq \varphi(c)]\end{array}\right\}. \tag{1.8}$$

Another kind of similar inclusions is equality

$$\left.\begin{array}{c}\operatorname{im}\varphi = \mathrm{id}_{B}\\ \varphi(A) = A\varphi = B\end{array}\right\}. \tag{1.9}$$

This is usually called *surjection* of set A *on* set B (or their *epimorphism*). Due to inequality $\mathrm{id}_B \neq \emptyset$ there may be written inequality $\varphi^{-1}(y) \circ y = \mathrm{id}_B \neq \emptyset$. It leads to conventional surjection determining

$$A = \operatorname{dom}\varphi \neq \emptyset. \tag{1.10}$$

But it doesn't deny the existence of surjection that has no range

$$\left.\begin{array}{c}\operatorname{im}\varphi = \mathrm{id}_{\emptyset}\\ \varphi\colon A\to\emptyset\end{array}\right\}. \tag{1.11}$$

If both equalities (1.6) and (1.9) are put together, then it is named *bijection* of sets or their *isomorphism*

$$\left.\begin{array}{c}\ker\varphi = \mathrm{id}_{A}\\ \operatorname{im}\varphi = \mathrm{id}_{B}\\ B\varphi^{-1} = A\\ \varphi A = B\end{array}\right\}. \tag{1.12}$$

Two last equalities lead to one-one equalities

$$\left.\begin{array}{c}B = \varphi B\varphi^{-1}\\ A = \varphi^{-1}A\varphi\end{array}\right\}. \tag{1.13}$$

It's clear that mapping of empty sets may be also observed as bijection[5]

$$\mathrm{id}_{\emptyset}\colon\emptyset\to\emptyset. \tag{1.14}$$

Obviously, this is direct corollary of equalities (1.8) and (1.11).

Seemingly, formulae (1.11) are particular for some general expressions containing inclusions instead of equalities. Unifying expressions (1.8) and (1.10) there may be written correspondence *functionality* common *criterion*

[5] Apparently, it explains why zero-factorial does be equal to one.

$$\left.\begin{array}{c}\ker\varphi \supseteq \mathrm{id}_A \\ \mathrm{im}\,\varphi \subseteq \mathrm{id}_B \\ B\varphi^{-1} \supseteq A \\ \varphi A \subseteq B\end{array}\right\}. \tag{1.15}$$

They are generalization for both equalities (1.12) and inclusions (1.4.15). But about the last one there may be said the following. As it has been shown, transitivity is the invariant with respect to inversion. The similar invariance is inherent to isomorphism too in spite of its criterion contains two different from each other diagonals. Now it is a little prematurely to talk about similar invariance for any functional correspondences – injections and surjections, particularly. For them such criteria may be written as

$$\left.\begin{array}{c}\ker\varphi = \mathrm{id}_A \\ \mathrm{im}\,\varphi \subset \mathrm{id}_B \\ B\varphi^{-1} = A \\ \varphi A \subset B\end{array}\right\}, \tag{1.16}$$

$$\left.\begin{array}{c}\ker\varphi \supset \mathrm{id}_A \\ \mathrm{im}\,\varphi = \mathrm{id}_B \\ B\varphi^{-1} \supset A \\ \varphi A = B\end{array}\right\}. \tag{1.17}$$

Instead of equalities (1.4) the only rigid inclusions may be written for them

$$\left.\begin{array}{c}\varphi A = \varphi B\varphi^{-1} \subset B \\ B\varphi^{-1} = \varphi^{-1}A\varphi \supset A\end{array}\right\}. \tag{1.18}$$

The first one concerns injection, the second – surjection. But this is still not yet an answer to question possibility of functionality to be invariant with respect to inversion.

Multiplying mappings π and φ, where π is surjective and φ is injective, their composition will be bijective – in this case the equality occurs

$$(\pi \circ \varphi)^{-1} = \varphi^{-1} \circ \pi^{-1} = \pi^{-1}[\varphi^{-1}(y)] \neq \emptyset.$$

It points out that such composition is surjective. In accordance with (1.5), there may be written formula

$$\pi \circ \varphi = \varphi[\pi(x)].$$

Because of φ is an injective, condition (1.8) occurs, in accordance with (1.8) there may be written formulae

$$\left.\begin{array}{c}[\pi(a) = \pi(b)] \to \{\varphi[\pi(a)] = \varphi[\pi(b)]\} \\ {}[\pi(a) \neq \pi(b)] \to \{\varphi[\pi(a)] = \varphi[\pi(b)]\}\end{array}\right\}.$$

It convinces of assumption’s rightness.

But it was achieved rather empirically – relevant criteria of whether or not some kind of composition leads to factorization still weren’t achieved. To get them it needs to point out following. Necessity condition of mapping multiplying is the idempotent intersection of indexing sets' presence – this is its only difference from function's multiplying. But, seemingly, it is not sufficient to be factorization's criterion. But for precedent indentation composition there may be written implication

$$\left.\begin{array}{r}\ker\pi\supset\mathrm{id}_A\\ \operatorname{im}\pi=\mathrm{id}_C\\ \ker\varphi=\mathrm{id}_C\\ \operatorname{im}\varphi\subset\mathrm{id}_B\end{array}\right\}\mapsto\begin{cases}\ker(\pi\circ\varphi)=\mathrm{id}_A\\ \operatorname{im}(\pi\circ\varphi)=\mathrm{id}_B\end{cases}. \tag{1.19}$$

Its features is that kernel of the first mapping coincides with image of the second one. If there is a *sequence* ... $Q\xrightarrow{\varphi_i}P\xrightarrow{\varphi_{i+1}}R$ ..., it is called *exact* since there is isomorphic coincidence for any of its participant

$$\left.\begin{array}{c}\ker\varphi_{i+1}\cong\operatorname{im}\varphi_i\\ \varphi_{i+1}\circ\varphi_{i+1}^{-1}\cong\varphi_i^{-1}\circ\varphi_i\end{array}\right\}. \tag{1.20}$$

Sequence (1.19) is exact; unifying it with composition of two bijections there may be written such criterion

$$\left.\begin{array}{r}\ker\pi\supseteq\mathrm{id}_A\\ \operatorname{im}\pi=\mathrm{id}_C\\ \ker\varphi=\mathrm{id}_C\\ \operatorname{im}\varphi\subseteq\mathrm{id}_B\end{array}\right\}\mapsto\begin{cases}\ker(\pi\circ\varphi)=\mathrm{id}_A\\ \operatorname{im}(\pi\circ\varphi)=\mathrm{id}_B\end{cases}. \tag{1.21}$$

In case of two injections' or surjections' compositions such criteria look like

$$\left.\begin{array}{r}\ker\pi=\mathrm{id}_A\\ \operatorname{im}\pi\subset\mathrm{id}_C\\ \ker\varphi=\mathrm{id}_C\\ \operatorname{im}\varphi\subset\mathrm{id}_B\end{array}\right\}\leftrightarrow\begin{cases}\ker(\pi\circ\varphi)=\mathrm{id}_A\\ \operatorname{im}(\pi\circ\varphi)\subset\mathrm{id}_B\end{cases}, \tag{1.22}$$

$$\left.\begin{array}{r}\ker\pi\supset\mathrm{id}_A\\ \operatorname{im}\pi=\mathrm{id}_C\\ \ker\varphi\supset\mathrm{id}_C\\ \operatorname{im}\varphi=\mathrm{id}_B\end{array}\right\}\leftrightarrow\begin{cases}\ker(\pi\circ\varphi)\supset\mathrm{id}_A\\ \operatorname{im}(\pi\circ\varphi)=\mathrm{id}_B\end{cases}. \tag{1.23}$$

These sequences are not exact; and, obviously, permutation of participants changes nothing here. Multiplying by bijection at the proper side (left or right – depends on another participant) doesn't change initial mapping too

$$\left.\begin{array}{r}\ker\pi\supseteq\mathrm{id}_A\\ \operatorname{im}\pi=\mathrm{id}_C\\ \ker\varphi=\mathrm{id}_C\\ \operatorname{im}\varphi=\mathrm{id}_B\end{array}\right\}\to\begin{cases}\ker(\pi\circ\varphi)\supseteq\mathrm{id}_A\\ \operatorname{im}(\pi\circ\varphi)=\mathrm{id}_B\end{cases}, \tag{1.24}$$

$$\left.\begin{array}{r}\ker\pi=\mathrm{id}_A\\ \operatorname{im}\pi=\mathrm{id}_C\\ \ker\varphi=\mathrm{id}_C\\ \operatorname{im}\varphi\subseteq\mathrm{id}_B\end{array}\right\}\to\begin{cases}\ker(\pi\circ\varphi)=\mathrm{id}_A\\ \operatorname{im}(\pi\circ\varphi)\subseteq\mathrm{id}_B\end{cases}. \tag{1.25}$$

Left part of inclusion (1.19) is correct that it contains the proper mapping succeed order. In contrary, permutation of participants will lead sequence not to be exact

$$\left.\begin{array}{r}\ker\varphi = \mathrm{id}_A\\ \operatorname{im}\varphi \subseteq \mathrm{id}_C\\ \ker\pi \supseteq \mathrm{id}_C\\ \operatorname{im}\pi = \mathrm{id}_B\end{array}\right\} \nrightarrow \begin{cases}\ker(\varphi\circ\pi) = \mathrm{id}_A\\ \operatorname{im}(\varphi\circ\pi) = \mathrm{id}_B\end{cases}. \tag{1.26}$$

Perhaps, such kind of compositions even exists but it is not functional.

In any case bijections are reversible. So, writing factorized expression for inverted one $\pi^{-1}: B \to C \wedge \varphi^{-1}: C \to A \Leftrightarrow \psi^{-1}: B \to A$, there may be achieved that inversion of surjection leads to injection and inside out

$$\left.\begin{array}{r}\left.\begin{array}{r}\ker\varphi = \mathrm{id}_A\\ \operatorname{im}\varphi \subset \mathrm{id}_B\end{array}\right\} \leftrightarrow \left.\begin{array}{r}\ker\varphi^{-1} \supset \mathrm{id}_A\\ \operatorname{im}\varphi^{-1} = \mathrm{id}_B\end{array}\right\}\\ \left.\begin{array}{r}\ker\varphi \supset \mathrm{id}_A\\ \operatorname{im}\varphi = \mathrm{id}_B\end{array}\right\} \leftrightarrow \left.\begin{array}{r}\ker\varphi^{-1} = \mathrm{id}_A\\ \operatorname{im}\varphi^{-1} \subset \mathrm{id}_B\end{array}\right\}\end{array}\right\}. \tag{1.27}$$

Thus, function's bijection is akin to relation's symmetry, injection and surjection are two forms of anti-symmetry that undistinguishing for relations because of existence of double-sided genuine identity there. Exactness criterion for relations looks like

$$\left.\begin{array}{r}\ker\tau_i \supseteq \mathrm{id}_A\\ \operatorname{im}\tau_i \subseteq \mathrm{id}_A\\ \ker\tau_j \supseteq \mathrm{id}_A\\ \operatorname{im}\tau_j \subseteq \mathrm{id}_A\end{array}\right\} \mapsto \begin{cases}\ker\left(\tau_i\circ\tau_j\right) \supseteq \mathrm{id}_A\\ \operatorname{im}\left(\tau_i\circ\tau_j\right) \subseteq \mathrm{id}_A\end{cases}.$$

Obviously, it wouldn't change if multiplicands rearranged – it should be the same – they would be still reversible

$$\left.\begin{array}{r}\ker\tau_j \supseteq \mathrm{id}_A\\ \operatorname{im}\tau_j \subseteq \mathrm{id}_A\\ \ker\tau_i \supseteq \mathrm{id}_A\\ \operatorname{im}\tau_i \subseteq \mathrm{id}_A\end{array}\right\} \mapsto \begin{cases}\ker\left(\tau_j\circ\tau_i\right) \supseteq \mathrm{id}_A\\ \operatorname{im}\left(\tau_j\circ\tau_i\right) \subseteq \mathrm{id}_A\end{cases}.$$

Therefore, unifying these criteria there may be written

$$\left.\begin{array}{r}\ker\tau_i \supseteq \mathrm{id}_A\\ \operatorname{im}\tau_i \subseteq \mathrm{id}_A\\ \ker\tau_j \supseteq \mathrm{id}_A\\ \operatorname{im}\tau_j \subseteq \mathrm{id}_A\end{array}\right\} \leftrightarrow \begin{cases}\ker\left(\tau_i\circ\tau_j\right) \supseteq \mathrm{id}_A\\ \operatorname{im}\left(\tau_i\circ\tau_j\right) \subseteq \mathrm{id}_A\end{cases}. \tag{1.28}$$

So, diagonals' distinction "range narrowing" is compensated by peculiar "range expansion" of exactness criteria applicability – i.e. coincidence (1.19) or (1.20) for relations to remain transitive is not obliged anymore.

2. Transitivity via functionality

As it was inspired at previous paragraph, in some cases there may be convenient to describe relations by using functional terms. Actually, such distinction itself between relations and correspondences is often very difficult – e.g. some subset of a set is a set too, but these are may be different sets as for each other.

First of all, in any set there exists diagonal that makes mapping which may be called sets *overlap*

$$\mathrm{id}_X : X \to X. \tag{2.1}$$

Surely, in any case this is bijective. Exactness criterion here looks like

$$\left.\begin{aligned} \ker(\mathrm{id}_X) = \mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X \\ \mathrm{im}(\mathrm{id}_X) = \mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X \end{aligned}\right\}. \tag{2.2}$$

In essence, sets staying distinct sides of the arrow may be quite different and even disjoint, but there is an isomorphism between them. As an example, they may distinct by a mode of *permutation* or *substitution*

$$\left.\begin{aligned} \begin{pmatrix} 1 & 2 & \dots n \\ i_1 & i_1 & \dots i_n \end{pmatrix} \ncong \begin{pmatrix} 1 & 2 & \dots n \\ j_1 & j_2 & \dots j_n \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & \dots n \\ i_1 & i_1 & \dots i_n \end{pmatrix} \cap \begin{pmatrix} 1 & 2 & \dots n \\ j_1 & j_2 & \dots j_n \end{pmatrix} = \emptyset \end{aligned}\right\}.$$

If inclusion exists between mapping sets, then formula (2.1) looks like

$$\mathrm{id}_{X'} = \mathrm{id}_{X'|X' \subseteq X} : X' \to X. \tag{2.3}$$

This diagonal realizes a mapping that may be called *embedding* of set X' *into* set X. In case of total sets distinction they are, surely, *incomparable* – an attempt to compare them leads to contradiction (1.3.8). Even more, one can write its middle part as

$$\mathrm{id}_X \neq \mathrm{id}_X \circ \mathrm{id}_Y = \{\langle x, y\rangle : \langle x, x\rangle \wedge \langle y, y\rangle\} \neq \mathrm{id}_Y.$$

There may be said that such “composition’ is neither invariant semantically, nor diagonal. Formally, this is due to diagonal *bases* – mapping sets themselves are *independent* on each other. But probable their inclusion may take away uncertainty, at least partially. In any case there may be said that for non-empty set such biconditional may occur

$$\left.\begin{aligned} X' \subseteq X \leftrightarrow \mathrm{id}_{X'} \subseteq \mathrm{id}_X \\ \emptyset \subset X \nleftrightarrow \mathrm{id}_\emptyset \subset \mathrm{id}_X \end{aligned}\right\}. \tag{2.4}$$

As any injection, sequential embedding composition of two mappings is embedding too

$$\mathrm{id}_{X'' \subseteq X'} \circ \mathrm{id}_{X' \subseteq X} = \mathrm{id}_{X'' \subseteq X' \subseteq X} = \mathrm{id}_{X'' \subseteq X}. \tag{2.5}$$

Converted inclusion allows defining that may be called sets *covering*

$$\mathrm{id}_X = \mathrm{id}_{X|X\supseteq X'}\colon X \to X'. \tag{2.6}$$

This is surjective and it may be shown formally

$$\left.\begin{array}{c}\ker\left(\mathrm{id}_{X|X\supseteq X'}\right) = \mathrm{id}_X \circ (\mathrm{id}_X)^{-1} = \mathrm{id}_X \supseteq \mathrm{id}_{X'}\\ \mathrm{im}\left(\mathrm{id}_{X|X\supseteq X'}\right) = (\mathrm{id}_X)^{-1} \circ \mathrm{id}_X = \mathrm{id}_X\end{array}\right\}. \tag{2.7}$$

As it was in case of embedding, similar expression may be written too for covering

$$\mathrm{id}_{X\supseteq X'} \circ \mathrm{id}_{X'\supseteq X''} = \mathrm{id}_{X\supseteq X'\supseteq X''} = \mathrm{id}_{X\supseteq X''}. \tag{2.8}$$

This may be interpreted as corollary of inclusion's transitivity.

Such discourses may be propagated to correspondences even though they are not connected with each other by inclusion. Anyway, there may be extracted range subset having diagonal that realize mapping

$$\mathrm{id}_{X'\subseteq X}\colon X' \to X.$$

In accordance with (1.3), in this case the unique functional sequential composition may correlate as

$$\left.\begin{array}{c}\mathrm{id}_{X'\subseteq X}\colon X' \to X \wedge \varphi\colon X \to Y \Rightarrow \varphi|X'\colon X' \to Y\\ \mathrm{id}_{X'\subseteq X} \circ \varphi \subseteq \varphi|X'\end{array}\right\}. \tag{2.9}$$

Mapping $\varphi|X'$ may be called *restriction* of mapping φ *to subset* X'. If mapping is injective, then its restriction to subset is injective too – i.e. one may write

$$\left.\begin{array}{c}\ker\varphi = \mathrm{id}_X\\ \mathrm{im}\,\varphi \subseteq \mathrm{id}_Y\end{array}\right\} \mapsto \left\{\begin{array}{c}\ker\varphi|X' = \mathrm{id}_{X'\subseteq X}\\ \mathrm{im}\,\varphi|X' \subseteq \mathrm{id}_Y\end{array}\right.. \tag{2.10}$$

Similar statement may be done concerning bijection. Particularly, returning to relations, in any set there may be pointed out mapping that is called *natural embedding*

$$\left.\begin{array}{c}\mathrm{id}_{X'\subseteq X} \circ \mathrm{id}_X \subseteq \mathrm{id}_X|X'\\ \ker\mathrm{id}_X|X' = \mathrm{id}_{X'\subseteq X}\\ \mathrm{im}\,\mathrm{id}_X|X' \subseteq \mathrm{id}_X\end{array}\right\}. \tag{2.11}$$

Formally, there may be observed case of improper inclusion too

$$\left.\begin{array}{c}\mathrm{id}_X\colon X \to X \wedge \varphi\colon X \to Y \Leftrightarrow \varphi|X\colon X \to Y\\ \mathrm{id}_X \circ \varphi = \varphi|X\end{array}\right\}. \tag{2.12}$$

But if mapping φ is surjective, it cannot be restricted to subset

$$\left.\begin{array}{c}\ker\varphi \supset \mathrm{id}_X\\ \mathrm{im}\,\varphi = \mathrm{id}_Y\end{array}\right\} \mapsto \left\{\begin{array}{c}\mathrm{id}_{X'\subseteq X}\colon X' \to X \wedge \varphi\colon X \to Y \nRightarrow \varphi|X'\colon X' \to Y\\ \mathrm{id}_{X'\subseteq X} \circ \varphi \nsubseteq \varphi|X'\end{array}\right.. \tag{2.13}$$

Due to apprehensive reason such correspondence is not functional.

The same time, there may be extracted range subset too, getting embedding

$$\mathrm{id}_{Y'\subseteq Y}\colon Y' \to Y.$$

It correlates to sequence

$$\left.\begin{array}{c}\varphi|Y'\colon X \to Y' \wedge \mathrm{id}_{Y'\subseteq Y}\colon Y' \to Y \Rightarrow \varphi\colon X \to Y \\ \varphi|Y' \circ \mathrm{id}_{Y'\subseteq Y} \subseteq \varphi\end{array}\right\}. \tag{2.14}$$

Then there may be said that mapping φ is *cut* (*constricted*) *down to* mapping $\varphi|Y'$. Due to apprehensive reason, this inclusion (implication) would, surely, occur if mapping φ were injective. If mapping φ is bijective, then its constriction is surjective and this expression becomes equality

$$\left.\begin{array}{c}\varphi|Y'\colon X \to Y' \wedge \mathrm{id}_{Y'\subseteq Y}\colon Y' \to Y \Leftrightarrow \varphi\colon X \to Y \\ \varphi|Y' \circ \mathrm{id}_{Y'\subseteq Y} = \varphi\end{array}\right\}. \tag{2.15}$$

This sequence is exact and correspondent criterion looks like

$$\left.\begin{array}{c}\ker \varphi|Y' \supseteq \mathrm{id}_X \\ \mathrm{im}\, \varphi|Y' = \mathrm{id}_{Y'} \\ \ker \mathrm{id}_{Y'\subseteq Y} = \mathrm{id}_{Y'} \\ \mathrm{im}\, \mathrm{id}_{Y'\subseteq Y} \subseteq \mathrm{id}_Y\end{array}\right\} \mapsto \left\{\begin{array}{c}\ker(\varphi|Y' \circ \mathrm{id}_{Y'\subseteq Y}) = \mathrm{id}_X \\ \mathrm{im}(\varphi|Y' \circ \mathrm{id}_{Y'\subseteq Y}) = \mathrm{id}_Y\end{array}\right.. \tag{2.16}$$

By apprehensive reason, constriction of surjection may be just improper

$$\left.\begin{array}{c}\varphi|Y\colon X \to Y \wedge \mathrm{id}_Y\colon Y \to Y \Leftrightarrow \varphi\colon X \to Y \\ \varphi|Y \circ \mathrm{id}_Y = \varphi\end{array}\right\}. \tag{2.17}$$

Mapping φ itself is an *extension* for both its restriction and its constriction.

If set X is indexed by some set $I = \{i\}$, then inclusion $X_i \subseteq \boldsymbol{X}$ exists. Thus, there is embedding

$$\mathrm{id}_{X_i\subseteq \boldsymbol{X}}\colon X_i \to \boldsymbol{X}.$$

Family indexing procedure may be written as mapping

$$\iota\colon I \to \boldsymbol{X}. \tag{2.18}$$

So, there is appeared a possibility to write sequence of form (2.14)

$$\left.\begin{array}{c}\iota|X_i\colon I \to X_i \wedge \mathrm{id}_{X_i\subseteq \boldsymbol{X}}\colon X_i \to \boldsymbol{X} \Leftrightarrow \iota\colon I \to \boldsymbol{X} \\ \iota|X_i \circ \mathrm{id}_{X_i\subseteq \boldsymbol{X}} = \iota\end{array}\right\}. \tag{2.19}$$

Just because we are talking about non-empty indexing, domain of its constriction is not empty too – that is why constriction is surjective and indexing of its own is bijective. So, one may write

$$\left.\begin{array}{c}\ker \iota = \mathrm{id}_I \\ \mathrm{im}\, \iota = \mathrm{id}_{\boldsymbol{X}}\end{array}\right\}. \tag{2.20}$$

But it is already comprehensive that existence of surjective indexing is well permissible. Actually, while it is surjective, its constriction just is not proper

$$\left.\begin{matrix} \iota|X: I \to X \wedge \mathrm{id}_{\boldsymbol{X}}: X \to \boldsymbol{X} \Leftrightarrow \iota: I \to \boldsymbol{X} \\ \iota|X \circ \mathrm{id}_{\boldsymbol{X}} = \iota \end{matrix}\right\}. \tag{2.21}$$

Defining Cartesian product on a set one assume an existence of inclusion

$$X_i \subseteq \prod_{i\in I} X_i. \tag{2.22}$$

According to (2.3) there is embedding

$$\mathrm{id}_{X_i \subseteq \prod_{i\in I} X_i}: X_i \to \prod_{i\in I} X_i. \tag{2.23}$$

Taking some of its projections one gets mapping

$$\mathrm{pr}_i: \prod_{i\in I} X_i \to X_i. \tag{2.24}$$

Expression (2.15) may be written here as

$$\left.\begin{matrix} \mathrm{pr}_i: \prod_{i\in I} X_i \to X_i \wedge \mathrm{id}_{X_i \subseteq \prod_{i\in I} X_i}: X_i \to \prod_{i\in I} X_i \Leftrightarrow \mathrm{id}_{\prod_{i\in I} X_i}: \prod_{i\in I} X_i \to \prod_i X_i \\ \mathrm{pr}_i \circ \mathrm{id}_{X_i \subseteq \prod_{i\in I} X_i} = \mathrm{id}_{\prod_{i\in I} X_i} \end{matrix}\right\}. \tag{2.25}$$

It may be seen that projection is constriction of Cartesian product "diagonal" down to correspondent multiplicand

$$\mathrm{pr}_i = \mathrm{id}_{\prod_{i\in I} X_i}|X_i. \tag{2.26}$$

Thus, this is surjective and correspondent criterion is

$$\left.\begin{matrix} \ker \mathrm{pr}_i \supset \mathrm{id}_{\prod_{i\in I} X_i} \\ \mathrm{im}\, \mathrm{pr}_i = \mathrm{id}_{X_i} \end{matrix}\right\}. \tag{2.27}$$

Surely, "diagonal" of Cartesian product is not determined. But it is not so important for this surjection because inclusion is not so determined as far as equality.

3. Partition and quotient-set

Representation of a set as a direct sum of one-element subsets, seemingly, is not the unique way to do this. But it is not so comprehensive yet – what else may play the role of its elements. To make it clear, it is good to find out some features of such representation by one-element sets, as an example. First of all, it needs to point out that all of them are just disjoint to each other – it does make possible to represent them as summands of disjoint union

$$(j \neq k) \in I \to \{a_j\} \cap \{a_k\} = \emptyset. \tag{3.1}$$

According to (1.2.7), such direct sum looks like

$$\boldsymbol{A} = A/A^2 = \coprod_{i=1}^{n} \{[a_i]_{A^2}\}. \tag{3.2}$$

So, in this case there may be said about partition of a set by one-element subsets. Elements of a set that are indexed via this mode may be determined as

$$\left.\begin{aligned}[a_i]_{A^2} \Leftrightarrow a_j : \langle a_j, a_i \rangle \in A^2 \\ \{[a_i]_{A^2}\} = \left\{a_j : \langle a_j, a_i \rangle \in A^2\right\}\end{aligned}\right\}. \tag{3.3}$$

In addition to, there may be pointed out that most important is only fact that such procedure totally determined by Cartesian square which is improper *equivalence*.

The other relation existing at every set and having similar properties is its diagonal – i.e. trivial *equivalence*. But instead of ordered set family, in this case unordered set occurs. Indeed, this is direct sum too, but containing the only summand

$$\boldsymbol{A} = A/\mathrm{id} = \{[a_i]_{\mathrm{id}}\} = \bigcup_{i=1}^{n}\{[a_i]_{\mathrm{id}}\}. \tag{3.4}$$

Here formulae (3.3) look like

$$\left.\begin{aligned}[a_i]_{\mathrm{id}} \Leftrightarrow a_i : \langle a_i, a_i \rangle \in \mathrm{id} \\ \{[a_i]_{\mathrm{id}}\} = \{a_i : \langle a_i, a_i \rangle \in \mathrm{id}\}\end{aligned}\right\}. \tag{3.5}$$

As oppose to previous case, here set is not separated to subsets[6] and its entire elements are *equivalent* each other[7]. To unify it with partition (3.2), such "partition" is represented as idempotent union in the last equality (3.4).

For further unification of this extremal partition cases[8], there may be pointed out that list of equivalences in multi-element set is not limited by these two that were enumerated above. Sequence $\mathrm{id} \subseteq A^2$ is not dense there and it may be compacted else. Taking some equivalence ε, sequence (2.3.2) goes to

$$\mathrm{id} \subseteq \varepsilon \subseteq A^2. \tag{3.6}$$

Formulae (3.3) and (3.5) go then to

$$\left.\begin{aligned}[a_i]_{\varepsilon} \Leftrightarrow a_j : \langle a_j, a_i \rangle \in \varepsilon \\ \{[a_i]_{\varepsilon}\} = \left\{a_j : \langle a_j, a_i \rangle \in \varepsilon\right\}\end{aligned}\right\}. \tag{3.7}$$

Collection, denoting by square brackets, is labeled as *class* of element a_i *by* the *equivalence* ε. To write direct sum, it needs to show that the only case of classes' intersection is that they totally coincide one another. In fact, if $\{[a_i]_{\varepsilon}\} \cap \{[b_i]_{\varepsilon}\} \neq \emptyset$, it means existence of some third element c_i satisfying to condition

$$c_i \in [a_i]_{\varepsilon} \wedge c_i \in [b_i]_{\varepsilon}.$$

Just because equivalence is always symmetric there may be written

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[6] In this case the only improper inclusion occurs.

[7] Seemingly, it allows considering *finite-generated* constructions as something akin to *finite* ones.

[8] There may be seemed that opposite situation occurs here – expansion (3.2) corresponds to partition by diagonal and expansion (3.4) corresponds to partition by Cartesian square. Later this proposal will be denied directly, but for a while there may be bounded by remark that in formulae (3.3) token of ordering is present and it is not in formulae (3.5). It circumstantially denies such proposal and confirms the main text version.

$$c_i \in [a_i]_\varepsilon \Leftrightarrow a_i \in [c_i]_\varepsilon.$$

Taking into account transitivity of equivalence, conjunction may be transformed as

$$a_i \in [c_i]_\varepsilon \land c_i \in [b_i]_\varepsilon \Rightarrow a_i \in [b_i]_\varepsilon.$$

It confirms statement's correctness and allows writing implication

$$\{[a_i]_\varepsilon\} \cap \{[b_i]_\varepsilon\} \neq \emptyset \rightarrow \{[a_i]_\varepsilon\} = \{[b_i]_\varepsilon\} = \{[a_i]_\varepsilon\} \cup \{[b_i]_\varepsilon\}. \tag{3.8}$$

Negation of the fact leads to statement that sets of different classes are disjoint

$$k \neq l \leftrightarrow \{[a_k]_\varepsilon\} \cap \{[a_l]_\varepsilon\} = \emptyset. \tag{3.9}$$

Comparison with (3.1) allows talking that partition by disjoint classes of non-equivalent each other elements occurs. Formula (3.2) goes then to

$$\boldsymbol{A} = A/\varepsilon = \coprod_{i=1}^{m \leq n} \{[a_i]_\varepsilon\}. \tag{3.10}$$

Thus, equivalence class is well-determined by one its own element that is labeled as its *representative*.

Last expression is the most common definition of set family with classes as elements; often it is denoted by symbol A/ε. Usually it is named as *quotient-set* of set A by equivalence ε. Taking into account the fact that it is determined by one of its representatives, allows defining something that is labeled as *natural mapping* or *identification* associated with equivalence ε

$$\left.\begin{array}{c} a_i \mapsto [a_i]_\varepsilon \\ \nu: A \rightarrow A/\varepsilon \end{array}\right\}. \tag{3.11}$$

Non-empty set identification's domain in any case is not empty. Thus, it is surjective, at least. It allows formulating some statements better known as *Noether isomorphism theorems*. Obviously, in any case equivalence ε is the kernel of identification. If between sets A and B there is bijection φ, then it may be factorized – i.e. there may be written expression something akin to formula (1.3)

$$[\nu: A \rightarrow (A/\ker \nu)] \land [\omega: (A/\ker \nu) \rightarrow B] \Leftrightarrow \varphi: A \rightarrow B. \tag{3.12}$$

Sequence in the right part is exact, thus, for identification's image there may be written expression of coincidence exact within *natural isomorphism*, in accordance with (1.20)

$$A/\ker \nu \cong \operatorname{im} \nu. \tag{3.13}$$

This is the sense of the *first theorem*[9]. But, as it is already clear, isomorphism is quite not obliged to be exact equality. To make it such, there may be introduced the notion of *co-image* with the help of formula

$$\left.\begin{array}{c} \operatorname{coim} \nu = A/\ker \nu \\ \operatorname{im} \nu \cong \operatorname{coim} \nu \end{array}\right\}. \tag{3.14}$$

[9] There are two others of them, but they won't be used in this description. So, they'll be missed.

The expression (3.14) may be “converted”, but, once again, by introducing the notion of *co-kernel* with the help of formula

$$\left.\begin{array}{r}\operatorname{coker}\nu = B/\operatorname{im}\nu\\ \ker\nu \cong \operatorname{coker}\nu\end{array}\right\}. \quad (3.15)$$

So, in spite of optionality of direct conversion from kernel to image and on the contrary exactly within equality, always there’s a possibility to do this exact within isomorphism.

Returning to beginning of this paragraph – confirmation of compliance between kind of partition and equivalence doing via, there may be noted that the only case while identification would be bijective, according to (3.13), were $\ker\nu = \mathrm{id}$, in another case it is surjective[10]. In accordance with the same, there may not be isomorphism between unordered set and something else, but ordered. So, there may be written formulae connecting trivial and improper equivalences exactly within isomorphism but not only by inclusion as it was before

$$\left.\begin{array}{r}A/\mathrm{id} \cong A\\ A/A^2 \cong \mathrm{id}\end{array}\right\}. \quad (3.16)$$

It confirms validity of accepted at the beginning proposal and denies the opposite version. It also may be confirmed by calculation of direct sums’ cardinalities, but it’s rather a matter of another research.

4. Actions and operations upon a set

According to (1.2.3), set family indexing may be done by using Cartesian product. Taking into account (2.19) it becomes clear that such mapping may be bijective. But writing bijection criterion it is necessary to write formulae similar to (2.20), which right part of first of them is assumed to be equal to “diagonal” $\mathrm{id}_{A\times I}$ and this is uncertain. Actually, in accordance with general definition of Cartesian product such set, probably, may be written as

$$\mathrm{id}_{A\times I} = \{\langle x,y\rangle : \langle x,x\rangle \in A \wedge \langle i,i\rangle \in I\} = \mathrm{id}_A \circ \mathrm{id}_I.$$

But this is contradictive due to rightness of inequality (1.3.8). On the other hand, in case of bijection such mapping may be represented by exact sequence, so, there may be written

$$\left.\begin{array}{r}\nu : A\times I \to A\otimes I \wedge \omega : A\otimes I \to \boldsymbol{A}_I \Leftrightarrow \iota : A\times I \to \boldsymbol{A}_I\\ \omega^{-1} : \boldsymbol{A}_I^* \to I\otimes A \wedge \nu^{-1} : I\otimes A \to I\times A \Leftrightarrow \iota^{-1} : \boldsymbol{A}_I^* \to I\times A\end{array}\right\}. \quad (4.1)$$

[10] There are two exceptions – one of them is one-element set (or structure generated by one element). There are only two equivalences here (trivial and improper one), but they are indistinguishing. The other exception is empty set – there is the only one equivalence – Cartesian square is not equivalence due to its anti-reflexivity.

Here symbol to denote set $A \otimes I$ is introduced. There may agree to call this new set by set's *tensor product*. Comparison between expressions (4.1) and (3.13) allows writing coincidence exact within isomorphism

$$A \otimes I \cong (A \times I)/\ker \nu. \tag{4.2}$$

To cancel even partially uncertainty with kernel of mapping ι, it needs to note that in any case mapping ν must be at least surjective, i.e.

$$\left.\begin{matrix} \ker \nu \supseteq \mathrm{id}_{A\times I} \\ \mathrm{im}\, \nu = \mathrm{id}_{A\otimes I} \end{matrix}\right\}. \tag{4.3}$$

Although left "diagonal" here is meaningless, it doesn't matter because for surjection it is not so important due to already comprehensive reason. Then, according to expression (3.15) in any case there is natural isomorphism

$$\ker \nu \cong (A \otimes I)/\mathrm{im}\, \nu = (A \otimes I)/\mathrm{id}_{A\otimes I}. \tag{4.4}$$

To satisfy to the first formula (3.16), it needs to request satisfying to condition too

$$\ker \nu \cong \mathrm{coker}\, \nu = (A \otimes I)/(\mathrm{id}_A \otimes \mathrm{id}_I) = A \otimes I = (A/\mathrm{id}_A) \otimes (I/\mathrm{id}_I). \tag{4.5}$$

It leads to equality

$$\left.\begin{matrix} \mathrm{id}_{A\otimes I} = \mathrm{id}_A \otimes \mathrm{id}_I \\ \mathrm{id}_{A\times I} \neq \mathrm{id}_A \circ \mathrm{id}_I \end{matrix}\right\}. \tag{4.6}$$

So, diagonal $\mathrm{id}_{A\otimes I}$ is the genuine identity element for tensor multiplication.

It's quite clear that tensor product, as Cartesian one too, non-commutative, but its transposed one is isomorphic to initial product

$$\left.\begin{matrix} A \otimes I \ncong I \otimes A \leftarrow\!\shortmid A \neq I \\ A \times I \ncong I \times A \leftarrow\!\shortmid A \neq I \end{matrix}\right\}. \tag{4.7}$$

It is also clear that sequence in the left part of (4.1) to be exact it needs mapping ω to be injective

$$\left.\begin{matrix} \ker \omega = \mathrm{id}_A \otimes \mathrm{id}_I \\ \mathrm{im}\, \omega \subset \mathrm{id}_{\boldsymbol{A}_I} \end{matrix}\right\}. \tag{4.8}$$

But, as it has been told, indexing is not quite obliged to be injection. If it is surjective, then expressions (4.1) look like

$$\left.\begin{matrix} \nu: A \times I \to A \otimes I \wedge \omega: A \otimes I \to \boldsymbol{A}_I \Rightarrow \iota: A \times I \to \boldsymbol{A}_I \\ \omega^{-1}: \boldsymbol{A}_I^* \to I \otimes A \wedge \nu^{-1}: I \otimes A \to I \times A \nRightarrow \iota^{-1}: \boldsymbol{A}_I^* \to I \times A \end{matrix}\right\}. \tag{4.9}$$

In addition, among surjections and injections there would be such that may be described by conditions (1.11) and (1.7); so, it is possible that non-empty multiplicands' tensor product may be empty – as opposed to Cartesian ones tensor multiplicands may be *non-empty divisors of empty set*

$$\left.\begin{array}{c} A \otimes I = \emptyset \ncong \boldsymbol{A}_I \\ A \times I \cong \boldsymbol{A}_I \ncong \emptyset \end{array}\right\}. \tag{4.10}$$

In this case expressions (4.1) look like

$$\left.\begin{array}{c} \nu: A \times I \to \emptyset \wedge \omega: \emptyset \to \boldsymbol{A}_I \Leftrightarrow \iota: A \times I \to \boldsymbol{A}_I \\ \omega^{-1}: \boldsymbol{A}_I^* \to \emptyset \wedge \nu^{-1}: \emptyset \to I \times A \Leftrightarrow \iota^{-1}: \boldsymbol{A}_I^* \to I \times A \end{array}\right\}. \tag{4.11}$$

Anyway, such mapping ι here is bijective, so, composition at the left part is always reversible, as opposed to composition that is described by the first of (4.9).

Also, for tensor product there may be written formulae describing its distributive properties with Cartesian product and direct sum

$$\left.\begin{array}{c} A \otimes (B \otimes C) \cong (A \otimes B) \otimes C \cong A \otimes B \otimes C \\ (A \otimes B) \times (C \otimes D) \cong (A \times C) \otimes (B \times D) \\ (A \otimes B) \oplus (C \otimes D) \cong (A \oplus C) \otimes (B \oplus D) \end{array}\right\}. \tag{4.12}$$

Because of non-commutativity sequential order is important here.

Semantic invariant version of expressions (4.1) looks like

$$\left.\begin{array}{c} \nu: A \times A \to A \otimes A \wedge \otimes_2: A \otimes A \to \boldsymbol{A} \Leftrightarrow *_2: A \times A \to \boldsymbol{A} \\ \otimes_2^{-1}: \boldsymbol{A} \to A \otimes A \wedge \nu^{-1}: A \otimes A \to A \times A \Leftrightarrow *_2^{-1}: \boldsymbol{A} \to A \times A \end{array}\right\}. \tag{4.13}$$

Here new symbols are introduced to denote mappings

$$\left.\begin{array}{c} *_2: A \times A \to \boldsymbol{A} \\ *_2^{-1}: \boldsymbol{A} \to A \times A \\ \iota: A \times I \to \boldsymbol{A}_I \\ \iota^{-1}: \boldsymbol{A}_I^* \to I \times A \end{array}\right\}. \tag{4.14}$$

This is called *algebraic operation* on set A – index beneath points at its arity – for binary case it will rather be missed. In addition, symbols are introduced to denote mappings

$$\left.\begin{array}{c} \otimes_2: A \otimes A \to \boldsymbol{A} \\ \otimes_2^{-1}: \boldsymbol{A} \to A \otimes A \\ \omega: A \otimes I \to \boldsymbol{A}_I \\ \omega^{-1}: \boldsymbol{A}_I^* \to I \otimes A \end{array}\right\}. \tag{4.15}$$

Apparently, it is appropriately to call *tensor operation* on a set. Other mappings shown here and defined earlier – here they are ι, ι^{-1} ω and ω^{-1}, are called *left* and *right actions* on a set. It is clear just operations, but not actions, may describe set semantically invariant. Operations index and order set automatically and, seemingly, such situation does occur in real nature, because it doesn't imply an existence of some supernatural subject who'd make it first and there's no need any meta-scientific reasons.

Most concerning actions will be propagated at operations, but first of all it needs to point some operations' features. Firstly, as oppose to actions operations are defined by Cartesian and tensor squares, those are commutative. So, there may assume that operations may be commutative too. Secondly, taking into account previous terminology, there may be said that, indeed, binary operation is ternary relation on a set

$$*_2 \subseteq A^3. \tag{4.16}$$

There may trace back some ties between this expression and definition of diagonal by formulae (1.2.3) in the meaning of Cartesian power. Familiar way there may be defined *arbitrary arity* algebraic operation

$$\left.\begin{matrix} *_n\colon A^n \to \boldsymbol{A} \\ *_n \subseteq A^{n+1} \end{matrix}\right\}. \tag{4.17}$$

Particularly, there may be defined *unary* algebraic operation

$$\left.\begin{matrix} *_1\colon A^1 \to \boldsymbol{A} \\ *_1 \subseteq A^2 \end{matrix}\right\}. \tag{4.18}$$

Comparing it with definition (2.1) allows to speak that, indeed, unary operation coincide with identity exact within isomorphism, at least,

$$*_1 \cong \mathrm{id}. \tag{4.19}$$

Expressions (4.13) here look like

$$\left.\begin{matrix} \nu\colon (A \times \mathrm{id}) \to (A \otimes \mathrm{id}) \wedge \otimes_1\colon (A \otimes \mathrm{id}) \to \boldsymbol{A} \Leftrightarrow *_1\colon (A \times \mathrm{id}) \to \boldsymbol{A} \\ \otimes_1^{-1}\colon \boldsymbol{A} \to (A \otimes \mathrm{id}) \wedge \nu^{-1}\colon (A \otimes \mathrm{id}) \to (A \times \mathrm{id}) \Leftrightarrow *_1^{-1}\colon \boldsymbol{A} \to (A \times \mathrm{id}) \\ *_1^{-1} \cong *_1 \end{matrix}\right\}. \tag{4.20}$$

Their reduced version is

$$\left.\begin{matrix} \nu\colon A \to A \wedge \otimes_1\colon A \to \boldsymbol{A} \Leftrightarrow *_1\colon A \to \boldsymbol{A} \\ \otimes_1^{-1}\colon \boldsymbol{A} \to A \wedge \nu^{-1}\colon A \to A \Leftrightarrow *_1^{-1}\colon \boldsymbol{A} \to A \\ *_1^{-1} \cong *_1 \end{matrix}\right\}. \tag{4.21}$$

Mapping $\otimes_1$ is what, that may be called *Kronecker delta prototype*, so, it is also identical but multiplication representing it is tensorial. To comprehend what it means and its distinction from Cartesian one, it needs to define delta itself but not its prototype in common with what as it is defined characteristic function (e.g. it may be seen in Appendix c.). Thus, there may be written mapping

$$\chi\colon A \times A \to \{0,1\}. \tag{4.22}$$

Co-product was changed by Cartesian square. It's clear that one-element summands of direct sum (c.7) characteristic function may be represented by characters[11] (or *column*)

$$\left.\begin{matrix} \mathbf{X}_1 = [\![\chi_1]\!] = [\![10 \ldots 0]\!] \\ \ldots \\ \mathbf{X}_n = [\![\chi_n]\!] = [\![00 \ldots 1]\!] \end{matrix}\right\}. \tag{4.23}$$

[11] Perhaps, it explains indirectly characters' orthogonality of *irreducible* group *representation* long before they may be determined.

Their direct addition, meaning isomorphism with delta, allows writing them as $n \times n$ *matrix*

$$\mathbf{\Delta} = \mathbf{\Delta^{-1}} = [\![\delta_{ij}]\!] = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}. \tag{4.24}$$

In the expression (4.22) Cartesian squaring is changed by tensorial one

$$\delta: A \otimes A \to \{0,1\}. \tag{4.25}$$

As it is described in formula (c.5), matrix component – *Kronecker symbol*, may be written as

$$\left.\begin{matrix} i = j \Rightarrow 1 \\ i \neq j \Rightarrow 0 \end{matrix}\right\} = \delta_{ij}. \tag{4.26}$$

Implying isomorphism (4.2) first of expressions (4.20) may be written as

$$\chi: A \times A \to \{0,1\} \wedge \delta: \{0,1\} \to A \otimes A \Leftrightarrow *_1: (A \times A)/\ker \nu \to \{0,1\}. \tag{4.27}$$

This expression also may be written in the form that is conventional in poly-linear algebra

$$\left.\begin{matrix} \mathbf{X} \circ \mathbf{\Delta} \cong \mathbf{X} \\ \sum_{i \in \mathbb{N}} \chi_i \delta_{ij} = \chi_j \\ \sum_{j \in \mathbb{N}} \chi_j \delta_{ji} = \chi_i \\ \delta_{ij} = \delta_{ji} \end{matrix}\right\}. \tag{4.28}$$

Meaning of isomorphism is what, that left and right parts of middle expressions contain various indices; distinction of these expressions from (1.3.6) ones in following. Here multiplicands are “almost”, i.e. isomorphic, idempotent, but there they’re “quite” idempotent. So, there may be written

$$\mathbf{X} \otimes \mathbf{X}^{+} \cong \mathbf{\Delta}. \tag{4.29}$$

If multi-element characteristic function as a character is denoted by symbol $\mathbf{X}$, then column is denoted by symbol $\mathbf{X}^{+}$, i.e. its *transposed* version. Also, as oppose to characters, having n elements, including zeroth, delta has n^2 elements, including zeroth. As it is already clear, all relations here, multiplicands and product, are equipotential to diagonal, so, their transitivity criterion looks like

$$\left.\begin{matrix} \ker \chi = \mathrm{id} \\ \mathrm{im}\, \chi = \mathrm{id} \\ \ker \delta = \mathrm{id} \\ \mathrm{im}\, \delta = \mathrm{id} \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \ker *_1 = \mathrm{id} \\ \mathrm{im} *_1 = \mathrm{id} \end{matrix}\right.. \tag{4.30}$$

Nullary algebraic operation is defined by expression

$$\left.\begin{matrix} *_0: A^0 \to A \\ *_0 \subseteq A^1 = A \end{matrix}\right\}. \tag{4.31}$$

Although, there may be chosen just one-element fragment delta prototype "matrix" – all the same – empty or not in accordance with the statement of empty set proper inclusion

$$\left.\begin{matrix}\emptyset = \bigotimes_0 \subset \omega \\ \emptyset \neq \bigotimes_a \subset \omega\end{matrix}\right\}. \tag{4.32}$$

Taking into account that $A^0 = \mathrm{id}_\emptyset \neq \emptyset$, there may be written implication of shape (4.20)

$$\left.\begin{matrix}\nu_a\colon \mathrm{id}_\emptyset \to \{a\} \wedge \bigotimes_a\colon \{a\} \to A \Rightarrow *_0\colon \mathrm{id}_\emptyset \to A \\ \nu_0\colon \mathrm{id}_\emptyset \to \emptyset \wedge \bigotimes_0\colon \emptyset \to A \Rightarrow *_0\colon \mathrm{id}_\emptyset \to A\end{matrix}\right\}. \tag{4.33}$$

In the first case mapping ν_a is bijective, and if set is multi-element, then mapping $\bigotimes_a$ is injective, thus nullary operation is also injective here. So, it is not reversible. It is explainable by the fact that diagonal fragments on their own are not transitive relations. If set is one-element, then first formula (4.33) becomes

$$\nu_a\colon \mathrm{id}_\emptyset \to \{a\} \wedge \bigotimes_a\colon \{a\} \to \{a\} \Leftrightarrow *_0\colon \mathrm{id}_\emptyset \to \{a\}. \tag{4.34}$$

Apparently, this operation is reversible, but, at the best, operation $*_0$ is bijective mapping but not transitive relation, because "transitivity" criterion contains diagonals of different sets

$$\left.\begin{matrix}\ker *_0 = \mathrm{id}_{\mathrm{id}_\emptyset} = \mathrm{id}_\emptyset \\ \mathrm{im} *_0 = \mathrm{id}_A\end{matrix}\right\}. \tag{4.35}$$

Obviously, it doesn't become equivalence even by using of empty delta prototype

$\nu_0\colon \mathrm{id}_\emptyset \to \emptyset \wedge \bigotimes_0\colon \emptyset \to \{a\} \Leftrightarrow *_0\colon \mathrm{id}_\emptyset \to \{a\}$. (4.36)

"Transitivity" criterion has similar form too

$$\left.\begin{matrix}\ker *_0 = \mathrm{id}_\emptyset \\ \mathrm{im} *_0 = \mathrm{id}_{\{a\}} \\ \mathrm{id}_\emptyset \ncong \mathrm{id}_{\{a\}}\end{matrix}\right\}. \tag{4.37}$$

Therefore, there are no sets where nullary operation might be equivalence. Actually, if set is even empty, then the second expression (4.33) has form

$$\nu_0\colon \mathrm{id}_\emptyset \to \emptyset \wedge \bigotimes_0\colon \emptyset \to \emptyset \Rightarrow *_0\colon \mathrm{id}_\emptyset \to \emptyset. \tag{4.38}$$

It shows that such operation is not equivalence too. In addition, those of them that are relations are equivalences, at the best, but not transitive ordering relations. Such description is not full – it doesn't lead it to linear ordering. Finally, it is clear that definition of nullary or unary "action" is not impossible – its arity cannot be less than two.

As it was among actions, nothing prevents even non-empty set tensor square to be empty, i.e. to be *nilpotent*[12], during binary algebraic operation's definition

[12] Seemingly, such property doesn't occur anywhere but among tensor product. There may be seemed that it may occur among disjoint unions, but it leads to contradiction due to necessity to observe non empty subsets of empty set.

$$\left.\begin{array}{c}\boldsymbol{A} \ncong A \otimes A = \emptyset \\ \boldsymbol{A} \cong A \times A \ncong \emptyset\end{array}\right\}. \qquad (4.39)$$

Then expression (4.13) goes over

$$\left.\begin{array}{c}\nu: A \times A \to \emptyset \wedge \otimes: \emptyset \to \boldsymbol{A} \Leftrightarrow *: A \times A \to \boldsymbol{A} \\ \otimes^{-1}: \boldsymbol{A} \to \emptyset \wedge \nu^{-1}: \emptyset \to A \times A \Leftrightarrow *^{-1}: \boldsymbol{A} \to A \times A\end{array}\right\}. \qquad (4.40)$$

Here transitivity criterion has form

$$\left.\begin{array}{c}\ker\nu \supseteq \mathrm{id}_{A\times A} \\ \operatorname{im}\nu = \mathrm{id}_{\emptyset} \\ \ker\otimes = \mathrm{id}_{\emptyset} \\ \operatorname{im}\otimes \subseteq \mathrm{id}_{\boldsymbol{A}}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\ker * \supseteq \mathrm{id}_{A\times A} \\ \operatorname{im} * \subseteq \mathrm{id}_{\boldsymbol{A}}\end{array}\right.. \qquad (4.41)$$

Such operations are reversible anyway.

Situation has led to expressions (4.9) may occur among operations too. Seemingly, there are no ways to disturb rule (1.3.8) because there is only double-sided diagonal, and it seems there may be written $\mathrm{id}_{A\times A} = \mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$. But due to reasons were described by formulae (3.16) such record is not always possible. If operation is surjective, then there may only be written

$$\left.\begin{array}{c}\ker\nu \supseteq \mathrm{id}_{A\times A} \\ \operatorname{im}\nu = \mathrm{id}_{A\otimes A} \\ \ker\otimes = \mathrm{id}_{A\otimes A} \\ \operatorname{im}\otimes = \mathrm{id}_{\boldsymbol{A}}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\ker * \supseteq \mathrm{id}_{A\times A} \\ \operatorname{im} * = \mathrm{id}_{\boldsymbol{A}}\end{array}\right.. \qquad (4.42)$$

Then, instead of formulae (4.13) there may be written

$$\left.\begin{array}{c}\nu: A \times A \to A \otimes A \wedge \otimes: A \otimes A \to \boldsymbol{A} \Rightarrow *: A \times A \to \boldsymbol{A} \\ \otimes^{-1}: \boldsymbol{A} \to A \otimes A \wedge \nu^{-1}: A \otimes A \to A \times A \nRightarrow *^{-1}: \boldsymbol{A} \to A \times A\end{array}\right\}. \qquad (4.43)$$

Contrary, there are no reasons to write such rigid exactness criterion for functions, because for transitive relation criterion (1.28) is sufficient and here it looks like

$$\left.\begin{array}{c}\ker\nu \supseteq \mathrm{id} \\ \operatorname{im}\nu \subseteq \mathrm{id} \\ \ker\otimes \supseteq \mathrm{id} \\ \operatorname{im}\otimes \subseteq \mathrm{id}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\ker * \supseteq \mathrm{id} \\ \operatorname{im} * \subseteq \mathrm{id}\end{array}\right.. \qquad (4.44)$$

It does make being possible reversible operations describing by formulae (4.13).

Appendices

a. Ordinal numbers

Seemingly, empty set is the “quite” disordered set. But, on the other hand, its power is not empty and it may be quite linearly ordered by inclusion. Apparently, it is due to its power quite coincides with zeroth Cartesian power. Such procedure may be continued, then there may be pointed out that their powers consist linear ordered natural numbers set

$$\left.\begin{array}{c} |\emptyset| = 0 \\ |\{\emptyset\}| = 1 \\ \left|\{\emptyset, \{\emptyset\}\}\right| = 2 \\ \left|\left\{\emptyset, \{\emptyset, \{\emptyset\}\}\right\}\right| = 3 \\ \ldots \\ \left|\underbrace{\left\{\emptyset \left\{\emptyset, \{\ldots, \{\emptyset\}\}\right\}\right\}}_{n}\right| = n \\ \ldots \\ \left|\underbrace{\left\{\emptyset \left\{\emptyset, \{\ldots, \{\emptyset\}\}\right\}\right\}}_{\aleph_0}\right| = \aleph_0 \end{array}\right\} \in \mathbb{Z}_+. \tag{a.1}$$

Actually, in this case there may be written sequence

$$0 < 1 < 2 < 3 < \ldots < n < \ldots < \aleph_0. \tag{a.2}$$

So, it is not a result of just an agreement. Symbol $\aleph_0$ denotes here cardinality of enumerable set. It may be introduced by formula

$$|\mathbb{N}| = |\mathbb{Z}_+| = \aleph_0. \tag{a.3}$$

Some notes about this set but not just of its cardinality – the relations of *precedence* are established between its elements that are denoted by symbol “$\prec$”; and it orders linearly all the set

$$\emptyset \prec \{\emptyset\} \prec \{\{\emptyset\}\} \prec \left\{\{\{\emptyset\}\}\right\} \prec \ldots \prec \underbrace{\left\{\{\{\ldots\{\emptyset\}\}\}\right\}}_{n} \prec \ldots \prec \underbrace{\left\{\{\{\ldots\{\emptyset\}\}\}\right\}}_{\aleph_0}. \tag{a.4}$$

Elements of sequence (a.4) are *ordinal numbers*. Instead of formulae (a.1) there may be written similar for ordinals

$$\left.\begin{array}{c} \emptyset \mapsto 0 \\ \emptyset \oplus \{\emptyset\} \mapsto \{0\} = 1 \\ \{\emptyset\} \oplus \{\{\emptyset\}\} \mapsto \{0,1\} = 2 \\ \{\{\emptyset\}\} \oplus \left\{\{\{\emptyset\}\}\right\} \mapsto \{0,1,2\} = 3 \\ \ldots \\ \underbrace{\left\{\emptyset \left\{\emptyset, \{\ldots, \{\emptyset\}\}\right\}\right\}}_{\aleph_0} \mapsto \{0,1,2,3, \ldots, n, \ldots, \aleph_0\} = \mathbb{Z}_+ \end{array}\right\}. \tag{a.5}$$

In contrary to cardinals, ordinals are neither one-element nor empty (surely, excluding two first of them from the list in (a.5)).

Using ordinal notion there may be shown that symbol “2” to denote base of power in Boolean has not just “quantitative” origin. Actually, the third of (a.5) may be written as

$$2^1 = 2^{\{0\}} = 2^{\{\emptyset\}} = \left\{\{\emptyset\}, \left\{\{\emptyset\}\right\}\right\} = \{0,1\} = 2. \tag{a.6}$$

The second of (a.5) describes one-element set – Boolean of empty set does be such set

$$2^0 = 2^{\emptyset} = \left\{\emptyset, \{\emptyset\}\right\} = \{0\} = 1. \tag{a.7}$$

It includes only empty set.

b. Inclusion – exclusion principle for set family

Formulae that express *inclusion – exclusion principle* for set family may be obtained as follows. In this case cardinality of set union decays by n summands

$$(-1)^{m-1} \sum_{1\le i<j<\ldots<k\le n}^{\binom{n}{m}} \left|\underbrace{X_i \cap X_j \cap \ldots \cap X_k}_{m}\right|.$$

Linear order $1 \le i < j <\ldots< k \le n$ is just for reading convenience sake – actually, it is not so important because of their commutativity. Binomial coefficient at the top of sum symbol is the amount of summands. So, there may be written expressions for cardinalities of set union and of symmetrical difference

$$\left.\begin{aligned} |\cup_{m=1}^{n} X_m| &= \sum_{m=1}^{n} (-1)^{m-1} \sum_{1\le i<j<\ldots<k\le n}^{\binom{n}{m}} \left|\underbrace{X_i \cap X_j \cap \ldots \cap X_k}_{m}\right| \\ |\ominus_{m=1}^{n} X_m| &= \sum_{m=1}^{n} (-1)^{m-1} 2^{m-1} \sum_{1\le i<j<\ldots<k\le n}^{\binom{n}{m}} \left|\underbrace{X_i \cap X_j \cap \ldots \cap X_k}_{m}\right| \end{aligned}\right\}. \tag{b.1}$$

These formulae are obtained by generalization of empirical induction result and by using of well-known formula for binomial coefficients

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}.$$

The first summand in first formula (b.1) is also not an exception due to intersection is idempotent. So, it may be written as

$$\sum_{l=1}^{n} |X_l| = \underbrace{|X_1 \cap X_1| + |X_2 \cap X_2| +\ldots+ |X_n \cap X_n|}_{\binom{n}{1} = \frac{n!}{1!(n-1)!} = n} = |\coprod_{l=1}^{n} X_l|. \tag{b.2}$$

This is generalization to calculate cardinality of disjoint set union.

c. Characteristics

Cardinality calculation procedure for enumerated set union, symmetric difference and its Boolean might be represented as some mapping χ that is usually called characteristic function (for brevity's sake in this text it often will be character[13]). To find the most common shape it needs to assume it by some amount, then, anyway, there may be written binomial expansion

$$(1-\chi)^n = \sum_{q=0}^{n}(-1)^q \binom{n}{q} \chi^q = 1 + \sum_{q=1}^{n}(-1)^q \binom{n}{q} \chi^q.$$

It may be transformed by follows

$$(1-\chi_1)(1-\chi_2)\dots(1-\chi_n) = 1 + \sum_{q=1}^{n}(-1)^q \sum_{1\le i<j<\dots<k\le n}^{\binom{n}{q}} \underbrace{\chi_i\chi_j\cdots\chi_k}_{q}.$$

This formula has a form that is something akin to shape of formulae (b.1), if set intersection correlates characters' product

$$\bigcap_{l=1}^{n} A_l \mapsto \prod_{l=1}^{n} \chi_l. \tag{c.1}$$

Actually, set union is matching with its character as

$$\bigcup_{l=1}^{n} A_l \mapsto \chi_{\bigcup_{l=1}^{n} A_l}. \tag{c.2}$$

Then there may be written formula

$$\chi_{\bigcup_{l=1}^{n} A_l} = 1-(1-\chi_1)(1-\chi_2)\dots(1-\chi_n) = 1-\prod_{l=1}^{n}(1-\chi_l). \tag{c.3}$$

If operands here are disjoint, then, due to formula (c.2), formula (c.3) may be written as

$$\chi_{\coprod_{l=1}^{n} A_l} = \sum_{l=1}^{n} \chi_l. \tag{c.4}$$

It coincides with cardinality of one-element sets' direct sum while characters are determined as

$$\left.\begin{matrix} l \in A \mapsto 1 \\ l \notin A \mapsto 0 \end{matrix}\right\} = \chi_{A_l} = \chi_l. \tag{c.5}$$

Such definition allows writing expressions for direct sums and symmetrical difference in terms of characters

$$\left.\begin{matrix} \left|\bigcup_{l=1}^{n} A_l\right| = \sum_{q=1}^{n}(-1)^q \sum_{1\le i<j<\dots<k\le n}^{\binom{n}{q}} \underbrace{\chi_i\chi_j\cdots\chi_k}_{q} \\ \left|\ominus_{l=1}^{n} A_l\right| = \left|\sum_{q=1}^{n}(-1)^q 2^{q-1} \sum_{1\le i<j<\dots<k\le n}^{\binom{n}{q}} \underbrace{\chi_i\chi_j\cdots\chi_k}_{q}\right| \end{matrix}\right\}. \tag{c.6}$$

One-element set characters realize mapping

$$\chi_l : \coprod_{l=1}^{n} A_l \to |\{0,1\}|. \tag{c.7}$$

Considering subsets' characters instead of one-element sets' ones there may be written formula

[13] Reasons of doing so may be comprehensive during reading of the terminal main paragraph of this text.

$$\chi_{X'\subseteq X}: X \to 2. \quad \text{(c.8)}$$

Subsets' characters may be written as similarly to one-element sets' ones

$$\left.\begin{aligned} x \in X' \subseteq X \mapsto 1 \\ x \notin X' \subseteq X \mapsto 0 \end{aligned}\right\} = \chi_{X'\subseteq X}. \quad \text{(c.9)}$$

It's important that all summands in (c.7) are disjoint. If to summarize by the similar way characters (c.9), then there will be achieved formula

$$\sum_{X'} \chi_{X'\subseteq X} : \coprod_{X'} X' \to 2^{|X|}. \quad \text{(c.10)}$$

The sets $\coprod_{X'} X'$ and $2^X$are equipotent, so there may be said that there is bijection between them writing formula

$$\vartheta : \coprod_{X'} X' \to 2^X. \quad \text{(c.11)}$$

In any case there may be written common bijection criterion that has the forms here

$$\left.\begin{aligned} \ker \vartheta = \mathrm{id}_{\coprod_{X'} X'} \\ \operatorname{im} \vartheta = \mathrm{id}_{2^X} \end{aligned}\right\}, \quad \text{(c.12)}$$

$$\left.\begin{aligned} \ker \vartheta^{-1} = \mathrm{id}_{2^X} \\ \operatorname{im} \vartheta^{-1} = \mathrm{id}_{\coprod_{X'} X'} \end{aligned}\right\}. \quad \text{(c.13)}$$

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