Equivalient condition of the Generalized Riemann Hypoythesis

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We prove next theorem about Dirichlet series χ.

Main theorem

$$\sum_{n=1}^{m} \mu(n)\chi(n) = O(\sqrt{m}log(m)) \Leftrightarrow G.R.H for \chi$$

The relation of mobius function and Riemann Hypothesis like this.

Theorem

$$\sum_{n=1}^{m} \mu(n) = O(\sqrt{m}log(m)) \Leftrightarrow R.H$$

Proof)
Define $M(x)$ like this

$$M(x) := \sum_{n=1}^{x} \mu(n)$$

$$\frac{1}{\zeta(s)} = \sum \frac{\mu(n)}{n^s}$$

$$\frac{1}{\zeta(s)} = \int_{x=1}^{\infty} \frac{1}{x^s} d(M(x))$$

$d(M(x))$ is Stieltjes integral.

$$= [M(x)x^{-s}] + s\int_{x=1}^{\infty} M(x)x^{-s-1}$$

$M(x) < O(\sqrt{x}) \Rightarrow$ This integral must convergence on $s(Re(s) = \frac{1}{2})$
$O(\sqrt{x}) < M(x) < O(\sqrt{x} log(x)) \Rightarrow$ This integral may not convergence on $s(Re(s) = \frac{1}{2})$ and must convergence not on $s(Re(s) = \frac{1}{2})$
q.e.d

We have got main theorem by rewrite $M(x)$ to $M_\chi(x) := \sum \mu(n)\chi(n)$