Orbiting particles' analytic time dilations correlated with the Sagnac formula and a general 'versed sine' satellite clock absolute dilation factor

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November 8, 2015

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Abstract

Eastward and westward orbiting plane time dilation formulae envisaged for *Hafele&Keating's* 1971 equatorial clocks experiment were incorrectly derived in the 2004 textbook *Relativity in Rotating Frames* although exact analytic expressions actually result directly from velocity composition—*provided* gravitational effects are disregarded. Nevertheless the same idealised equations together yield the classic formula for Sagnac's analogous 1913 experiment where interference fringe patterns from monochromatic light waves emitted in opposite directions around a rotating wheel, *shifted* in accordance with rotation rate—an observation misinterpreted by some as challenging special relativity theory. Although only approximately correct for the 1971 experiment, the resulting formulae also yield—*independently of general relativity theory*—a notable *exact* formula for a rotating satellite's clock dilation. The factor's inverse equals *the versed sine* of the angle whose *sine* equals the satellite's peripheral speed scaled for unit limit speed—the cubic root of the product of the Earth's mass, the universal gravitational constant and the orbit's rate of rotation.

1 Hafele-Keating's experiment idealised

The first—indirect—substantiation of own-time 'stretchings' was achieved in 1938 by *Ives & Stilwell* and later in 1963 [2] by statistical comparisons of time stretched 'half lives' of high speed ($\approx 0.9c$) muon particles generated by cosmic rays from outer space which were measured at high altitude and at sea level. *Direct* observations of time dilation were not made until 1971 [3], as described in [6]: *"In the real world, …an experiment [was performed in 1971]… by Hafele and Keating … by means of commercial flights (carrying caesium beam atomic clocks) around the world in eastwards and westward directions….Strangly enough, an exact theoretical prediction [of clock differences] in full theory seems*

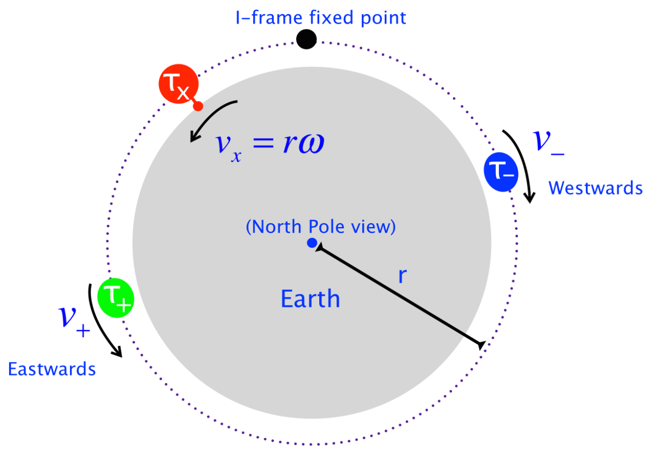

Figure 1: Equatorial clocks experiment

to be lacking (or maybe lost somewhere in the literature)....” (*Relativity in Rotating Frames* textbook).

We consider an *idealised* form[1] of the 1971 experiment. A clock mounted in a balloon X firmly anchored above the rotating Earth's equator, traverses an equatorial ring fixed in a reference frame $\underline{I}$. As the Sun is so distant, frame $\underline{I}$ may be assumed to be effectively nonrotating and *inertial*.

Referring to Figure 1, balloon X's *tangential* speed in frame $\underline{I}$ is $v_x = r\omega$, where r is X's distance from the Earth's centre and ω its rate of axial rotation in radians per second (both as measured in inertial frame $\underline{I}$). A precision clock τ_x attached to X is zero-synchronised with the identical clocks τ_+ and τ_- of two planes which simultaneously fly past point X in the eastwards anticlockwise $(+)$ and westwards clockwise $(-)$ directions respectively.

The two planes orbit around the Earth at constant *tangential* positive speed s_+ and negative speed s_- relative to the co-moving inertial frame of the $\underline{I}$-ring's momentarily 'local' elevated equatorial point. They return to X at slightly different times. It need not be assumed that s_- and $-s_+$ are equal. On the eastward plane's return to point X, its clock 'circum-time' $\tilde{\tau}_x$ [2]is compared with X's then current clock time $\tilde{\tau}_{x+}$ (the tilde 'overline' symbolizes *round trip* values). Likewise the westward plane's circum-time $\tilde{\tau}_-$ is compared with the corresponding X clock reading $\tilde{\tau}_{x-}$. Of interest are the two circum-time *differences* $\tilde{\tau}_+ - \tilde{\tau}_{x+}$ and $\tilde{\tau}_- - \tilde{\tau}_{x-}$, in terms of measurable parameters r, $v_x = r\omega$, s_+ and s_-.

As demonstrated by the clock worldlines in Figure 2, the eastwards $(+)$ plane moving ahead of X at $\underline{I}$-frame relative orbital speed v_+ (not measured) at about two-thirds (say) of the circumference point X's speed v_x, meets X again after

[1]Disregarding for the moment the question of clocks being differently affected by gravity.
[2]We refer to the three clocks themselves as τ_x, τ_+ and τ_-. Actual clock round trip *'circum-values'* are designated with *tilde overheads*.

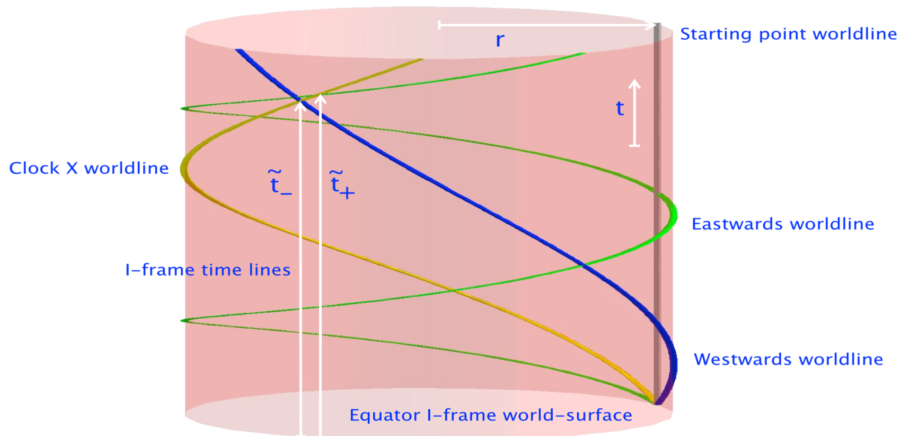

Figure 2: Equatorial I-frame world-lines on a 'world-surface'

X has progressed through somewhat over half an $\underline{I}$-frame circumference. It will by then have traversed (in frame $\underline{I}$) one whole circumference more than X. The westwards ($-$) plane moving 'backwards' from X in the opposite direction at $\underline{I}$-frame tangential speed $v_{\_}$ meets up with X (as we shall see) slightly before the eastwards plane does, and its $\underline{I}$-frame trajectory added to that of X's makes up one whole $\underline{I}$-frame circumference.

1.1 I-frame circum-times

[3] *Unmeasured* plane $\underline{I}$-frame circum-distances $\tilde{d}_{\pm}$ and circum-times $\tilde{t}_{\pm}$ relate as: $\tilde{d}_{+} = v_{+}\tilde{t}_{+} = v_x\tilde{t}_{+} + 2\pi r$ and $\tilde{d}_{\_} = v_{\_}\tilde{t}_{\_} = v_x\tilde{t}_{\_} - 2\pi r$ i.e.

$$\underline{I}\textit{-frame circum-times: } \tilde{t}_{\pm} = \pm 2\pi r / \big(v_{\pm} - v_x\big)\,. \tag{1}$$

Relativistic velocity composition gives us the *unmeasured* orbital speeds of the planes in frame $\underline{I}$:

$$v_{\pm} = \big(v_x + s_{\pm}\big) \,/\, \Big(1 + \frac{v_x s_{\pm}}{c^2}\Big)\,. \tag{2}$$

According to *the 'triple gammas' relationship*[4]:

$$\frac{\gamma_{\pm}}{\gamma_x \gamma_{s_{\pm}}} = 1 + \frac{v_x s_{\pm}}{c^2}. \tag{3}$$

As ${\gamma_x}^2 = 1/(1 - v_x^2/c^2)$, substituting (2) and then (3) in (1) gives:

$$\pm 2\pi r = \left(\frac{v_x + s_{\pm}}{1 + v_x s_{\pm}/c^2} - v_x\right)\tilde{t}_{\pm} = s_{\pm}\left(\frac{1 - v_x^2/c^2}{1 + v_x s_{\pm}/c^2}\right)\tilde{t}_{\pm} = \frac{s_{\pm}\tilde{t}_{\pm}/{\gamma_x}^2}{1 + v_x s_{\pm}/c^2} = \frac{s_{\pm}\tilde{t}_{\pm}}{{\gamma_x}^2}.\frac{\gamma_x\gamma_{s_{\pm}}}{\gamma_{\pm}} \qquad \text{i.e.}$$

[3] Using 'plus/minus' subscripts to avoid writing each time almost identical equations *twice*.

[4] $\frac{1}{\gamma_w^2} = 1 - \frac{w^2}{c^2} = \frac{1 + \frac{2vu}{c^2} + (\frac{vu}{c^2})^2 - (\frac{u}{c})^2 - \frac{2vu}{c^2} - (\frac{v}{c})^2}{(1+\frac{vu}{c^2})^2} = \frac{(1-(\frac{u}{c})^2)(1-(\frac{v}{c})^2)}{(1+\frac{vu}{c^2})^2} = \frac{1/(\gamma_u^2\gamma_v^2)}{(1+\frac{vu}{c^2})^2}$.

$$\underline{I}\text{-frame circum-times: } \tilde{t}_{\pm} = \pm 2\pi r \frac{\gamma_x \gamma_{\pm}}{s_{\pm} \gamma_{s_{\pm}}}. \tag{4}$$

2 Orbiting clock differences

By virtue of equatorial symmetry, the time dilation formula $\Delta t = \gamma \Delta\tau$[5] generally relates the unmeasured $\underline{I}$-times (no $\underline{I}$-frame t-clocks are actually deployed) as: $\tilde{t}_{\pm} = \tilde{\tau}_{\pm}\gamma_{\pm} = \tilde{\tau}_{x\pm}\gamma_x$. Accordingly:

$$\tilde{\tau}_{\pm} - \tilde{\tau}_{x\pm} = \tilde{\tau}_{\pm}\left(1 - \frac{\tilde{\tau}_{x\pm}}{\tilde{\tau}_{\pm}}\right) = \frac{\tilde{t}_{\pm}}{\gamma_{\pm}}\left(1 - \frac{\gamma_{\pm}}{\gamma_{\times}}\right). \tag{5}$$

Substituting (4) into (5):

$$\tilde{\tau}_{\pm} - \tilde{\tau}_{x\pm} = \pm\frac{2\pi r \gamma_x}{s_{\pm}\gamma_{s_{\pm}}}\left[1 - \frac{\gamma_{\pm}}{\gamma_x}\right].$$

Substituting from (3), the desired analytic formulae are:

Eastward/Westward 'provisional' clock relative dilations

$$\tilde{\tau}_{\pm} - \tilde{\tau}_{x\pm} = \pm\frac{2\pi r \gamma_x}{s_{\pm}}\left[\frac{1}{\gamma_{s_{\pm}}} - (1 + \frac{v_x s_{\pm}}{c^2})\right]. \tag{6}$$

These equations in terms of measurable parameters r, v_x, r, s_+ and s_-, constitute the hitherto 'missing' theoretical (provisional) relationship.

3 The 'equivalent' Sagnac equation

If plane speeds were the same i.e. $s_{+} = -s_{-} = s$, then from (6)

$$\left[\tilde{\tau}_{-} - \tilde{\tau}_{x-}\right] - \left[\tilde{\tau}_{+} - \tilde{\tau}_{x+}\right] = \frac{2\pi r \gamma_x}{s}\left[\frac{1}{\gamma_s} - (1 - \frac{v_x s}{c^2})\right] - \frac{2\pi r \gamma_x}{s}\left[\frac{1}{\gamma_s} - (1 + \frac{v_x s}{c^2})\right].$$

Arbitrary symmetric speeds clock dilations differential—the Sagnac formula

$$\Delta\tilde{\tau}_{+/-} = \frac{4\pi r^2 \omega \gamma_x}{c^2} = \frac{4A\omega\gamma_x}{c^2} = \frac{4A\omega}{\sqrt{c^2 - r^2\omega^2}}. \tag{7}$$

$A = \pi r^2$ is the cross sectional area of the idealised 'ring' equator (and $v_x = r\omega$). Equation (7) is independent of plane speed s. Hence if the 'planes' are replaced by 'photons' it must be and is identical to the Sagnac formula.[6]

The Sagnac experiment constitutes an analogous analytically equivalent 'cosmic limit speed' case of the idealised Hafele-Keating experiment.

[5]Obtained from the Lorentz transformation for a home frame observer of a moving clock.
[6]Also the RRF book's equation (10.25) where $\Omega = \omega$ and $R = r$.

4 Experimental conclusions

Hafele and Keatings atomic clocks, which were transported in commercial aircraft taking off and landing several times and not actually travelling along the equator, provided only approximate results. These however were broadly consistent with expected time gain and time loss and, as mentioned, were the first ever direct observations of time 'dilations'. Sagnac experiment results on the other hand, are fully in accordance with the Sagnac formula [4].

The landmark experiment performed in 1913 by *George Sagnac*[7] led to the French physicist claiming that the speed of light is *not* constant. Sagnac's reasoning was seized upon by some academics to challenge basic special relativity theory. Ironically nevertheless, actual *absence* of such direction-dependent interference fringe shifts would invalidate special relativity—*contrary to Sagnac's original reasoning.* Paradoxically, observed Sagnac experiment's 'apparent' rotation-dependent speed of light manifest in the interference fringe shifts, indeed CONFIRMS limit speed constancy. This inevitable fact is further corroborated by the correlation between the Sagnac equation and the analytically established orbiting clocks dilation formulae.

Having zero tangential acceleration our rotating clocks, being at an idealised equal altitude, are all identically subject to the same gravitational *radial* pull which however is *differently* reduced by centrifugal accelerations since they orbit at *differing* constant rotation rates.[8] Nevertheless this problem does not arise in the case of orbiting *satellites* (geostationary or otherwise) where, as outlined below, gravitational pull is exactly cancelled by the respective outward centrifugal force. Hence satellite clocks, like our ubiquitously imaged inertial frame $\underline{I}$ clocks, are not subject to *any* radial or tangential forces.

5 A rotating satellite's clock's time dilation

Dilation equations (6) will also apply in general to circular trajectories of constant rotation speed particles. So if $s_{\_} = -v_x$ i.e. $v_{\_} = 0$, $\gamma_{s_{\_}} = \gamma_x$ and the westward plane remains *stationary* in frame $\underline{I}$ at the starting point, then (6) gives us the rotating X clock's dilation:

$$\tilde{\tau}_{\_} - \tilde{\tau}_{x-} = \frac{2\pi r \gamma_x}{v_x}\left[\frac{1}{\gamma_x} - (1 - \frac{{v_x}^2}{c^2})\right] = \frac{2\pi r}{v_x}\left[1 - \frac{1}{\gamma_x}\right].$$

Dividing this by X's I-frame circum-time $\tilde{t}_{x} = 2\pi r/v_{x}$ we then obtain for scaled relative velocity $v_{x}/c = \sin\alpha$:

An individual rotating clock's proportional inverse dilation

$$\frac{\tilde{\tau}_{\_} - \tilde{\tau}_{x-}}{2\pi r/v_x} = 1 - \frac{1}{\gamma_x} = 1 - \cos\alpha \ . \tag{8}$$

[7] *Franz Harrass* had done similar work earlier in Jena.
[8] This was not properly accounted for in the earlier paper versions.

Newton's gravitational law for the inwards pull on a uniformly orbiting satellite is $F = mMG/r^2$, m being the satellite's mass, M that of the Earth, G the gravitational constant and r the satellite's distance from the Earth's centre. For the satellite not to fall to Earth, or 'fall away' from Earth, this force must be exactly balanced by the outward centrifugal force $m.v^2/r = m.r\omega^2$ where the satellite's orbital rate of rotation $\omega = v/r$. Accordingly $\omega = \sqrt{MG/r^3}$ and $v = \sqrt[3]{MG\omega}$.[9] Hence for an earth-bound *satellite's* absolute dilation (*relative to a nonrotating clock in the background inertial frame* $\underline{I}$), using equation (8):

An Earth-bound satellite's dilation inverse factor

$$1 - \cos(\arcsin \frac{\sqrt[3]{MG\omega}}{c^2}). \tag{9}$$

This compares with the dilation inverse factors for a gravity-free comet or a gravity-free accelerating rocket. From the *inverse Lorentz transformation*:

A gravity-free comet's dilation inverse factor

$$\frac{d\tau}{dt} = \frac{1}{\gamma} = \sqrt{1 - v^2/c^2}. \tag{10}$$

From the familiar equations for a rocket with fixed own-acceleration η where $t = \sinh \eta\tau/\eta$, $\gamma = \cosh \eta\tau$ and $v = \tanh \eta\tau$:

A gravity-free accelerating rocket's dilation inverse factor

$$\frac{d\tau}{dt} = \frac{1}{\gamma} = \sqrt{1 - v^2/c^2} = \sqrt{1 - (\tanh^2 \eta\tau)/c^2}. \tag{11}$$

Significantly, equation (9) has been obtained without recourse to general relativity theory.

6 A journal's unsavoury U-turn

The origin of this paper, a retitled slightly enhanced version of a reviewed but unpublished 2008 paper [9], has a curious background and a surprising sequel which, even considering the paper's initial overlooking of gravitational effects, perhaps deserves being put on record.

In the 1990's, papers supporting Sagnac's denial of relativity theory were published by the late *Al Kelly*.[10] In 2008, a second fellow Dubliner (physicist/researcher *Roy Johnston*) requested the present author to investigate this 'delicate' matter. As this was clearly an interesting subject, the task was taken

[9] For a *geostationary* equatorial orbit of course, the satellite's orbital rate of rotation is that the of the Earth.

[10] Supported by the late Italian physicist *Franco Selleri*, author of the RRF book's chapter 4, whose opinions were not shared by its other contributors. In an *Irish Times* letter in 1996, Trinity College (Dublin University) physics faculty put on record that Alphonsus Kelly's lecture held at that university had been instigated by a student group—not by college officials.

up enthusiastically. Widely covered in the literature, the Sagnac experiment (and *the Sagnac formula*) turned out to be fairly easy to understand. However the analogous *Hafele & Keating* equatorial clocks experiment—in idealised form—did not appear to be *analytically* treated anywhere.

Unaware then of the above quoted RRF 2004 book [6], the present author derived dilation formulae (6) and also a 'representative' eastwards/westwards clocks difference of $268ns$ (nanoseconds). Shortly afterwards a web reference to the RRF book was spotted and a copy obtained. That book also presented a $268ns$ clock difference for the same idealised scenario, but its equations (5.19/20), which *differed* from equations (6), turned out to be incorrect.[11]

A paper submitted to the *Foundations of Physics* in August 2008 was *not* accepted due to a second reviewer's negative yet thoroughly confused remarks—in spite of a highly positive first reviewer's assessment:

> *"This very short paper deals with the Hafele - Keating experiment, and gives a simple and thorough analysis, according to the principles of special relativity. The underlying ideas are very simple, and it is useful that they are recalled, since often this experiment is the origin of misunderstandings on the principles of special relativity, as the author says. Accordingly, I believe that the paper is correct and can be published as is."*

A request to the RRF book co-editors for their opinion on this elicited a reply which very generously *fully and unreservedly* endorsed the positive FoP reviewer's assessment. An appeal to FoP challenging its negative reviewer's inept and unsubstantiated comments failed. Referring to *"the large amount of formulas and the somewhat confusing figure"*[12], FoP then ventured a U-turn 'disclaimer': *"We often have to reject correct and interesting papers such as yours"*. Thus instead of publishing a paper acknowledged (without noticing or referring to the now clarified gravity effects issue) as having corrected the RRF book's important indeed highlighted yet erroneous dilation formulae, the FoP journal choose to ignore it. Ironically both the RRF book and the FoP journal had the *same* publisher—*Kluwer* (Dordrecht/Boston/London).

A later attempt in 2011 to post the (retitled) paper on arXiv was blocked with the 'anonymous' statement: "Our moderators maintain that your article is not of interest to the [physics] community."[13].

[11]As graciously later acknowledged by an RRF book co-editor. Rates of rotation had been added *linearly* (equation (5.15)) without taking relativistic velocity composition into account, a simple mistake unthinkable had the velocity addition path been adopted. On the other hand, formulae (6) above and the incorrect RRF equations (5.19/20), though different, produced *arithmetical* results identical to within 20 places of decimals.

[12]The paper had less than half the number of equations in the RRF chapter and was a quarter the size. The diagram was a carefully generated computer graphic—Figure 2.

[13]arXiv.org eMail 23rd Sept. 2011, reference *submit/0320889*.

Acknowledgements

Thanks are due to Dublin physicists *Roy Johnston* for his suggestion to take issue with local confusion [5] on the subject, and *Ian Elliott* for helpful comments. In particular, I am grateful for constructive comments kindly offered by *David Mermin* which led to a title change. Gracious encouragements by *Matteo Ruggiero* ([6]) and *Moses Fayngold* ([10]) with regard to a 2008 version of the present paper, are also much appreciated.

References

[1] Sagnac G 1913 *L'éther lumineux demontré par l'effet du vent relatif d'éther dans un interféromètre en rotation uniforme (The luminiferous aether manifested by the relative aether wind in a uniformly rotating interferometer)* Comptes Rendu **157** 708-710. Paris 1913

[2] Frisch D, Smith J 1963 *Measurement of the relativistic Time-Dilation Using Mu-mesons* Am.J.Phys. **31** 342-355 http://physics.gmu.edu/~rubinp/courses/122/readings/AJP000342.pdf

[3] Hafele J C, Keating R E 1972 *Around-the-World Atomic Clocks: Predicted/Observed Relativistic Time Gains* Science **177**, 166-168

[4] Anderson R, Bilger H R, Stedman G E 1994 *'Sagnac' effect: A century of Earth-rotated interferometers* Am.J.Phys. **62** (11)

[5] Kelly A G 1998 *Open Questions in Relativistic Physics* F. Selleri (editor) Apeiron Press, Montreal. Kelly's (and Selleri's) 'absolute space' theory is firmly countered in several of [6]'s chapters and round table discussion.

[6] Rizzi G, Ruggiero M Editors 2004 *Relativity in Rotating Frames* Kluwer

[7] Rizzi G, Serafini A 2004 *Synchronization and desynchronization on rotating platforms*—in [6] chapter 5. Equation (5.15) inadvertently implied that the planes' tangential speed equals the linear sum/difference of the local Earth position's speed and the plane's speed relative thereto.

[8] Selleri F 2004 *Sagnac Effect: End of the Mystery*—in [6]

[9] Coleman B *Sagnac dilation expressions exactly and simply established* unpublished submission to *Foundation of Physics* FOOP-S-08-00393/FoP653 (*"We often have to reject correct and interesting papers such as yours."*)

[10] Fayngold M, 2008 *Special Relativity and how it works* Wiley Weinheim. This book also came close to deriving the proper dilation equations.

[11] Coleman B 2009 *Time 'dilation' and length 'contraction' - abuse of the present tense in special relativity and missed formulae such as for the Hafele-Keating/'Sagnac effect' experiment* Seminar April 24th *Institut Non Linéaire de Nice* Sophia Antipolis