. Research on the Riemann Hypothesis

. Wenlong Du

Abstract: This paper studies the relationship between the prime divisor and Stirling's approximation. We get prime number theorem and its corrected value. We get bound for the error of the prime number theorem. Riemann hypothesis is established.

Key words: the Riemann hypothesis, prime number, error of the prime number theorem, prime divisor

1. INTRODUCTION

In mathematics, the Riemann hypothesis, proposed by Bernhard Riemann (1859), is a conjecture that the non-trivial zeros of the Riemann zeta function all have real part 1/2.

The Riemann zeta function is defined for complex *s* with real part greater than 1 by the absolutely convergent infinite series[1]

$$\zeta(s)=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+......=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

The Riemann hypothesis is one of the most important conjectures in mathematics[2]. Hilbert listed the Riemann Hypothesis as one of his 23 problems for mathematicians of the twentieth century to work on[3].

Von Koch (1901) showed that the Riemann hypothesis is equivalent to[4]:

$$\pi(x)=Lix+\mathrm{O}\left(\sqrt{x}\log x\right)$$

2. RESEARCH ON THE PRIME NUMBER THEOREM

If P is a prime number, the number of prime divisor P which is contained by the natural number is less than N is:

The number of natural number which contains more than one prime divisor P is $\left[\frac{N}{P}\right]$.

The number of natural number which contains more than two prime divisors P is $\left[\frac{N}{P^2}\right]$.

The number of natural number which contains more than three prime divisors P is $\left[\frac{N}{P^3}\right]$.

………………

The number of natural number which contains more than m prime divisors P is $\left[\frac{N}{P^m}\right]$.

$N \ge P^m$, $N \le P^{m+1}$.

So the number of prime divisors P which is contained by the natural number less than N is:

$$\left[\frac{N}{P}\right]+\left[\frac{N}{P^2}\right]+\left[\frac{N}{P^3}\right]+......+\left[\frac{N}{P^m}\right]$$
$$\approx N\frac{\frac{1}{P}\left[1-\left(\frac{1}{P}\right)^m\right]}{1-\frac{1}{P}}$$
$$=N\frac{1-\frac{1}{P^m}}{P-1}$$
$$\approx\frac{N-1}{P-1}$$

We assume that the primes less than N is:

$P_1,P_2,......,P_m$, $P_1<P_2<......<P_m$.

We assume that the number of prime factor P which is contained[5] by the natural number less than N is α .

so $N!=P_1^{\alpha_1}P_2^{\alpha_2}......P_m^{\alpha_m}$

We can get $N!\approx P_1^{\frac{N-1}{P_1-1}}P_2^{\frac{N-1}{P_2-2}}P_3^{\frac{N-1}{P_3-3}}......P_m^{\frac{N-1}{P_m-1}}$,because of $\alpha\approx\frac{N-1}{P-1}$.

Prime number density is $\rho(n)$

$$P_1^{\frac{N-1}{P_1-1}}P_2^{\frac{N-1}{P_2-2}}P_3^{\frac{N-1}{P_3-3}}......P_m^{\frac{N-1}{P_m-1}}$$
$$\approx 2^{\rho(2)\frac{N-1}{2-1}}3^{\rho(3)\frac{N-1}{3-1}}4^{\rho(4)\frac{N-1}{4-1}}......N^{\rho(N)\frac{N-1}{N-1}}$$
$$=\prod_{n=2}^{N}n^{\rho(n)\frac{N-1}{n-1}}$$

$$\prod_{n=2}^{N}n^{\rho(n)\frac{N-1}{n-1}}=N!$$

(1) $$\prod_{n=2}^{N}n^{\rho(n)\frac{1}{n-1}}=\left[N!\right]^{\frac{1}{N-1}}$$

(2) $$\prod_{n=2}^{N+1} n^{\rho(n)\frac{1}{n-1}} = \left[(N+1)!\right]^{\frac{1}{N}}$$

（2）÷ (1)

$$(N+1)^{\rho(N+1)\frac{1}{N}} = \frac{\left[(N+1)!\right]^{\frac{1}{N}}}{\left[N!\right]^{\frac{1}{N-1}}} = \left[N!\right]^{\frac{1}{N}-\frac{1}{N-1}}(N+1)^{\frac{1}{N}}$$

$$\frac{\rho(N+1)}{N}\ln(N+1) = (\frac{1}{N}-\frac{1}{N-1})\ln(N!) + \frac{1}{N}\ln(N+1)$$

$$\rho(N+1) = \frac{N}{\ln(N+1)}\left[(\frac{1}{N}-\frac{1}{N-1})\ln(N!) + \frac{1}{N}\ln(N+1)\right]$$

$$= \frac{N}{\ln(N+1)}\left[\frac{-1}{N(N-1)}(\ln\sqrt{2\pi N} + N\ln N - N + \frac{\theta}{12N}) + \frac{1}{N}\ln(N+1)\right]$$

$$= \frac{N}{\ln(N+1)}\left\{\left[-\frac{\ln N}{N-1} + \frac{1}{N}\ln(N+1)\right] - \frac{\ln\sqrt{2\pi N} + \frac{\theta}{12N}}{N(N-1)} + \frac{1}{N-1}\right\} \quad (3)$$

$$-\frac{\ln N}{N-1} + \frac{1}{N}\ln(N+1)$$
$$= \frac{\ln(N+1)^{N-1} - \ln N^N}{N(N-1)}$$
$$= \frac{\ln\frac{(N+1)^{N-1}}{N^N}}{N(N-1)}$$
$$= \frac{\ln\left[\left(\frac{N+1}{N}\right)^N \frac{1}{N+1}\right]}{N(N-1)}$$
$$= \frac{1-\ln(N+1)}{N(N-1)}$$
$$<< \frac{1}{N-1}$$

$$\frac{\ln\sqrt{2\pi N} + \frac{\theta}{12N}}{N(N-1)} << \frac{1}{N-1}$$

$$(3) \approx \frac{N}{\ln(N+1)}\frac{1}{N-1} \approx \frac{1}{\ln(N+1)}$$

So $\rho(N+1)\approx\frac{1}{\ln(N+1)}$, $\rho(N)\approx\frac{1}{\ln N}$.

3. DICUSSION ON THE NUMBER OF PRIME NUMBERS' EXPECTED VALUE

We assume that the Prime number density is $\frac{1}{\ln N}$. The product of all prime factors is

$$
\begin{aligned}
&P_1^{\frac{N-1}{P_1-1}}P_2^{\frac{N-1}{P_2-2}}P_3^{\frac{N-1}{P_3-3}}\ldots\ldots P_m^{\frac{N-1}{P_m-1}}\\
&\approx 2^{\frac{N-1}{\ln 2(2-1)}}3^{\frac{N-1}{\ln 3(3-1)}}4^{\frac{N-1}{\ln 4(4-1)}}\ldots\ldots N^{\frac{N-1}{\ln(N-1)N-1}}\\
&=\prod_{n=2}^{N} n^{\frac{N-1}{\ln n(n-1)}}\\
&=\prod_{n=2}^{N} e^{\frac{N-1}{(n-1)}}\\
&=\left[e^{\int_M^N\frac{1}{n-1}dn}\right]^{N-1}\\
&=\left(\frac{N-1}{M-1}\right)^{N-1}
\end{aligned}
$$

Because of $\left(\frac{N}{N-1}\right)^{N-1}\approx e$, we get $\left(\frac{N-1}{M-1}\right)^{N-1}=\frac{N^{N-1}}{e(m-1)^{N-1}}$.

Error of the prime number theorem which is caused by $\frac{e^{\frac{\theta}{12N}}(m-1)^{N-1}}{e^{N-1}}\sqrt{2\pi}N^{\frac{3}{2}}$ is small. We can ignore the error.

The product of all prime divisor P is $P^{\frac{N-1}{P-1}}$, but The actual value is less than $P^{\frac{N-1}{P-1}}$. When $P>\sqrt{N}$, the product is about $P^{\frac{N-1}{P-1}-\frac{1}{2}}$. When $\sqrt{N}>P>\sqrt[3]{N}$, the product is about $P^{\frac{N-1}{P-1}-1}$……

The decrease of prime divisor product is about $e^N e^{\sqrt{N}} e^{\frac{1}{3}\sqrt[3]{N}} e^{\frac{1}{4}\sqrt[4]{N}}$……. The decrease of Prime number is

$$\int\frac{1}{2\sqrt{n}\ln n}dn+\int\frac{1}{2\sqrt[3]{n}^{\frac{2}{3}}\ln n}dn+\int\frac{1}{2\sqrt[4]{n}^{\frac{3}{4}}\ln n}dn+\ldots\ldots$$

Prime number which is corrected is closer to the actual value.

4. DISCUSS ON THE ERROR OF THE PRIME NUMBER THEOREM

The product of all prime factors P is $P^{\frac{N-1}{P-1}}$. The maximum error is N. We assume that the primes is $P_1, P_2, P_3, \ldots\ldots P_l$. The remainder that N is divided by $P_1, P_2, P_3, \ldots\ldots P_l$ is $\delta_1, \delta_2, \delta_3, \ldots\ldots \delta_l$. The value range of δ_1 is $(0, P_1 - 1)$. The value range of δ_2 is $(0, P_2 - 1)$ ……. When $N = P_1 P_2 P_3 \ldots\ldots P_l$, $\delta_1, \delta_2, \delta_3, \ldots\ldots \delta_l$ contain all different values, and each value is the unique. When N is a certain value, $\delta_1, \delta_2, \delta_3, \ldots\ldots \delta_l$ is a fixed value. With the increase of N, the value of $\delta_1, \delta_2, \delta_3, \ldots\ldots \delta_l$ change, and tend to disorder. When P is much less than N, δ is random.When P is not much less than N, δ is not random. With the increase of N, $\delta^* (P^* \ll m)$ increases and decreases periodically. So location of P is not random. The error of the prime number theorem will be reduced.

The changes of Prime's location change the number of prime numbers. The product of all prime factors P is $P^{\frac{N-1}{P-1}}$. When P is increased, the product is reduced, so we need more prime. When P is reduced, the product is increased, so we need to reduce the number of prime. δ increases and decreases periodically. Prime does not change the position more than two times or less than 1/2. Therefore, the number of prime numbers will not change more than two times or less than 1/2. So it can't bring magnitude changes.

We analyze the random error. The maximum error is N. We can assume that the probability that error is N is 0.5 and the probability that error is 0 is 0.5. Mean square error of the number of prime numbers is about $\frac{N^{\frac{1}{2}}}{2(\ln N)^{\frac{1}{2}}}$. The maximum number of primes' error is about $\frac{N^{\frac{1}{2}}}{\sqrt{2}}$. The probability that error is greater than $3N^{\frac{1}{2}}$ tends to 0. The level of the actual value is not more than $\frac{N^{\frac{1}{2}}}{\sqrt{2}}$.

Von Koch (1901) showed that the Riemann hypothesis is equivalent to:

$$\pi(x) = Lix + \mathrm{O}\left(\sqrt{x} \log x\right)$$

So Riemann hypothesis is established.

5. SUMMARY

This paper analyses error of the prime number theorem. Riemann hypothesis is established. Through the further analysis, we may obtained the error function. This method is effective on the twin prime conjecture, Goldbach's conjecture and Mersenne Primes conjecture.

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