

Catalysis, Heavy Fermions, Solitons, Cold Fusion, Low Energy Nuclear Reactions (LENR) and all that

M. W. Kalinowski
IMDIK PAN
Pracownia Bioinformatyki
ul. Pawińskiego 5, 02-106 Warsaw, Poland
e-mails: markwkalbioexploratorium.pl, mkalinowskiimdik.pan.pl,
phone: +48228499358

*To the memory
of my friend
Stanisław Radzki,
a chemist with wide horizons*

Abstract. We consider in the paper an idea of a soliton and heavy fermion catalysis for a cold fusion similar to a muon catalysis. This catalysis is achieved via quasi-chemical bonds for heavy fermions and solitons as well. We consider also a soliton catalysis (for KP-solutions), which is quite different. This kind of catalysis is similar to enzymatic catalysis. In the paper we construct a model for a cold fusion reactor based on Onsager–Prigogine irreversible thermodynamics.

We give examples of several compounds with heavy fermions (heavy electrons) which are hydrogen storages. Samples of those compounds can be (in principle) cold fusion reactors if filled with a deuter. It is necessary to do several experiments (described in the paper) in order to find a proper compound which will be a base for a battery device. We consider also a case with cold plasma (e.g. in metals) filled with a deuter. Solitons in a plasma can catalyse a fusion in two regimes: as quasiparticles and in enzymatic-like regime.

Key words: *solitons, heavy fermions, cold fusion, catalysis, hydrogen storage, low energy nuclear reactions, chemically assisted nuclear reactions, quasi-chemical bonds, KdV equation, plasma physics, KP equations, Onsager–Prigogine principle, deuter, quasiparticles.*

1 Introduction

In the paper we consider three types of catalysis for a cold fusion: heavy fermion, 1-soliton and n -soliton.

In the paper we give an idea of a soliton and heavy fermion (heavy electron) catalysis for a (super)cold fusion. The idea is based on two pillars: muon catalysis (Refs [1]–[6]) and cold fusion in palladium-hydrogen systems (Refs [7]–[9]) and exact solution of nonlinear PDE (Partial Differential Equations) in plasma physics (KdV — Korteweg–de

Vries equation). In this approach we want to use heavy fermions from strongly electron correlated materials (Refs [10]–[23]) as muons in muon catalysis, in some materials used for a hydrogen storage (Refs [24]–[28]). In that way we want to get a fusion reaction catalysed by a heavy fermion and a situation where a density of deuterium is very high as in a hydrogen storage. Thus we need a material with heavy fermions and with a high affinity to hydrogen. We consider also solitons in solid state (metals) electron plasma in order to catalyse ($d + d$) fusion.

It is interesting that the idea of heavy fermions and low energy nuclear reactions appears in Ref. [29].

In the paper we consider a kinetics of low energy nuclear reactions in a condensed phase. This involves many different domains of physics and chemistry, e.g. chemically assisted nuclear reactions. This approach can be even extended to biology in a living cell via biochemistry. On the molecular level there is no difference between biology, chemistry and physics. In some sense everything is physics. Moreover, there is no physics and chemistry without mathematics. The catalysis caused by heavy fermions in a condensed phase connects many aspects of solid state physics, nuclear physics and chemistry. Some considerations connected to free energy (free enthalpy) production due using of irreversible chemical thermodynamics can also be inspiring for researchers not particularly involved in a subject covered by the paper. The paper is primarily of a theoretical nature. Moreover, we give some ideas of an experimental setup to check the theory. Summing up, the paper comments many domains of physics, chemistry and even biology from both theoretical and experimental points of view.

The paper is divided into five sections. In the second we consider heavy fermion catalysis. In the third thermodynamics of fusion reactor, in the fourth soliton catalysis. In the fifth we consider a cylindrical sample as an example.

2 Heavy fermion catalysis

In this section we develop a heavy fermion catalysis. We call this idea supercold fusion for a possibility that such materials can have strongly correlated electrons (coming to heavy fermion systems) not only in an ambient temperature but even in low temperatures (of liquid helium or liquid nitrogen). (This is possible for some of those materials are superconductors.) The possibility of heavy fermion systems is very exciting for an effective mass of a quasiparticle (a quasifermion) can be so high as $1000m_e$ (electron mass). In this way we consider a quasiparticle (a heavy fermion) as a source of a quasi-chemical bond.

The idea of quasiparticles is a very powerful tool in physics from Hawking radiation theory (Bogolyubov transformation approach) to condensed matter physics and nuclear physics or even biophysics. It is very well known that in solid state physics electrons and holes can form bounded states—excitons. Quite recently we meet also new bounded states—trions formed from two holes and an electron or two electrons and a hole (see Ref. [30]). They are analogues of H^- or H_2^+ states. Some of researchers communicate to

discover in experiments condensation of exciton gas to a liquid or even Bose–Einstein condensation of this gas. Excitons and trions can be considered as quasiparticles as well.

Our idea is to use quasiparticles approach to form more exotic molecules—bounded systems of two deuterons and one heavy fermion in a hydrogen storage material in order to proceed a controlled nuclear fusion in ambient temperature (maybe in low temperatures).

According to an international discussion on a cold fusion in palladium no one was able to repeat results from Refs [7]–[8]. However some anomalous excesses in a neutron production have been reported.

Quasiparticles concept is strongly connected to condensed matter physics, e.g. solid state physics. They are related to highly complicated behaviour in a solid state by an analogy of a very simple picture, e.g. molecules forming a perfect gas in a container. Quasiparticles are usually considered as fermions (sometimes as bosons, but in this case they are usually called collective excitations). One can find very simple physical concepts of quasiparticles in Ref. [31]. In a solid state even electrons (which are “real” particles) should be considered as quasiparticles. Quasiparticles can interact. Moreover, the general idea is such that they interact weakly. Quasiparticles can interact with ordinary “real” particles. A typical example is an interaction (a scattering) of quasiparticles in a superfluid (a liquid helium 2) with neutrons discovered by L. D. Landau. The second very well known example is a scattering of photons (real particles) and phonons (quasiparticles) as a description of Raman effect in solids. Our approach is to use interactions of quasiparticles in a sample of a solid state material with ordinary particles like deuterons, protons etc. Moreover, there is a different approach (see Ref. [32]) where real interactions are considered between deuterons induced by a lattice palladium (PdH) or Ti lattice TiH_2 . In this approach effective potentials between Pd and H

$$V_{\text{Pd-H}} = \frac{22.2}{r} \text{ eV} \cdot \text{\AA} \exp\left(-\frac{r}{0.42\text{\AA}}\right) \quad (2.1)$$

and between Ti and H

$$V_{\text{Ti-H}} = \frac{31}{r} \text{ eV} \cdot \text{\AA} \left(\exp\left(-\frac{r}{0.51\text{\AA}}\right) + \left(\frac{r}{1.095\text{\AA}}\right)^{10.57} \right) \exp\left(-\frac{r}{0.23\text{\AA}}\right) \quad (2.2)$$

have been derived.

Due to these potentials it is possible to derive effective potentials between H and H (D and D). These potentials consist of two parts: repulsive and attractive

$$V_{\text{H-H}}(r) = V_{\text{R}}(r) + V_{\text{A}}(r) \quad (2.3)$$

$$V_{\text{R}}(r) = \frac{e^2}{\varepsilon r} \left(\exp\left(-\frac{r}{D_S}\right) + \left(\frac{r}{r_{\text{R}}}\right)^p \exp\left(-\frac{r}{D_i}\right) \right) \quad (2.4)$$

$$V_{\text{A}}(r) = -\frac{e^2}{\varepsilon r} \left(\frac{r}{r_{\text{A}}}\right)^q \exp\left(-\frac{r}{D_{\text{A}}}\right) \quad (2.5)$$

($\varepsilon, D_S, r_R, r_A, D_A, p, q$ are parameters). This approach is known as a dense plasma approach for a cold fusion. In this approach a tunnel effect is considered in the potential (2.3) for two deuterons to get a cold fusion.

Our approach is different for we consider “quasichemical bonds” in a sample of a solid state matter caused by a “heavy fermion”—a quasiparticle. Due to this “quasichemical bond” a quasimolecule of two deuterons is formed and a tunnel effect is similar as in a muon catalysis. In some sense this description is dual to the effective potential between two deuterons for a heavy fermion—heavy electron exists due to lattice interactions with real electrons.

In Ref. [32] the few-particle processes are considered as a cause of a cold fusion in contradiction to binary processes which are rather typical for a hot fusion. Our approach is also a few-particle process. Moreover one of the particles is a quasiparticle. Quasiparticle is a collective effect in condensed matter physics. In this paper we consider also the few-particle processes but with two stages. The first is a quasiparticle formation process and the second an interaction of a quasiparticle with real particles. The similarity with muon catalysis is evident and an analogue with chemical bond is natural. In this way our fusion process is a cold fusion process also in the terminology of Ref. [32].

Let us consider a hydrogen in heavy fermions (heavy electron) systems. In this way we consider an ionized hydrogen molecule bound by a heavy fermion (electron). This is similar to the μ^- hydrogen molecule. Let us denote a heavy fermion by F^- and let a mass of the particle be m_F . Let us suppose that we have to do with a heavy isotope of hydrogen D. In this way we have to do with three bodies interactions F^-, d^+, d^+ or F^-, d^+, p^+ , where d^+ denotes a deuterium nucleus (deuteron) and p^+ is a proton. We need a bounded system of $(d^+ d^+ F^-)^+$ or $(d^+ p^+ F^-)^+$ being an analogue of H_2^+ , HD^+ or $(D_2^+, (d^+ p^+ e)^+, (d^+ p^+ \mu^-)^+, (d^+ d^+ \mu^-)^+)$ as in muon catalysis. In general in heavy fermion systems we have to do with $m_F \gtrsim m_\mu = 207m_e$. In this way a_F is an analogue of a Bohr radius

$$a_F = \varepsilon \left(\frac{m_e}{m_F} \right) a_B = \left(\frac{m_e}{\frac{m_F}{\varepsilon}} \right) a_B \quad (2.6)$$

$$a_B = 0.5292 \cdot 10^{-10} \text{ m}, \quad (2.7)$$

where ε is a dielectric constant of the material. Thus we can apply the theory known as mesomolecules theory simply putting in the place of m_μ (a mass of a muon) an effective mass (from the point of view of the tunnel effect) of a heavy fermion divided by the dielectric constant of the material — $\frac{m_F}{\varepsilon}$ (see Refs [1]–[6]). In this way we can consider bound states of two deuterons or proton and deuteron bind by a quasiparticle (a heavy fermion) in a solid state lattice. This bond is very similar to ordinary chemical bond in H_2^+ or $(DH)^+$. Moreover because of a relation (2.6) it is much more tight. Both nuclei are closer in the quasimolecule. Because of this a tunnel effect (through a Coulomb barrier), which will cause a nuclear reaction, is much more possible than in the case of D_2^+ or $(DH)^+$. According to Refs [1]–[6] we get using Gamow theory of a tunnel

effect:

$$B_F \simeq \exp\left(-\frac{2}{\hbar}(2Me^2a_F)^{1/2}\right) \cong \exp\left(-K\sqrt{\frac{M\varepsilon}{m_F}}\right) \quad (2.8)$$

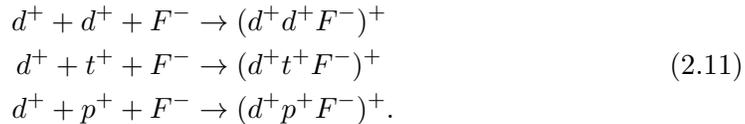
(B_F is a probability of a tunnel effect), where $K \simeq 3 \div 3.3$, M is a mass of a nucleus (d^+ , p^+ or t^+) or

$$B_F \simeq \exp\left(-K\sqrt{\frac{M}{m_e}} \cdot \sqrt{\frac{m_e\varepsilon}{m_F}}\right) = \left(\exp\left(-K\sqrt{\frac{M}{m_e}}\right)\right)^{\sqrt{\varepsilon m_e/m_F}}. \quad (2.9)$$

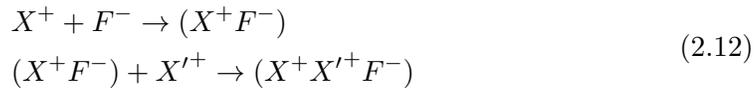
It is easy to write (2.9) in terms of B_μ —a probability of a muon catalysed nuclear fusion

$$B_F \simeq (B_\mu)^{\sqrt{\frac{\varepsilon m_\mu}{m_F}}}. \quad (2.10)$$

In this way we get a higher probability of a tunnel effect and a higher probability of a nuclear reaction. However we should take under consideration a different dielectric constant of a material with heavy fermions. In this way we get something which can be called a heavy fermion catalysis for low energy nuclear reactions. We can repeat all the considerations of Refs [1]–[6] concerning muon catalysis. If the mass of a heavy fermion divided by a dielectric constant of the material is m_μ the numerical results are exactly the same. Moreover we do not need any muon factory and some negative results of Refs [1]–[6] are not applicable. We need only a heavy fermion system with an affinity for a hydrogen (a different dielectric constant of this material should be taken into account). Thus we should consider a material with a high affinity for hydrogen with heavy electrons. In that material we should have a high density of hydrogen storage. Thus we consider the following processes

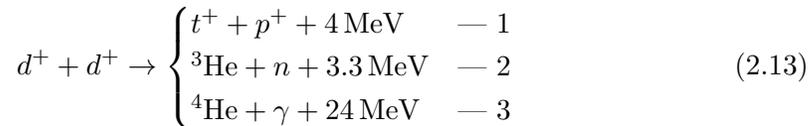


These three bodies processes can go via two stages



where $X^+, X'^+ = p^+, d^+, t^+$.

In those quasimolecular processes we get quasichemical bonds (quasimolecules) of d^+, t^+ or p^+ . Due to this we can expect higher rates of nuclear reactions



coming to cold-fusion with heavy fermion catalysis.

Is it possible to realize this idea in practice? In order to realize it we should satisfy two conditions:

- 1) We should find a material with heavy fermions (as high mass as possible) and with a high affinity for hydrogen. Possibly as high as possible i.e. a hydrogen storage.
- 2) To find a condition for a work of this system (hydrogen storage + heavy fermion systems). Probably in a low temperature where we have to do with a high density of heavy fermions.

What kind of materials have we on the market? First of all we have heavy fermion superconductors: CeCu_2Si_2 , UPt_3 , $\text{U}(\text{Be}_{1-x}\text{Th}_x)_{13}$ ($0 \leq x \leq 1$), UPd_2Al_3 , UNi_2Al_3 , URu_2Si_2 , CeCu_2Ge_2 , CeRh_2Si_2 , CeNi_2Ge_2 , CeIn_3 , ThCr_2Si_2 , CeTIn_3 ($T = \text{Co, Rh, Ir}$), UGe_2 , PuCoGa_5 .

Those materials are divided into two classes—the so-called Ce-based compounds and U-based compounds.

The effective mass of an electron is here several hundred times larger than a mass of a free electron due to strong electron correlations.

Some of those materials possess ferromagnetic or antiferromagnetic properties. The heavy fermion (heavy electron quasiparticle) is in general mixture of d -band or f -band electron with ordinary conduction electron. For example for CeCu_2Si_2 $m_F \simeq 220m_e$, for UPt_3 $m_F \simeq 200m_e$, for UBe_{13} $m_F \simeq 192m_e$ (see Refs [10]–[23]).

The second group is a heavy fermion semiconductors group: SmB_6 , YbB_{12} , CeBi_4Pt_3 , CeNiSm , CeRhSb , UNiSm , $\text{Ce}_3\text{Sb}_4\text{Pt}_3$, CeSb_4Au_3 , $\text{CeFe}_4\text{P}_{12}$, CeRhAs , $\text{U}_3\text{Sb}_4\text{Pt}_3$, $\text{U}_3\text{Sb}_4\text{Pd}_3$, $\text{UFe}_4\text{P}_{12}$, FeSi .

They also have ferromagnetic properties.

The physical origin of an appearance of a large quasiparticle—a heavy electron—is the same as for the first group. For example for FeSi , $m_F \simeq 14m_e$ (see Refs [10]–[23]).

All of those materials are intermetallic compounds involving lanthanides (rare earth elements) and actinides or Pd (palladium), Pt (platinum). It is very well known that rare earth elements, actinides, Pd, Pt, Ni and their intermetallic compounds possess high affinity for hydrogen. For example $\text{YNi}_2\text{B}_2\text{C}$ (see [33]). Some of them are used as hydrogen storage: Pd (as $\text{PdH}_{0.6}$), LaNi_5 (as LaNi_5H_6), Mg_2Ni (Mg_2NiH_2). In brackets we give formulae for hydrides of those intermetallics (Refs [24]–[28]). For example Pd can absorb 0.56 mass% of hydrogen and LaNi_5 1.37. The unusual properties of palladium-hydrogen system has been mentioned (Refs [24]–[28]). The hydrogen absorption in intermetallic compounds for lanthanides and actinides has been reviewed in Refs [27]–[28]. Thus we should look for such materials among lanthanides' and actinides' intermetallics (also for Pd and Pt). However in the case $\text{PdH}_{0.6}$ $m_F \approx 2 \div 3m_e$ which is quite small and maybe because of this cold fusion in palladium-hydrogen system is not effective. The Palladium Hydride is an ordinary low temperature superconductor with

a reverse isotope effect (see Refs [34]–[44]). The superconductivity for Pd H system has been observed by T. Skośkiewicz (see Ref. [45]).

We can consider also different hydrogen-storage compounds MgH_2 , NaAlH_4 , LiAlH_4 , TiFeH_2 , NaBH_4 (10.8 mass%), LiNH_2 , LiBH_2 , NaBH_4 (with unknown properties of an effective mass of an electron) and polymeric $\text{C}_6\text{H}_{10}\text{O}_5$, fulleren and its derivatives, nanotubes and graphen.

Let us notice that quasimolecular processes (2.11) can be described by a very well developed quantum-mechanical formalism applied for meso-atoms and meso-molecules. In this formalism we can calculate reaction rates calculating crosssections for those processes (see Refs [1]–[6]). The interesting point (if this idea works) is to use practically energy from (2.13). From the third one it would be possible to get it via a cascade of ionization processes and pair creation. In this way due to pair creation of e^+e^- and afterwards electron-hole creation in semiconductors we can try to get an energy in some kind of a battery device. The same can be said for the first reaction of (2.13). In the case of the second one we can expect only a heat production.

Eventually we mention the following problem: the existence of a bound state of two nuclei and a quasiparticle. In the case of muon catalysis this problem has been solved (see Refs [46]–[50]) and such a bound state really exists. The existence of the state has been confirmed in experiments (see Refs [46]–[52]). Using the same formalism as in the approach of Ponomarev et al. we can prove the existence of the bound state in our case. Thus we get a quasichemical bond of two deuterons and a heavy fermion. The problem of a stickness of a muon which plagues a muon catalysis does not concern us for we have to do with many heavy fermions in a material. Thus we have to do with chemically assisted nuclear reactions.

It is worth to mention that an idea of screening of a Coulomb potential in low energy nuclear reactions by electrons in metals gives an enhancement of deuteron fusion reactions (see Refs [53]–[54]). We will consider it later in a context of a soliton catalysis.

Let us consider a fusion process in a hydrogen storage medium. First of all we should supply this medium with deuterium and tritium mixture (D_2 and DT) or with pure deuterium (D_2). In the medium we should determine a rate of d^+F^- “atom” production and to find a sign of the existence of those “atoms”. Such an atom according to QM can be in excited state and emits (for absorbs) an electromagnetic radiation (a photon) of a frequency

$$h\nu_{nm} = E_m - E_n, \quad n < m \quad (2.14)$$

where

$$E_n = -\frac{\bar{R}_\infty}{n^2} \cdot \frac{1}{1 + \frac{m_F}{M_D}} \quad (2.15)$$

$$\bar{R}_\infty = \frac{m_F e^4}{32\pi^2 \varepsilon_0^2 \hbar^2 \varepsilon^2} \quad (2.16)$$

where M_D is the mass of a nucleus of deuterium and e an elementary charge. We need d^+F^- in a ground state $n = 1$ and of course for $l = 0$.

Let us consider Eqs (2.14)–(2.16). One gets

$$\nu_{nm}(m_F, \varepsilon, M_D) = \nu_{nm}(H) \frac{\left(1 + \frac{m_e}{m_p}\right)}{\left(1 + \frac{m_F}{M_D}\right)} \left(\frac{m_F}{m_e}\right) \frac{1}{\varepsilon^2} \quad (2.17)$$

or

$$\lambda_{nm}(m_F, \varepsilon, M_D) = \lambda_{nm}(H) \frac{\left(1 + \frac{m_F}{M_D}\right)}{\left(1 + \frac{m_e}{m_p}\right)} \left(\frac{m_e}{m_F}\right) \varepsilon^2 \quad (2.18)$$

where $\nu_{nm}(m_F, \varepsilon, M_D)$ means a frequency of a spectrum of d^+F^- “atom”, $\nu_{nm}(H)$ a frequency of a spectrum of hydrogen atom, $\lambda_{nm}(m_F, \varepsilon, M_D)$ and $\lambda_{nm}(H)$ the corresponding lengths and m_p a mass of a proton. Eqs (2.17)–(2.18) can be rewritten:

$$\nu_{nm}(m_F, \varepsilon, M_D) = \nu_{nm}(H) \frac{1.0005446 m_F}{(1 + 2.7233 \times 10^{-4} m_F) \varepsilon^2} \quad (2.19)$$

$$\lambda_{nm}(m_F, \varepsilon, M_D) = \lambda_{nm}(H) \frac{0.999456(1 + 2.7233 \times 10^{-4} m_F) \varepsilon^2}{m_F} \quad (2.20)$$

where now m_F is measured in m_e units and we use values of M_D , m_p , m_e from physical data. It is easy to see that usually $\lambda_{nm}(H) > \lambda_{nm}(m_F, \varepsilon, M_D)$. In this way visual spectrum of hydrogen corresponds to UV spectrum of our d^+F^- atom. Thus Paschen ($n = 3$) or even Brackett ($n = 4$), Pfund ($n = 5$), Humphreys ($n = 6$) spectral series correspond to visual or infrared spectral series of d^+F^- atom. In a future experiment with a tunable laser radiation we should look for infrared spectral lines of d^+F^- atom corresponding to higher spectral series ($n \geq 4$). An emission or an absorption process is of an electric dipole nature, i.e. E1.

Moreover, from a practical point of view we should consider the following strategy. First of all we should fill our sample of a material with ordinary hydrogen and to look for the existence of $p^+F^- = H^+F^-$ atom. The methodology is exactly the same as for d^+F^- atom. We are using a tunable laser for looking for an absorption spectrum of H^+F^- . Moreover, the spectrum will be a little different

$$\nu_{nm}(m_F, \varepsilon) = \nu_{nm}(H) \frac{\left(1 + \frac{m_e}{m_p}\right)}{\left(1 + \frac{m_F}{m_p}\right)} \left(\frac{m_F}{m_e}\right) \frac{1}{\varepsilon^2} \quad (2.21)$$

or

$$\lambda_{nm}(m_F, \varepsilon) = \lambda_{nm}(H) \frac{\left(1 + \frac{m_F}{m_p}\right)}{\left(1 + \frac{m_e}{m_p}\right)} \left(\frac{m_e}{m_F}\right) \varepsilon^2 \quad (2.22)$$

where $\nu_{nm}(m_F, \varepsilon)$ means a frequency of a spectrum of H^+F^- “atom”, $\lambda_{nm}(m_F, \varepsilon)$ the corresponding length.

Equations (2.21)–(2.22) can be rewritten:

$$\nu_{nm}(m_F, \varepsilon) = \nu_{nm}(\text{H}) \frac{1}{\varepsilon^2} \frac{1.0005446 m_F}{(1 + 5.446 \times 10^{-4} m_F)} \quad (2.23)$$

$$\lambda_{nm}(m_F, \varepsilon) = \lambda_{nm}(\text{H}) \varepsilon^2 \frac{0.999456(1 + 5.446 \times 10^{-4} m_F)}{m_F} \quad (2.24)$$

where m_F is measured in m_e units.

It is easy to see that

$$\lambda_{nm}(H) > \lambda_{nm}(m_F, \varepsilon) > \lambda_{nm}(m_F, \varepsilon, M_D). \quad (2.25)$$

As in the case of d^+F^- atom we should look for $n > 3$ spectral series to be in visual or infrared region.

Let us notice that in the case of H^+F^- and d^+F^- “atoms” we can expect also so-called Rydbergs atoms as in the case of excitons (as in Cu_2O case), i.e. in a very high quantum number n . The absorption spectrum of those “atoms” would not be Lorentzian (not symmetric). The spectrum will be distorted due to an overall absorption and a shape will be described by a Fano curve.

The next step is a formation of an analogue H_2^+ (ionized hydrogen molecule), i.e. $p^+p^+F^- = \text{H}^+\text{H}^+F^-$. In the case of muon catalysis there are some resonances which allow us to form a $\mu^-d^+d^+$ molecule. In our case such resonance can appear for $F^-H^+H^+$ or $F^-d^+d^+$ quasimolecules. Thus we should find some signals for an existence of such a quasimolecule in a piece of a material of heavy fermion systems filled with an ordinary hydrogen. This can be achieved only via spectroscopy of such a molecule.

Let us sketch some properties of such a molecule from the point of view of QM (Quantum Mechanics). A Schrödinger equation for such a three body system is as follows

$$H\psi = E\psi \quad (2.26)$$

where

$$H = -\frac{\hbar^2}{2m_F} \nabla^2 + \frac{q^2}{4\pi\varepsilon_0\varepsilon} \left(\frac{Z_A}{r_a} + \frac{Z_B}{r_b} \right). \quad (2.27)$$

Usually we suppose $Z_A = Z_B$. In this approach we neglect a motion of nuclei and consider a motion of heavy electron in a two-fixed-center potential. In atomic units Eq. (2.26) can be rewritten

$$\left(\nabla^2 + 2 \left(\frac{\tilde{Z}_A}{r_a} + \frac{\tilde{Z}_B}{r_b} \right) + 2\tilde{E} \right) \psi = 0 \quad (2.28)$$

where

$$\tilde{Z}_A = \frac{m_F}{\varepsilon} \cdot Z_A, \quad \tilde{Z}_B = \frac{m_F}{\varepsilon} \cdot Z_B, \quad \tilde{E} = m_F E. \quad (2.29)$$

m_F is measured in m_e units. r_a and r_b are distances of an heavy electron for two nuclei.

According to Ref. [55], Eq. (2.28) can be separated in prolate-spheroidal coordinates

$$\begin{aligned}\xi &= (r_a + r_b)/R, & 1 \leq \xi \leq \infty, \\ \eta &= (r_a - r_b)/R, & -1 \leq \eta \leq 1,\end{aligned}\tag{2.30}$$

$$\begin{aligned}Q_1 &= R(\tilde{Z}_A - \tilde{Z}_B) \\ Q_2 &= R(\tilde{Z}_A + \tilde{Z}_B)\end{aligned}\tag{2.31}$$

$$0 \leq \varphi \leq 2\pi.\tag{2.32}$$

φ is an azimuth angle, R is the distance between nuclei,

$$\psi(\xi, \eta, \varphi) = \Lambda(\xi)M(\eta, \varphi) = \Lambda(\xi)G(\eta)e^{\pm im\varphi}\tag{2.33}$$

(see Ref. [55]).

One gets

$$\left[\frac{\partial}{\partial \eta} \left((1 - \eta^2) \frac{\partial}{\partial \eta} \right) - \frac{m^2}{1 - \eta^2} - p^2 \eta^2 - Q_1 \eta - A \right] M(\eta, \varphi) = 0\tag{2.34}$$

$$\left[\frac{\partial}{\partial \xi} \left((\xi^2 - 1) \frac{\partial}{\partial \xi} \right) - \frac{m^2}{\xi^2 - 1} - p^2 \xi^2 - Q_2 \xi + A \right] \Lambda(\xi) = 0,\tag{2.35}$$

A is a separation constant. Thus we reduce the problem $H^+H^+F^-$ to the ordinary H_2^+ problem with some rescaling. If we denote a solution of Eq. (2.28) by $\Psi_{\tilde{Z}_A, \tilde{Z}_B, \tilde{E}}(\xi, \eta, \varphi)$ we get for $H^+H^+F^-$, $\Psi_{m_F, m_F, m_F E}(\xi, \eta, \varphi)$, where E is an eigenvalue of an energy for $H^+H^+F^-$ molecule. According to Ref. [55] the solution of Eq. (2.34) should be expanded into spherical harmonics

$$M(\eta, \varphi) = \sum_{k=m}^{\infty} f_m^k Y_k^m(\eta, \varphi)\tag{2.36}$$

$$\Lambda(\xi) = e^{-p(\xi-1)} [p(\xi-1)]^{m/2} \sum_{n=m/2}^{\infty} C_{n-(m/2)} \mathcal{L}_{n-(m/2)}^m [2p(\xi-1)].\tag{2.37}$$

For A and p^2 we have

$$\lim_{R \rightarrow 0} A = -l(l+1), \quad \lim_{R \rightarrow 0} p^2 = 0\tag{2.38}$$

(a united atom limit), $\mathcal{L}_k^m(x)$ are Laguerre polynomials.

Quantum numbers n, l, m are good quantum numbers only in the limit of a united atom (we are using them). In general only m is a good quantum number

$$\tilde{E} = \tilde{E}_{\text{elec}} = -2 \frac{p^2}{R^2},\tag{2.39}$$

p^2 depends also on \tilde{Z}_A and \tilde{Z}_B . In our case $\tilde{Z}_A = \tilde{Z}_B = m_F$. There are a lot of computer programs which can solve equations for any $\tilde{Z}_A = \tilde{Z}_B = m_F$ getting \tilde{E} which

is equal to electron energy (see Ref. [55] and references therein). Moreover, we should add an energy of nuclei equal to $\frac{1}{\varepsilon R}$ and calculate the minimum with respect to R . This gives us a ground state energy. We should remember that we have to do with symmetric and antisymmetric states. The ground state of H_2^+ is denoted by $X^2\Sigma_g^+$ or $1s\sigma_g$ and is symmetric. The first excited state $A^2\Sigma_u^+$ or $2p\sigma_u$ is antisymmetric. Thus we can expect similar states for $H^+H^+F^-$ molecule but rescaled (also $^2\Pi_u$ or $^2\Pi_g$) by m_F and ε . This means we should look for more energetic spectrum using a tunable laser method in order to prove the existence of $H^+H^+F^-$.

Afterwards we should consider a $d^+F^-d^+$ or $d^+F^-t^+$ molecule formation. The most important feature is an interaction between an atom d^+F^- (dF) and d^+ or t^+ . We can suppose it is a harmonic potential or a Morse potential and similarly as in muon catalyzed fusion we expect $(ddF)^+$ for $K = 1, v = 1$, where K is an angular momentum quantum number for a molecule and v is a vibrational number (see Ref. [56]). In the case of $(dFt)^+$ we can employ similar mechanism based on E1 transition. We need $(dFd)^+$ and $(dFt)^+$ in a ground state. In the ground state a tunnel effect causes a fusion very efficiently. For a muon catalyzed fusion we have 10^{-12} s for a nuclear fusion. Using formulas (2.9) and (2.10) we get

$$10^{-12} \sqrt{\frac{m_F}{\varepsilon m_\mu}} \text{ s}$$

which is smaller than 10^{-12} s (for a muon catalyzed fusion) if $\frac{m_F}{\varepsilon} > m_\mu$. Let us estimate a rate of a fusion in our hydrogen storage medium per a unit of volume.

One gets

$$Y = \varphi \cdot \lambda_{(dF)} \lambda_{(dFd)^+} B_F \cdot \rho_H \quad (2.40)$$

where φ is a density of D in a hydrogen storage normalized to the density of a liquid hydrogen (ρ_H is a density of a liquid hydrogen measured in atoms on cm^3 , $\rho_H = 4.25 \times 10^{22}$ atoms/ cm^3). This value can be greater than one.

$\lambda_{(dF)}$, $\lambda_{(dFd)^+}$ are respectively probabilities of an atom (dF) formation in a ground state and quasi-molecules formation $(dFd)^+$. B_F is a probability of a tunnel effect (see similar formulas for a muon catalyzed fusion Ref. [57]). We neglect an existence of tritium in a sample.

Let us apply Eq. (2.17) in the case of reactions (2.13). One gets

$$e_i = \varphi \lambda_{(dF)} \lambda_{(dFd)^+} B_F \rho_H B_i E_i, \quad i = 1, 2, 3, \quad (2.41)$$

where e_i is an energy production in one second per the unit of volume for three reactions in (2.13). E_i is an energy for a reaction number i , $E_1 = 4 \text{ MeV}$, $E_2 = 3.3 \text{ MeV}$, $E_3 = 24 \text{ MeV}$. B_i , $i = 1, 2, 3$, are branching ratios of these reactions: $B_1 \simeq B_2 \simeq 50\%$ and B_3 is very small, i.e. $B_3 \simeq 10^{-6}$. From practical point of view 1 and 3 are very important for they are ‘‘aneutronic’’ reactions. They produce charged particles and for this they can be a source of an electrical energy without conversion from thermal to electrical energy. For the process of an energy conversion is more efficient.

One can criticize our approach to cold fusion, i.e. a presence of a strong “heavy electron” screening of the Coulomb barrier which would prevent cold fusion. One argues the physical basis of the claim of strong electron screening on a length scale much less than an angström is false.

This claim is based on the following reasoning. In the crystal with “heavy electron” band we have periodic band energy

$$E(\vec{k}) = E_0 - \sum_{\vec{M} \neq 0} H(\vec{M}) \cos(\vec{k} \cdot \vec{M}) \quad (2.42)$$

where $H(\vec{M})$ is the Hamiltonian for the electron to go from one crystal cell to another, where \vec{M} is a lattice vector connecting two cells. If matrix elements of the Hamiltonian $|H(\vec{M})| \ll 1$ eV the electron behaves as heavy for its velocity is small

$$\vec{v}(\vec{k}) = \frac{1}{\hbar} \left(\frac{\partial E}{\partial \vec{k}} \right) = \frac{1}{\hbar} \sum_{\vec{M} \neq 0} \vec{M} H(\vec{M}) \sin(\vec{k} \cdot \vec{M}). \quad (2.43)$$

Those electrons are trapped in a localized Wannier state which can have wide spatial dispersion. In this way the bound deuteron–deuteron–electron wave function can be of order ~ 1 Å size as in the vacuum which makes impossible to screen the Coulomb barrier and afterwards a bigger probability of a tunnel effect.

Moreover, we can make a localized Wannier state narrower making “heavy electron” band localized, which is possible in some mentioned materials, i.e. in materials which are superconductors (heavy fermion superconductors). Some boundaries of a sample can help to get localized heavy electron states (surface states). In this way we have to do with so-called CMNS (condensed matter nuclear science).

Thus the mentioned mechanism can work and only the experiment can settle the controversy.

How to design an experiment? First of all we should choose a hydrogen storage with heavy fermions and afterwards fill it with deuterium. (It is reasonable first to fill a sample of a hydrogen storage with heavy fermion material with an ordinary hydrogen H, in order to find which of them (i.e. materials) are able to form quasi-atoms H^+F^- . This can be achieved by spectroscopic methods using a tunable laser. Only such materials should be filled afterwards with deuterium.) Then we should look for an absorption spectrum of (d^+F^-) (see Eqs (2.14)–(2.16)) to be sure that in our piece of material filled with D_2 there is an interaction between d^+ and F^- . This is a first step of the experiment. If we confirm this step we can go to the second step of the experiment. Now we should look for some products of low energy nuclear reactions, i.e. for p^+ (protons), γ —photons and n (neutrons). If we detect them on the level higher than the background radiation we can be sure that the mechanism works. We should work with several materials by trial and error method.

Thus we should use a tunable laser to obtain the required frequency of light (Eqs (2.14)–(2.15)) to observe an absorption and emission of photons and detectors of photons. According to a common practice for such an experiment we are required to put

a device with deuterium inside a calorimeter (e.g. flow-type calorimeter—SRI International flow calorimeter) in order to measure heat production. The energetic particles can be detected by CR-39, a plastic that suffers local damage by passage of charge particles or neutrons being a track detectors (solid-state nuclear track detectors). We can also observe tracks on X-ray films. The detectors should record the history of nuclear emission from an experiment and signal-to-noise ratios. The important point is to keep out of a possible chemical reaction heat sources, first of all to keep out of a possible reaction $2D_2 + O_2 \rightarrow 2D_2O$ which could be a source of undesired heat. For we have to do with many intermetallic materials, we should also keep out of phase transitions among several phases which can be a source or sink of heat.

3 Thermodynamical description of a fusion reactor (a battery device)

The interesting point in these investigations is a problem of thermodynamical description of a device: 1) a piece of material with heavy fermion system, 2) a flow of deuterium in the calorimeter. It seems that we should use non-equilibrium thermodynamics. Moreover, it is natural to consider non-equilibrium (irreversible) thermodynamics close to equilibrium. Such thermodynamics uses Onsager relations and a principle of minimal entropy production. Thus we should identify flows and thermodynamic forces and consider a possible free energy (enthalpy) production locally. In our case we have to do with chemical-like reactions as in Eq. (2.13). Considering Eq. (2.13) as a chemical reaction we get

$$J_i = \frac{\varphi}{N_A} \lambda_{(dF)} \lambda_{(dFd)^+} B_F \rho_H B_i$$

we should also add an additional reaction

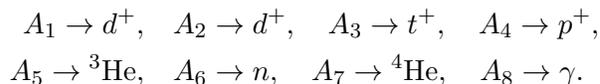


and

$$J_4 = \frac{\varphi}{N_A} (1 - \lambda_{(dF)} \lambda_{(dFd)^+} B_F) \rho_H. \quad (3.2)$$

Thus we define functions of a progress of reactions J_j , $j = 1, 2, 3, 4$. N_A is the Avogadro number.

Let us define



Thus we have 8 components in four reactions.

We define stochiometric coefficients a j component of i reaction ξ_{ij} . We get

$$\xi_{ij} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.3)$$

We have

$$\sum_{i=1}^8 \xi_{ij} A_i = 0, \quad j = 1, 2, 3, 4. \quad (3.4)$$

Let us denote by M_i a molar mass of the component A_i . Thus we have

$$\begin{aligned} M_1 &= M_2 = M_{d^+}, & M_3 &= M_{t^+}, & M_4 &= M_{p^+}, \\ M_5 &= M_{3\text{He}}, & M_6 &= M_n, & M_7 &= M_{4\text{He}}, & M_8 &= M_\gamma. \end{aligned}$$

A typical conservation law known in chemistry (a conservation of substance) can be written only as

$$\sum_{i=1}^8 \xi_{ij} M_i \approx 0 \quad (3.5)$$

for we have an energy production and this has an influence on the total mass.

We get a source term due to chemical reactions

$$M_i \sum_j \xi_{ij} J_j. \quad (3.6)$$

Let us define a balance equation for the i -th component

$$\frac{\partial \rho_j}{\partial t} = -\text{div}(\rho_j \vec{v}_i) - \text{div} \vec{j}_j + M_i \sum_i \xi_{ji} J_i, \quad (3.7)$$

where ρ_j , \vec{v}_j and \vec{j}_j are density, velocity and diffusion flux of the j -th component. It is easy to see that for $j = 1, 2$ we have the same balance of deuterium, for $j = 8$ $\vec{v}_j = 0$.

One can derive the following equation for an internal energy density

$$\rho \frac{du}{dt} = -\text{div}(\rho u \vec{v}) - p \text{div} \vec{v} + \frac{\rho}{N_A} \sum_{i=1}^3 e_i. \quad (3.8)$$

For a density of an entropy one gets

$$\rho \frac{ds}{dt} = -\text{div} \left(\frac{1}{T} \left(\rho u \vec{v} - \sum_{\substack{i=1 \\ i \neq 2}}^8 \mu_i \vec{j}_i \right) \right) - \frac{\rho u}{T^2} \vec{v} \cdot \vec{\nabla} T - \sum_{\substack{i=1 \\ i \neq 2}}^8 \vec{j}_i \cdot \vec{\nabla} \left(\frac{\mu_i}{T} \right) + \frac{1}{T} \sum_{j=1}^4 \bar{A}_j J_j. \quad (3.9)$$

where

$$\bar{A}_j = \sum_{\substack{i=1 \\ i \neq 2}}^8 \mu_i \xi_{ij} M_i$$

and μ_i is a chemical potential of the i -th component. This is basically a thermodynamics of our system.

Moreover, we are interested in a free energy production. One gets for f

$$\rho f = \rho(u - Ts) \quad (3.10)$$

$$\begin{aligned} \rho \frac{df}{dt} = & -\operatorname{div}(\rho u \vec{v}) + T \operatorname{div} \left(\frac{1}{T} \left(\rho u \vec{v} - \sum_{\substack{i=1 \\ i \neq 2}}^8 \mu_i \vec{j}_i \right) \right) + \frac{\rho u}{T} \vec{v} \cdot \vec{\nabla} T \\ & + T \sum_{\substack{i=1 \\ i \neq 2}}^8 \vec{j}_i \cdot \vec{\nabla} \left(\frac{\mu_i}{T} \right) - \sum_{j=1}^4 \bar{A}_j J_j - p \operatorname{div} \vec{v} + \frac{\rho}{N_A} \sum_{i=1}^3 e_i. \end{aligned} \quad (3.11)$$

We can also consider a density of a free enthalpy production:

$$g = u - Ts + \frac{pV}{\rho} \quad (3.12)$$

getting

$$\rho \frac{dg}{dt} = \rho \frac{df}{dt} + \rho \frac{d}{dt} \left(\frac{pV}{\rho} \right) \quad (3.13)$$

where p is a pressure, V —a volume, \vec{v} —a velocity, T —a temperature, ρ —density, \vec{j}_i —a diffusion flux, \bar{A}_j —a chemical affinity.

Let us notice that g is important for isothermo-isobaric processes and f for isothermo-isochoric processes

$$\vec{j}_i = -D_i \vec{\nabla} C_i \quad (3.14)$$

where D_i is a diffusion constant and c_i are concentration.

Eventually let us define two important functions Φ and Ψ :

$$\Phi = \rho u \vec{v} \cdot \vec{\nabla} \left(\frac{1}{T} \right) - \sum_{\substack{j=1 \\ j \neq 2}}^8 \vec{j}_j \cdot \vec{\nabla} \left(\frac{\mu_j}{T} \right) + \frac{1}{T} \sum_{j=1}^4 A_j J_j \quad (3.15)$$

$$\Psi = T\Phi \quad (3.16)$$

Φ is an entropy production function and Ψ is a dissipation function. We neglect a heat conduction in a sample supposing it is small in size.

Let us consider Eq. (3.15) in more details. One gets

$$\begin{aligned}
\Phi = & \rho u \vec{v} \cdot \vec{\nabla} \left(\frac{1}{T} \right) + D_{d^+} \vec{\nabla} C_{d^+} \vec{\nabla} \left(\frac{\mu_{d^+}}{T} \right) + D_{t^+} \vec{\nabla} C_{t^+} \cdot \vec{\nabla} \left(\frac{\mu_{t^+}}{T} \right) \\
& + D_{p^+} \vec{\nabla} C_{p^+} \cdot \vec{\nabla} \left(\frac{\mu_{p^+}}{T} \right) + D_{^3\text{He}} \vec{\nabla} C_{^3\text{He}} \vec{\nabla} \left(\frac{\mu_{^3\text{He}}}{T} \right) + D_n \vec{\nabla} C_n \cdot \vec{\nabla} \left(\frac{\mu_n}{T} \right) \\
& + D_{^4\text{He}} \vec{\nabla} C_{^4\text{He}} \vec{\nabla} \left(\frac{\mu_{^4\text{He}}}{T} \right) + D_\gamma \vec{\nabla} C_\gamma \cdot \vec{\nabla} \left(\frac{\mu_\gamma}{T} \right) + \frac{\varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{T N_A} \\
& \cdot \left[B_1 (\mu_{t^+} M_{t^+} + \mu_{p^+} M_{p^+}) + B_2 (\mu_{^3\text{He}} M_{^3\text{He}} + \mu_n M_n) \right. \\
& \left. + B_3 (\mu_{^4\text{He}} M_{^4\text{He}} + \mu_\gamma M_\gamma) - 2\mu_{d^+} M_{d^+} \right]
\end{aligned} \tag{3.17}$$

where μ, M, D, C are respectively chemical potentials, molar mass, coefficients of diffusion and concentrations of all components, i.e. $d^+, t^+, p^+, n, ^3\text{He}, ^4\text{He}$ and γ .

The general principle of a nonequilibrium thermodynamics, formulated by Onsager and Prigogine, consists in a minimal entropy production $\int \int_V \Phi dv = \min$. This is satisfied in a stationary case. In this way we go to the variational principle

$$\delta \int \int \int_V \Phi dv = 0 \tag{3.18}$$

where a variation is taken with respect to all mechanical, thermodynamical and chemical quantities. In particular with respect to $T, \rho, \vec{v}, u, C_{d^+}, \mu_{d^+}, C_{t^+}, \mu_{t^+}, C_{p^+}, \mu_{p^+}, C_{^3\text{He}}, \mu_{^3\text{He}}, C_n, \mu_n, C_{^4\text{He}}, \mu_{^4\text{He}}, C_\gamma, \mu_\gamma, \varphi$ established stationary boundary conditions for all quantities.

The variational problem can be written in a form of a system of partial differential equations for every function in the following way:

$$\begin{aligned}
\vec{\nabla} \frac{\partial \Phi}{\partial \vec{\nabla} \left(\frac{1}{T} \right)} - \frac{\partial \Phi}{\partial \left(\frac{1}{T} \right)} = 0 & \quad \vec{\nabla} \frac{\partial \Phi}{\partial \vec{\nabla} \mu} - \frac{\partial \Phi}{\partial \mu} = 0 \\
\vec{\nabla} \frac{\partial \Phi}{\partial \vec{\nabla} C} - \frac{\partial \Phi}{\partial C} = 0 & \quad \vec{\nabla} \frac{\partial \Phi}{\partial \vec{\nabla} \rho} - \frac{\partial \Phi}{\partial \rho} = 0 \\
\vec{\nabla} \frac{\partial \Phi}{\partial \vec{\nabla} u} - \frac{\partial \Phi}{\partial u} = 0 & \quad \text{etc.}
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\vec{\nabla}(\rho u \vec{v}) + \text{div} \left(D_{d^+} \vec{\nabla} C_{d^+} \mu_{d^+} + D_{t^+} \vec{\nabla} C_{t^+} \mu_{t^+} + D_{p^+} \vec{\nabla} C_{p^+} \mu_{p^+} \right. \\
+ D_{^3\text{He}} \vec{\nabla} C_{^3\text{He}} \mu_{^3\text{He}} + D_n \vec{\nabla} C_n \mu_n + D_{^4\text{He}} \vec{\nabla} C_{^4\text{He}} \mu_{^4\text{He}} + D_\gamma \vec{\nabla} C_\gamma \mu_\gamma \left. \right) \\
- \left(D_{d^+} \vec{\nabla} C_{d^+} \cdot \vec{\nabla} \mu_{d^+} + D_{t^+} \vec{\nabla} C_{t^+} \cdot \vec{\nabla} \mu_{t^+} + D_{p^+} \vec{\nabla} C_{p^+} \cdot \vec{\nabla} \mu_{p^+} \right. \\
+ D_{^3\text{He}} \vec{\nabla} C_{^3\text{He}} \cdot \vec{\nabla} \mu_{^3\text{He}} + D_n \vec{\nabla} C_n \cdot \vec{\nabla} \mu_n + D_{^4\text{He}} \vec{\nabla} C_{^4\text{He}} \cdot \vec{\nabla} \mu_{^4\text{He}} \\
+ D_\gamma \vec{\nabla} C_\gamma \cdot \vec{\nabla} \mu_\gamma \left. \right) - \frac{\varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} \cdot \left[B_1 (\mu_{t^+} M_{t^+} + \mu_{p^+} M_{p^+}) \right. \\
\left. + B_2 (\mu_{^3\text{He}} M_{^3\text{He}} + \mu_n M_n) + B_3 (\mu_{^4\text{He}} M_{^4\text{He}} + \mu_\gamma M_\gamma) - 2\mu_{d^+} M_{d^+} \right] = 0
\end{aligned} \tag{3.20}$$

$$\vec{\nabla} \frac{\partial \Phi}{\partial \vec{\nabla} \mu_{d^+}} - \frac{\partial \Phi}{\partial \mu_{d^+}} = 0 \quad (3.21)$$

$$\operatorname{div} \left(\frac{D_{d^+}}{T} \vec{\nabla} C_{d^+} \right) - D_{d^+} \vec{\nabla} \left(\frac{1}{T} \right) \vec{\nabla} C_{d^+} + \frac{2M_{d^+} \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{TN_A} = 0 \quad (3.22)$$

and for remaining concentrations

$$\operatorname{div} \left(\frac{D_{t^+}}{T} \vec{\nabla} C_{t^+} \right) - D_{t^+} \vec{\nabla} \left(\frac{1}{T} \right) \vec{\nabla} C_{t^+} - \frac{B_1 M_{t^+} \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{TN_A} = 0 \quad (3.23)$$

$$\operatorname{div} \left(\frac{D_{p^+}}{T} \vec{\nabla} C_{p^+} \right) - D_{p^+} \vec{\nabla} \left(\frac{1}{T} \right) \vec{\nabla} C_{p^+} - \frac{B_1 M_{p^+} \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{TN_A} = 0 \quad (3.24)$$

$$\operatorname{div} \left(\frac{D_{^3\text{He}}}{T} \vec{\nabla} C_{^3\text{He}} \right) - D_{^3\text{He}} \vec{\nabla} \left(\frac{1}{T} \right) \vec{\nabla} C_{^3\text{He}} - \frac{B_2 M_{^3\text{He}} \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{TN_A} = 0 \quad (3.25)$$

$$\operatorname{div} \left(\frac{D_{^4\text{He}}}{T} \vec{\nabla} C_{^4\text{He}} \right) - D_{^4\text{He}} \vec{\nabla} \left(\frac{1}{T} \right) \vec{\nabla} C_{^4\text{He}} - \frac{B_3 M_{^4\text{He}} \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{TN_A} = 0 \quad (3.26)$$

$$\operatorname{div} \left(\frac{D_n}{T} \vec{\nabla} C_n \right) - D_n \vec{\nabla} \left(\frac{1}{T} \right) \vec{\nabla} C_n - \frac{B_2 M_n \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{TN_A} = 0 \quad (3.27)$$

$$\operatorname{div} \left(\frac{D_\gamma}{T} \vec{\nabla} C_\gamma \right) - D_\gamma \vec{\nabla} \left(\frac{1}{T} \right) \vec{\nabla} C_\gamma - \frac{B_3 M_\gamma \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{TN_A} = 0. \quad (3.28)$$

For chemical potentials one gets

$$\operatorname{div} \left(D_{d^+} \vec{\nabla} \left(\frac{\mu_{d^+}}{T} \right) \right) = 0 \quad (3.29)$$

$$\operatorname{div} \left(D_{t^+} \vec{\nabla} \left(\frac{\mu_{t^+}}{T} \right) \right) = 0 \quad (3.30)$$

$$\operatorname{div} \left(D_{p^+} \vec{\nabla} \left(\frac{\mu_{p^+}}{T} \right) \right) = 0 \quad (3.31)$$

$$\operatorname{div} \left(D_{^3\text{He}} \vec{\nabla} \left(\frac{\mu_{^3\text{He}}}{T} \right) \right) = 0 \quad (3.32)$$

$$\operatorname{div} \left(D_n \vec{\nabla} \left(\frac{\mu_n}{T} \right) \right) = 0 \quad (3.33)$$

$$\operatorname{div} \left(D_{^4\text{He}} \vec{\nabla} \left(\frac{\mu_{^4\text{He}}}{T} \right) \right) = 0 \quad (3.34)$$

$$\operatorname{div} \left(D_\gamma \vec{\nabla} \left(\frac{\mu_\gamma}{T} \right) \right) = 0, \quad (3.35)$$

for a velocity one gets

$$\rho u \vec{\nabla} \left(\frac{1}{T} \right) = 0. \quad (3.36)$$

It means $T = \text{const.}$ For a parameter φ one gets

$$B_1 (\mu_{t^+} M_{t^+} + \mu_{p^+} M_{p^+}) + B_2 (\mu_{^3\text{He}} M_{^3\text{He}} + \mu_n M_n) + B_3 (\mu_{^4\text{He}} M_{^4\text{He}} + \mu_\gamma M_\gamma) = 2\mu_{d^+} M_{d^+}, \quad (3.37)$$

which is a constraint for chemical potentials.

Variations with respect to ρ and u do not give us any equations. They are satisfied identically for $T = \text{const}$.

We should supply all the equations with boundary data on a boundary of a sample V . This depends on the shape of the sample.

For concentrations one gets after some simplifications

$$D_{d^+} \vec{\nabla}^2 C_{d^+} + \frac{2M_{d^+} \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} = 0 \quad (3.38)$$

$$D_{t^+} \vec{\nabla}^2 C_{t^+} - \frac{B_1 M_{t^+} \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} = 0 \quad (3.39)$$

$$D_{p^+} \vec{\nabla}^2 C_{p^+} - \frac{B_1 M_{p^+} \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} = 0 \quad (3.40)$$

$$D_{^3\text{He}} \vec{\nabla}^2 C_{^3\text{He}} - \frac{B_2 M_{^3\text{He}} \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} = 0 \quad (3.41)$$

$$D_{^4\text{He}} \vec{\nabla}^2 C_{^4\text{He}} - \frac{B_3 M_{^4\text{He}} \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} = 0 \quad (3.42)$$

$$D_n \vec{\nabla}^2 C_n - \frac{B_2 M_n \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} = 0 \quad (3.43)$$

$$D_\gamma \vec{\nabla}^2 C_\gamma - \frac{B_3 M_\gamma \varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} = 0. \quad (3.44)$$

For chemical potentials simply

$$\vec{\nabla}^2 \mu_{d^+} = \vec{\nabla}^2 \mu_{t^+} = \vec{\nabla}^2 \mu_{p^+} = \vec{\nabla}^2 \mu_{^3\text{He}} = \vec{\nabla}^2 \mu_n = \vec{\nabla}^2 \mu_{^4\text{He}} = \vec{\nabla}^2 \mu_\gamma = 0. \quad (3.45)$$

Eqs (3.38)–(3.44) can be simply written as

$$\vec{\nabla}^2 C_i = \kappa K_i \quad (3.46)$$

where

$$\kappa = \frac{\varphi B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} \quad (3.47)$$

and $i = d^+, t^+, p^+, {}^3\text{He}, {}^4\text{He}, n, \gamma$,

$$K_{d^+} = -\frac{2M_{d^+}}{D_{d^+}} \quad (3.48)$$

$$K_{t^+} = \frac{B_1 M_{t^+}}{D_{t^+}} \quad (3.49)$$

$$K_{p^+} = \frac{B_1 M_{p^+}}{D_{p^+}} \quad (3.50)$$

$$K_{{}^3\text{He}} = \frac{B_2 M_{{}^3\text{He}}}{D_{{}^3\text{He}}} \quad (3.51)$$

$$K_{{}^4\text{He}} = \frac{B_3 M_{{}^4\text{He}}}{D_{{}^4\text{He}}} \quad (3.52)$$

$$K_n = \frac{B_2 M_n}{D_n} \quad (3.53)$$

$$K_\gamma = \frac{B_3 M_\gamma}{D_\gamma} \quad (3.54)$$

We suppose that D_i are constant (coefficients of diffusion).

For chemical potentials satisfy Laplace equations the constraint (3.37) really means a constraint on a boundary data for those equations.

We put a solution of a variation problem to the equation (3.11) to get a free energy production. The important point is to calculate the second variation in order to check if we have to do with a minimum.

Thus one gets

$$\begin{aligned} \rho \frac{df}{dt} = & -\text{div}(\rho u \vec{v}) \\ & + T \text{div} \left(\frac{1}{T} (\rho u \vec{v} + \mu_{d^+} D_{d^+} \vec{\nabla} C_{d^+} + \mu_{t^+} D_{t^+} \vec{\nabla} C_{t^+} + \mu_{p^+} D_{p^+} \vec{\nabla} C_{p^+} \right. \\ & \left. + \mu_{{}^3\text{He}} D_{{}^3\text{He}} \vec{\nabla} C_{{}^3\text{He}} + \mu_n D_n \vec{\nabla} C_n + \mu_{{}^4\text{He}} D_{{}^4\text{He}} \vec{\nabla} C_{{}^4\text{He}} + \mu_\gamma D_\gamma \vec{\nabla} C_\gamma) \right) \\ & + \frac{\rho u}{T} \vec{v} \cdot \vec{\nabla} T \\ & - T \left(D_{d^+} \vec{\nabla} C_{d^+} \vec{\nabla} \left(\frac{\mu_{d^+}}{T} \right) + D_{t^+} \vec{\nabla} C_{t^+} \vec{\nabla} \left(\frac{\mu_{t^+}}{T} \right) + D_{p^+} \vec{\nabla} C_{p^+} \vec{\nabla} \left(\frac{\mu_{p^+}}{T} \right) \right. \\ & \left. + D_{{}^3\text{He}} \vec{\nabla} C_{{}^3\text{He}} \vec{\nabla} \left(\frac{\mu_{{}^3\text{He}}}{T} \right) + D_n \vec{\nabla} C_n \vec{\nabla} \left(\frac{\mu_n}{T} \right) \right. \\ & \left. + D_{{}^4\text{He}} \vec{\nabla} C_{{}^4\text{He}} \vec{\nabla} \left(\frac{\mu_{{}^4\text{He}}}{T} \right) + D_\gamma \vec{\nabla} C_\gamma \vec{\nabla} \left(\frac{\mu_\gamma}{T} \right) \right) \\ & - \frac{\varphi}{N_A} \rho_H B_F \lambda_{(dF)} \lambda_{(dFd)^+} \left(B_1 (\mu_{t^+} M_{t^+} + \mu_{p^+} M_{p^+}) \right. \\ & \left. + B_2 (\mu_{{}^3\text{He}} M_{{}^3\text{He}} + \mu_n M_n) + B_3 (\mu_{{}^4\text{He}} M_{{}^4\text{He}} + \mu_\gamma M_\gamma) - 2\mu_{d^+} M_{d^+} \right) \\ & - p \text{div} \vec{v} + \frac{\rho}{N_A} \varphi \lambda_{(dF)} \lambda_{(dFd)^+} B_F \rho_H (B_1 E_1 + B_2 E_2 + B_3 E_3). \end{aligned} \quad (3.55)$$

In this formula E_i , $i = 1, 2, 3$, are measured in joules. The total free energy production is given by the formula

$$\tilde{F} = \iiint_V \frac{df}{dt} dv \quad (3.56)$$

where all quantities are solutions of variational problem.

Using some results from our investigations, especially $T = \text{const}$, one gets for $\frac{df}{dt}$

$$\begin{aligned} \frac{df}{dt} = & \frac{1}{\rho} \left(\text{div}(\rho u \vec{v}) + \text{div} \left(\mu_{d^+} D_{d^+} \vec{\nabla} C_{d^+} + \mu_{t^+} D_{t^+} \vec{\nabla} C_{t^+} + \mu_{p^+} D_{p^+} \vec{\nabla} C_{p^+} \right. \right. \\ & + \mu_{^3\text{He}} D_{^3\text{He}} \vec{\nabla} C_{^3\text{He}} + \mu_n D_n \vec{\nabla} C_n + \mu_{^4\text{He}} D_{^4\text{He}} \vec{\nabla} C_{^4\text{He}} + \mu_\gamma D_\gamma \vec{\nabla} C_\gamma \left. \right) \\ & - \left(D_{d^+} \vec{\nabla} C_{d^+} \vec{\nabla} \mu_{d^+} + D_{t^+} \vec{\nabla} C_{t^+} \vec{\nabla} \mu_{t^+} + D_{p^+} \vec{\nabla} C_{p^+} \vec{\nabla} \mu_{p^+} \right. \\ & + D_{^3\text{He}} \vec{\nabla} C_{^3\text{He}} \vec{\nabla} \mu_{^3\text{He}} + D_n \vec{\nabla} C_n \vec{\nabla} \mu_n + D_{^4\text{He}} \vec{\nabla} C_{^4\text{He}} \vec{\nabla} \mu_{^4\text{He}} + D_\gamma \vec{\nabla} C_\gamma \vec{\nabla} \mu_\gamma \left. \right) \\ & - \frac{\varphi \rho_H B_F \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} \left(B_1 (\mu_{t^+} M_{t^+} + \mu_{p^+} M_{p^+}) \right. \\ & + B_2 (\mu_{^3\text{He}} M_{^3\text{He}} + \mu_n M_n) + B_3 (\mu_{^4\text{He}} M_{^4\text{He}} + \mu_\gamma M_\gamma) - 2\mu_{d^+} M_{d^+} \left. \right) \\ & \left. - p \text{div} \vec{v} \right) + \frac{1}{N_A} \varphi \lambda_{(dF)} \lambda_{(dFd)^+} B_F \rho_H (B_1 E_1 + B_2 E_2 + B_3 E_3). \quad (3.57) \end{aligned}$$

Let us denote by $V \subset \mathbb{R}^3$ a set of a shape of a sample and by ∂V its boundary. Let us notice that V means also a volume of a sample. Moreover, it does not result in any misunderstanding. Let us define boundary conditions of our set of elliptic equations (3.38)–(3.44), (3.45) and (3.46) with (3.47), Eqs (3.48)–(3.54), i.e.

$$c_i|_{\partial V} = \tilde{c}_i \quad (3.58)$$

$$\mu_i|_{\partial V} = \tilde{\mu}_i \quad (3.59)$$

where the functions \tilde{c}_i and $\tilde{\mu}_i$ are defined on ∂V . One also gets

$$\begin{aligned} B_1 (\tilde{\mu}_{t^+} M_{t^+} + \tilde{\mu}_{p^+} M_{p^+}) + B_2 (\tilde{\mu}_{^3\text{He}} M_{^3\text{He}} + \tilde{\mu}_n M_n) \\ + B_3 (\tilde{\mu}_{^4\text{He}} M_{^4\text{He}} + \tilde{\mu}_\gamma M_\gamma) = 2\tilde{\mu}_{d^+} M_{d^+}. \quad (3.60) \end{aligned}$$

Solutions of Laplace equations can be obtained via elementary analytical or numerical methods. Afterwards we should consider Eqs (3.38)–(3.44). The best way to solve these equations is to use a Green function method. The Green function for V with ∂V is defined as follows:

$$\vec{\nabla}^2 G = -\delta(x - \xi, y - \eta, z - \zeta) \quad (3.61)$$

if $(x, y, z) \in V$ and $G = 0$ if $(x, y, z) \in \partial V$, where $G(x, y, z, \xi, \eta, \zeta)$ is a function of 6 variables singular if

$$x \rightarrow \xi \text{ or } y \rightarrow \eta \text{ or } z \rightarrow \zeta.$$

The solutions for Eqs (3.45) are given by

$$\mu_i(\xi, \eta, \zeta) = \iint_{\partial V} \frac{\partial G}{\partial n} \tilde{\mu}_i ds \quad (3.62)$$

where $\frac{\partial G}{\partial n}$ is the normal derivative of G on ∂V .

For Eq. (3.46) one gets

$$C_i(\xi, \eta, \zeta) = \kappa K_i \iiint_V G dv + \iint_{\partial V} \frac{\partial G}{\partial n} \tilde{c}_i ds. \quad (3.63)$$

Moreover, we need C_i also as a function of φ

$$C_i(\varphi, \xi, \eta, \zeta) = \varphi \tilde{K}_i \iiint_V G dv + \iint_{\partial V} \frac{\partial G}{\partial n} \tilde{c}_i ds, \quad (3.64)$$

where

$$\tilde{K}_i = \frac{B_F \rho_H \lambda_{(dF)} \lambda_{(dFd)^+}}{N_A} K_i. \quad (3.65)$$

We solve Eqs (3.38)–(3.44) for various values of φ and put all the solutions to Eq. (3.20) in order to get equations for

$$\rho u \vec{v} = \vec{w} \quad (3.66)$$

$$\operatorname{div} \vec{w} = \vec{f}(\vec{r}) \quad (3.67)$$

$$\vec{r} = (x, y, z) \in V \subset \mathbb{R}^3$$

One easily gets

$$\begin{aligned} \vec{f}(\vec{r}) = & -\varphi \left[\frac{D_{d^+} \tilde{K}_{d^+}}{2M_{d^+}} \iint_{\partial V} ds \frac{\partial G}{\partial n} \left(B_1(\tilde{\mu}_{t^+} M_{t^+} + \tilde{\mu}_{p^+} M_{p^+}) + B_2(\tilde{\mu}_{3\text{He}} M_{3\text{He}} + \tilde{\mu}_n M_n) \right. \right. \\ & + B_3(\tilde{\mu}_{4\text{He}} M_{4\text{He}} + \tilde{\mu}_\gamma M_\gamma) \left. \left. + D_{p^+} \tilde{K}_{p^+} \iint_{\partial V} ds \frac{\partial G}{\partial n} \tilde{\mu}_{p^+} \right. \right. \\ & + D_{3\text{He}} \tilde{K}_{3\text{He}} \iint_{\partial V} ds \frac{\partial G}{\partial n} \tilde{\mu}_{3\text{He}} + D_n \tilde{K}_n \iint_{\partial V} ds \frac{\partial G}{\partial n} \tilde{\mu}_n + D_{t^+} \tilde{K}_{t^+} \iint_{\partial V} ds \frac{\partial G}{\partial n} \tilde{\mu}_{t^+} \\ & \left. \left. + D_{4\text{He}} \tilde{K}_{4\text{He}} \iint_{\partial V} ds \frac{\partial G}{\partial n} \tilde{\mu}_{4\text{He}} + D_\gamma \tilde{K}_\gamma \iint_{\partial V} ds \frac{\partial G}{\partial n} \tilde{\mu}_\gamma \right] \quad (3.68) \end{aligned}$$

We suppose a potential flow

$$\vec{v} = \vec{\nabla} R, \quad (3.69)$$

where R is a velocity potential.

Thus one gets

$$\operatorname{div}(\rho u \vec{\nabla} R) = \tilde{f}(\vec{r}). \quad (3.70)$$

Simultaneously we have a continuity equation

$$\operatorname{div}(\rho \vec{v}) = 0 \quad (3.71)$$

$$\text{or } \operatorname{div}(\rho \vec{\nabla} R) = 0. \quad (3.72)$$

Supposing equation

$$u = u(T, \rho) \quad (3.73)$$

one eventually gets

$$\begin{cases} \rho \frac{\partial u}{\partial \rho} \vec{\nabla} \rho \vec{\nabla} R = \tilde{f}(\vec{r}) \\ \vec{\nabla} \rho \vec{\nabla} R + \vec{\nabla}^2 R = 0. \end{cases} \quad (3.74)$$

We can solve system Eqs (3.74) under boundary conditions

$$R|_{\partial V} = \tilde{R}, \quad \rho|_{\partial V} = \tilde{\rho}$$

where \tilde{R} and $\tilde{\rho}$ are functions defined on ∂V . Afterwards we put solutions and Eq. (3.74) to Eq. (3.57) with several values of φ getting from Eq. (3.56) a total free energy production, \tilde{F} . We suppose

$$p = p(T, \rho). \quad (3.75)$$

From Eq. (3.57) one easily gets

$$\frac{df}{dt} = -\frac{p}{\rho} \vec{\nabla}^2 R + \frac{1}{N_A} \varphi \lambda_{(dF)} \lambda_{(dFd)+} B_F \rho_H (B_1 E_1 + B_2 E_2 + B_3 E_3). \quad (3.76)$$

Using Eq. (3.68) one easily gets

$$\begin{aligned} \frac{df}{dt} = \varphi \left\{ \frac{1}{N_A} \lambda_{(dF)} \lambda_{(dFd)+} B_F \rho_H (B_1 E_1 + B_2 E_2 + B_3 E_3) \right. \\ - \iint_{\partial V} ds \frac{\partial G}{\partial n} \left[\frac{D_{d+} \tilde{K}_{d+}}{2M_{d+}} \cdot (B_1 (\tilde{\mu}_{t+} M_{t+} + \tilde{\mu}_{p+} M_{p+}) + B_2 (\tilde{\mu}_{3\text{He}} M_{3\text{He}} + \tilde{\mu}_n M_n) \right. \\ \left. \left. + B_3 (\tilde{\mu}_{4\text{He}} M_{4\text{He}} + \tilde{\mu}_\gamma M_\gamma) \right) + D_{p+} \tilde{K}_{p+} \tilde{\mu}_{p+} + D_{3\text{He}} \tilde{K}_{3\text{He}} \tilde{\mu}_{3\text{He}} + D_n \tilde{K}_n \tilde{\mu}_n \right. \\ \left. \left. + D_{t+} \tilde{K}_{t+} \tilde{\mu}_{t+} + D_{4\text{He}} \tilde{K}_{4\text{He}} \tilde{\mu}_{4\text{He}} + D_\gamma \tilde{K}_\gamma \tilde{\mu}_\gamma \right] \right\}. \quad (3.77) \end{aligned}$$

Eqs (3.56), (3.73), (3.74), (3.75), (3.77) give us a simple theory of our cold fusion reactor (together with all boundary conditions). The system of Eqs (3.74) forms a nonlinear system of pde. In this way even if it is elliptic we can expect some interesting phenomena for \tilde{F} connecting to soliton-like solutions. This demands more investigations and will be done elsewhere.

For a total free energy production in a second one gets from Eq. (3.56) and Eq. (3.77)

$$\begin{aligned}
\tilde{F} = & \iiint_V dv \varphi \left\{ \frac{1}{N_A} \lambda_{(dF)} \lambda_{(dFd)^+} B_F \rho_H (B_1 E_1 + B_2 E_2 + B_3 E_3) \right. \\
& - \iint_{\partial V} ds \frac{\partial G}{\partial n} \left[\frac{D_{d^+} \tilde{K}_{d^+}}{2M_{d^+}} \cdot \left(B_1 (\tilde{\mu}_{t^+} M_{t^+} + \tilde{\mu}_{p^+} M_{p^+}) + B_2 (\tilde{\mu}_{^3\text{He}} M_{^3\text{He}} + \tilde{\mu}_n M_n) \right) \right. \\
& + B_3 (\tilde{\mu}_{^4\text{He}} M_{^4\text{He}} + \tilde{\mu}_\gamma M_\gamma) \left. \right) + D_{p^+} \tilde{K}_{p^+} \tilde{\mu}_{p^+} + D_{^3\text{He}} \tilde{K}_{^3\text{He}} \tilde{\mu}_{^3\text{He}} + D_n \tilde{K}_n \tilde{\mu}_n \\
& \left. \left. + D_{t^+} \tilde{K}_{t^+} \tilde{\mu}_{t^+} + D_{^4\text{He}} \tilde{K}_{^4\text{He}} \tilde{\mu}_{^4\text{He}} + D_\gamma \tilde{K}_\gamma \tilde{\mu}_\gamma \right] \right\}. \quad (3.78)
\end{aligned}$$

Functions u and p depend on the material of a sample and can be obtained from physico-chemical tables.

Let us notice that we can consider also a bound system of deuteron–deuteron–quasiparticle in such a case where a charge quasiparticle is screening a Coulomb barrier. In this way a tunnel effect has higher probability. The mentioned quasiparticle could have a different statistics than Fermi–Dirac statistics, i.e. Böse–Einstein statistics. This situation is similar to π^- -atoms and π^- -molecules. (We should prevent π -meson to interact strongly with nuclei.) There are also some interests in hardonic atoms (see Refs [58]–[62]). One can also try to design 2-dimensional systems interacting with deuterons. In this case we can have to do with anyons as the third particle in three-bodies interacting bound system. The important point is to find such materials where such charged quasiparticles are heavy and their states are narrow. They should screen the Coulomb barriers to make a tunnel effect more probable.

There is also a possibility to employ negatively charged solitons known in solid state physics in order to lower a Coulomb barrier in a three bodies interactions (i.e. deuteron–deuteron–soliton). In superconducting materials such solitons are possible to exist. In calculation of binding energy of such quasimolecules we can follow e.g. Ref. [63].

4 Soliton catalysis for LENR (Low Energy Nuclear Reactions)

Quasiparticles can be also considered as solutions (solitons) of some PDEs. A soliton in physics is a model of an individuum. In plasma in solid material (e.g. metal) KdV equation can be applied. This is for a collisionless plasma composed by cold ions and warm electrons (see Ref. [64]). They are ion acoustic solitons in a plasma.

Let us give some details. The basic equations are as follows

$$\frac{\partial n}{\partial \bar{t}} + \frac{\partial(n\bar{u})}{\partial \bar{x}} = 0 \quad (4.1)$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} = -\frac{\partial \psi}{\partial \bar{x}} \quad (4.2)$$

$$\frac{\partial^2 \psi}{\partial \bar{x}^2} = n_e - n \quad (4.3)$$

$$n_e = \exp \psi \quad (4.4)$$

where

$$n = \frac{\tilde{n}_i}{n_0}, \quad n_e = \frac{\tilde{n}_e}{n_0}, \quad \bar{u} = \tilde{u}(kT_e/m)^{-1/2} \quad (4.5)$$

are the dimensionless ion number density, electron number density, ion velocity and electro-static potential, respectively. (m is an ion mass and T_e an electron gas temperature.) Dimensionless space-time variable are measured by the Debye distance $(kT_e/4\pi e^2 n_0)^{1/2}$, and the ion plasma frequency is $(4\pi e^2 n_0/m)^{1/2}$ (see Refs [65]–[68]).

After some manipulations and simplifications we get KdV equation

$$\frac{\partial \zeta}{\partial \tau} + \frac{1}{2} \frac{\partial^3 \zeta}{\partial \xi^3} + \zeta \frac{\partial \zeta}{\partial \xi} = 0, \quad (4.6)$$

where

$$\begin{aligned} \xi &= \varepsilon^{1/2}(\bar{x} - \bar{t}) \\ \tau &= \varepsilon^{3/2}\bar{t} \end{aligned}$$

are so-called stretched variables, ε is an expansion parameter which is used for basic equation such that

$$\begin{aligned} n &= 1 + \varepsilon n^{(1)} + \dots \\ u &= \varepsilon u^{(1)} + \dots \\ \psi &= \varepsilon \zeta \end{aligned} \quad (4.7)$$

We are supposing boundary conditions

$$n = 1, \quad \psi = 0, \quad \bar{u} = 1 \quad \text{if } |\bar{x}| \rightarrow \infty. \quad (4.8)$$

For first order expansion one gets

$$\zeta = n^{(1)} = u^{(1)} = n_e^{(1)} \quad (4.9)$$

and KdV equation.

Let us consider the following transformations:

$$\zeta = 3\sqrt[3]{4}u, \quad \tau = t, \quad \xi = \frac{1}{\sqrt[3]{2}}x. \quad (4.10)$$

After these transformations our equation will be in a canonical form

$$u_t + u_{xxx} + 6uu_x = 0. \quad (4.11)$$

This equation has an important solution

$$u = \frac{2\kappa^2}{\cos h^2(\kappa(x - 4\kappa^2 t - x_0))}. \quad (4.12)$$

This solution is known as one-soliton solution (κ is a constant). The solution can be considered as the third particle interacting with two deuterons forming quasi-chemical bond between them. Due to electrostatic interaction a soliton is screening Coulomb interaction between deuterons to lower a Coulomb barrier. This is a one-dimensional model (in one dimension there are no fermions). Moreover, in higher dimensions, e.g. in 3-dimensional space, we can consider such an interaction as a quasi-chemical bond and an effect of this as a soliton catalysis.

We have also different equations: Boussinesq equation, Benjamin–Bona–Mahony equation, KdV–Burgers equation (KdV equation with dissipation). We can also consider spherical and cylindrical solitons. Some of mentioned equations are strictly connected. These equations have multiple soliton solutions (also infinite soliton solutions, so-called cn-waves). Moreover, we are interested only in one-soliton solutions. It happens that someone observed enhancement of deuterons-fusion reactions in metals (see Refs [53]–[54]). These results cannot be explained by the Debye–Hückel model that has been proposed to explain the influence of the electron screening on nuclear reactions. It seems that nonlinear effects coming to soliton solutions and an application of this idea to three bodies interactions (quasi-chemical bonds) can explain these results. Let us notice that a mass of a soliton is an energy of the soliton (as it can be calculated in a field theory of u). Thus all our calculations concerning a tunnel effect (a Gamow formula) are valid in the case of a soliton catalysis. Simultaneously all formulas obtained for heavy fermion catalysis are also valid for a soliton catalysis. (I mean free energy (enthalpy) production with application of Onsager–Prigogine irreversible thermodynamics). Let us consider KdV as a toy model.

In order to do this let us consider two important characterizations of a soliton charge and an energy (mass). A charge is simply equal to

$$Q = \int_{-\infty}^{+\infty} u \, dx. \quad (4.13)$$

A mass

$$\frac{E(u)}{c^2} = \frac{1}{c^2} \int_{-\infty}^{+\infty} \left[\frac{u_x^2}{2} - u^3 \right] dx \quad (4.14)$$

where c is the velocity of light and $E(u)$ is an energy of a soliton obtained from the Hamiltonian

$$H = \int_{-x}^x \left[\frac{u_x^2}{2} - u^3 \right] dx. \quad (4.15)$$

One easily gets

$$m_{\text{sol}} = -\frac{112\kappa}{5c^2} \sqrt[3]{2} n_0^2 \quad (4.16)$$

and

$$Q_{\text{sol}} = 12\kappa \sqrt[3]{4} n_0 e. \quad (4.17)$$

(e is an elementary charge). For $Q_{\text{sol}} < 0$ (negatively charged) $\kappa < 0$ and

$$m_{\text{sol}} = -\frac{112\kappa}{5c^2} \sqrt[3]{2} n_0 > 0.$$

In this way in our toy model for a Gamow formula one gets

$$B_{\text{sol}} \simeq \exp\left(-\frac{2}{h} (2m_{\text{sol}} Q_{\text{sol}}^2 a_{\text{sol}})^{1/2}\right) \quad (4.18)$$

where a_{sol} is a Bohr radius analogue for a soliton,

$$a_{\text{sol}} = \frac{4\pi\epsilon\epsilon_0\hbar^2}{m_{\text{sol}} Q_{\text{sol}}^2}. \quad (4.19)$$

Moreover, in order to consider a realistic 3-dimensional model we should consider the KP (Kadomtsev–Petviashvili) equation

$$\frac{\partial}{\partial x} (u_t + 6uu_x + u_{xxx}) = -3\beta^2 \Delta_{\perp} u \quad (4.20)$$

where

$$\Delta_{\perp} = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad (4.21)$$

This equation possesses soliton solutions, β is a constant.

A Hamiltonian for KP equation looks like

$$H_{\text{KP}} = \int_{-x}^x \int_{-y}^y \int_{-z}^z \left(\frac{u_x^2}{2} - \frac{3\beta^2}{2} (\vec{\nabla}_{\perp} w)^2 - u^3 \right) dV \quad (4.22)$$

in such a way that

$$w_x = u \quad (4.23)$$

$$\vec{\nabla}_{\perp} = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \quad (4.24)$$

One considers multisoliton solutions in order to get an enhancement of nuclear reactions. Moreover, an analogy of a catalysis breaks down. The screening of Coulomb interaction between deuterons and lowering a Coulomb barrier by multisoliton solutions of KP equations will be very important in metals filled with deuterons.

We can still use a Gamow formula, moreover, now an analogy of binding deuterons together cannot be maintained. Moreover, it seems that one can consider more general

configuration of ions surrounded by electron plasma governed by KP equations. This problem demands more investigations. In this way a Hamiltonian approach to soliton theory has been applied to cold fusion problem (see Ref. [69]).

Eventually one can calculate a ratio $\left| \frac{Q_{\text{sol}}}{m_{\text{sol}}} \right|$ for our toy model with KdV equation

$$\left| \frac{Q_{\text{sol}}}{m_{\text{sol}}} \right| = \frac{15 \sqrt[3]{2} c^2 e}{28 n_0}. \quad (4.25)$$

We can translate a probability of a tunnel effect from Gamow formula to a cross-section of a scattering of two deuterons. It follows that solitons can cause some kind of resonance effects.

In the case of soliton catalysis in cold plasma with solitons one gets similarly

$$\begin{aligned} \frac{df}{dt} = \varphi \left\{ \frac{1}{N_A} \sigma_{d+d^+} \rho_H (B_1 E_1 + B_2 E_2 + B_3 E_3) \right. \\ - \iint_{\partial V} ds \frac{\partial G}{\partial n} \left[\frac{D_{d^+} \tilde{K}_{d^+}}{2M_{d^+}} \cdot \left(B_1 (\tilde{\mu}_{t^+} M_{t^+} + \tilde{\mu}_{p^+} M_{p^+}) + B_2 (\tilde{\mu}_{3\text{He}} M_{3\text{He}} + \tilde{\mu}_n M_n) \right. \right. \\ \left. \left. + B_3 (\tilde{\mu}_{4\text{He}} M_{4\text{He}} + \tilde{\mu}_\gamma M_\gamma) \right) + D_{p^+} \tilde{K}_{p^+} \tilde{\mu}_{p^+} + D_{3\text{He}} \tilde{K}_{3\text{He}} \tilde{\mu}_{3\text{He}} + D_n \tilde{K}_n \tilde{\mu}_n \right. \\ \left. \left. + D_{t^+} \tilde{K}_{t^+} \tilde{\mu}_{t^+} + D_{4\text{He}} \tilde{K}_{4\text{He}} \tilde{\mu}_{4\text{He}} + D_\gamma \tilde{K}_\gamma \tilde{\mu}_\gamma \right] \right\}. \quad (4.26) \end{aligned}$$

Eqs (3.56), (3.73), (3.74), (3.75), (4.26) give us a simple theory of our plasma cold fusion (soliton catalysis) reactor (together with all boundary conditions, which are the same as in the heavy fermion case). The system of Eqs (3.74) forms a nonlinear system of PDE. In this way even if it is elliptic we can expect some interesting phenomena for \tilde{F} connecting to soliton-like solutions. This demands more investigations and will be done elsewhere. (Those solitons have nothing to do with solitons of KdV or KP.)

For a total free energy production in a second one gets (similarly as in heavy fermions case)

$$\begin{aligned} \tilde{F} = \iiint_V dv \varphi \left\{ \frac{1}{N_A} \sigma_{d+d^+} \rho_H (B_1 E_1 + B_2 E_2 + B_3 E_3) \right. \\ - \iint_{\partial V} ds \frac{\partial G}{\partial n} \left[\frac{D_{d^+} \tilde{K}_{d^+}}{2M_{d^+}} + \left(B_1 (\tilde{\mu}_{t^+} M_{t^+} + \tilde{\mu}_{p^+} M_{p^+}) + B_2 (\tilde{\mu}_{3\text{He}} M_{3\text{He}} + \tilde{\mu}_n M_n) \right. \right. \\ \left. \left. + B_3 (\tilde{\mu}_{4\text{He}} M_{4\text{He}} + \tilde{\mu}_\gamma M_\gamma) \right) + D_{p^+} \tilde{K}_{p^+} \tilde{\mu}_{p^+} + D_{3\text{He}} \tilde{K}_{3\text{He}} \tilde{\mu}_{3\text{He}} + D_n \tilde{K}_n \tilde{\mu}_n \right. \\ \left. \left. + D_{t^+} \tilde{K}_{t^+} \tilde{\mu}_{t^+} + D_{4\text{He}} \tilde{K}_{4\text{He}} \tilde{\mu}_{4\text{He}} + D_\gamma \tilde{K}_\gamma \tilde{\mu}_\gamma \right] \right\}. \quad (4.27) \end{aligned}$$

We can introduce a total cross-section for deuteron scattering in two cases:

$$\sigma_{d+d^+} = \lambda_{(dF)} \lambda_{(dFd)} + B_F \quad (4.28)$$

for heavy fermion catalysis and

$$\sigma_{d+d^+} = \lambda_{(dS)}\lambda_{(dSd)}B_{\text{sol}} \quad (4.29)$$

in a soliton catalysis, where B_{sol} is given by the formula (4.18) and $\lambda_{(dS)}$ and $\lambda_{(dSd)}$ are probabilities of formations of a bound system soliton deuteron and deuteron soliton deuteron. The last quasi-molecule should be formed from dS and a deuteron by an electrostatic attraction. We should also remember that we can get soliton solutions also in the case of a crystal lattice (for more recent results see Ref. [70]).

Moreover, we can calculate σ_{d+d^+} in a cold plasma filled with deuterons in the following way. In the case of 3-dimensional KP-equation we get a potential between two deuterons screening by a soliton solution of KP-equation

$$V(X, Y, Z, t) = \frac{e}{4\pi\epsilon_0\epsilon} \left(\frac{1}{\sqrt{X^2 + Y^2 + Z^2}} - 3\sqrt[3]{4}n_0 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx dy dz \frac{u(x, y, z, t)}{\sqrt{(X-x)^2 + (Y-y)^2 + (Z-z)^2}} \right) \quad (4.30)$$

where $u(x, y, z, t) = U(\vec{k} \cdot \vec{r} - \omega t)$ is a one-soliton solution of KP-equations.

In two-dimensional case

$$V(x, y, t) = \frac{1}{2} a^2 \operatorname{sech}^2 \left(\frac{1}{2} a(x - by - \omega \frac{t}{a} - x_0) \right) \quad (4.31)$$

and also

$$u(x, y, t) = 4 \frac{-(x + ay + 3(a^2 - b^2)t)^2 + b^2(y + 6at)^2 + \frac{1}{b^2}}{[(x + ay + 2(a^2 - b^2)t)^2 + b^2(y + 6at)^2 + \frac{1}{b^2}]} \quad (4.32)$$

which is a so-called lump solution (also a two-dimensional solitary wave).

A transmission coefficient (a transition probability) for a deuteron (a quantum tunneling) is:

$$P(E) \cong \exp \left(-2 \int \int \int_{V_0(t)} \sqrt{\frac{2M_D}{\hbar} (V(\vec{r}, t) - E)} d^3\vec{r} \right) \quad (4.33)$$

where E is an energy of a deuteron. Moreover, $E \simeq 0$ and we get

$$\sigma_{d+d^+} = P(0) \cong \exp \left(-2\sqrt{2} \sqrt{\frac{M_D}{\hbar}} \int \int \int_{V_0(t)} \sqrt{V(\vec{r}, t)} d^3\vec{r} \right). \quad (4.34)$$

The integral is over a potential barrier ($V_0(t)$) which depends on a time. In this way we get (even in multisoliton solutions for KdV or KP equations) something which is similar to catalysis. Multisoliton solutions (n -soliton solutions or even cn-like waves) can lower a Coulomb barrier between two deuterons resulting in higher rates of low energy nuclear

reactions. These phenomena can be called a “soliton catalysis”. We have not here any analogue to muon catalysis. This analogy breaks down as we mention above. There is no analogy also to heavy fermion catalysis or even to one-soliton catalysis. Moreover, we call this possible enhancement of nuclear reaction rates a soliton catalysis. This catalysis is similar to enzymatic catalysis in biochemistry.

This seems to be a promising approach. Moreover, we should fill a sample of metal (maybe a metal foil) with a deuter. This could be done using low energy beams of deuterons. Simultaneously the metal involved here should be a hydrogen storage (a high affinity to hydrogen). We can of course try Al, C, Ta, Zr or metal oxides (as mentioned in Refs [53]–[54]).

Recently there is an interest in some exotic molecules with some different particles than electrons and nuclei. It means molecules with μ^+ particles in place of protons and also with positrons e^+ except very well known mesoatoms (see Refs [71]–[72]). Our approach to nuclear fusion is based on similar ideas. Moreover, we consider molecules with quasiparticles in solid state physics coupled to ordinary nuclei.

More advanced quantum chemistry treatment of exotic molecules appropriate for further studies can be found in Ref. [73].

Low Energy Nuclear Reactions (LENR) sometimes called Chemically Assisted Nuclear Reactions are still under investigations all around the world (see Refs [74]–[79]).

5 Example (a cylindrical cold fusion reactor)

Let us consider a sample in a shape of a cylinder with radius ρ_0 and height h (see Fig. 1). The Green function for such a set is a solution to the following equation

$$\bar{\nabla}^2 G = \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right) G = -\frac{\delta(\rho - \rho')}{\rho} \delta(\varphi - \varphi') \delta(z - z') \quad (5.1)$$

with boundary conditions

$$G = 0 \text{ for } \rho = \rho_0, 0 \leq \varphi \leq 2\pi, 0 \leq z \leq h \text{ and } z = 0, 0 \leq \rho \leq \rho_0, 0 \leq \varphi < 2\pi, \\ z = h, 0 \leq \rho \leq \rho_0, 0 \leq \varphi < 2\pi, \quad (5.2)$$

where ρ , φ and z are ordinary cylindrical coordinates. δ is the one-dimensional Dirac delta distribution.

One can find the following solution to this Dirichlet problem (see Ref. [80]).

$$G(\rho, \rho', \varphi, \varphi', z, z') \\ = \frac{1}{\pi \rho_0^2} \sum_{m=-\infty}^{m=+\infty} \sum_{n=1}^{\infty} \frac{J_m\left(\frac{x_{mn}\rho}{\rho_0}\right) J_m\left(\frac{x_{mn}\rho'}{\rho_0}\right) (z + z' - |z - z'|) \cos(m(\varphi - \varphi'))}{J_{m+1}^2(x_{mn})}. \quad (5.3)$$

J_m is the Bessel function of the m -th order and x_{mn} is the n -th root of the Bessel function of the m -th order.

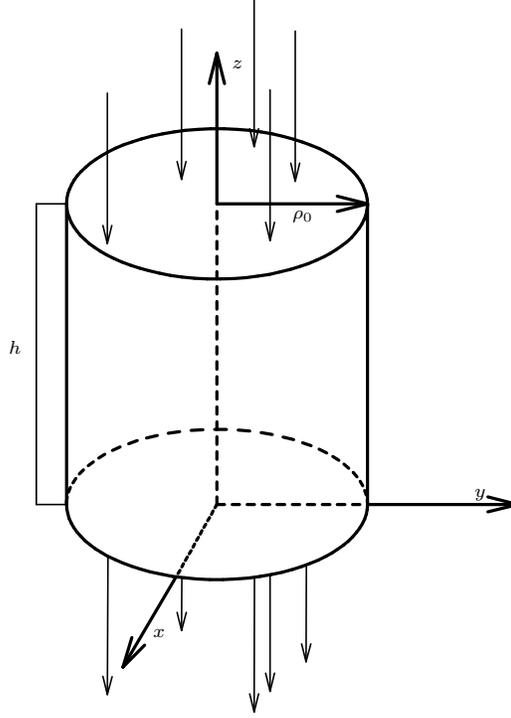


Fig. 1. A cylindrical sample

One gets

$$\iiint_V G dv = 0 \quad (5.4)$$

and

$$\begin{aligned} C_i = \iint_{\partial V} \frac{\partial G}{\partial n} \tilde{\mu}_i ds &= \frac{z}{\pi \rho_0^2} \sum_{m=-\infty}^{m=+\infty} \sum_{n=1}^{\infty} \frac{J_m\left(\frac{x_{mn}\rho}{\rho_0}\right)}{J_{m+1}^2(x_{mn})} \int_0^{2\pi} d\varphi' \int_0^{\rho_0} \rho' d\rho' \\ &\quad \times J_m\left(\frac{x_{mn}\rho'}{\rho_0}\right) \cos(m(\varphi - \varphi')) (\tilde{C}_i|_{z'=0} - \tilde{C}_i|_{z'=h}) \\ &+ \frac{1}{\pi \rho_0} \sum_{m=-\infty}^{m=+\infty} \sum_{n=1}^{\infty} \frac{dJ_m}{dx}(x_{mn}) \frac{J_m\left(\frac{x_{mn}\rho}{\rho_0}\right)}{J_{m+1}^2(x_{mn})} \int_0^{2\pi} d\varphi' \int_0^h dz' \\ &\quad \times (z + z' - |z - z'|) \cos(m(\varphi - \varphi')) \tilde{C}_i|_{\rho'=\rho_0}. \end{aligned} \quad (5.5)$$

$$\begin{aligned} \mu_i = \iint_{\partial V} \frac{\partial G}{\partial n} \tilde{c}_i ds &= \frac{z}{\pi \rho_0^2} \sum_{m=-\infty}^{m=+\infty} \sum_{n=1}^{\infty} \frac{J_m\left(\frac{x_{mn}\rho}{\rho_0}\right)}{J_{m+1}^2(x_{mn})} \int_0^{2\pi} d\varphi' \int_0^{\rho_0} \rho' d\rho' \\ &\quad \times J_m\left(\frac{x_{mn}\rho'}{\rho_0}\right) \cos(m(\varphi - \varphi')) (\tilde{\mu}_i|_{z'=0} - \tilde{\mu}_i|_{z'=h}) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\pi \rho_0} \sum_{m=-\infty}^{m=+\infty} \sum_{n=1}^{\infty} \frac{dJ_m}{dx}(x_{mn}) \frac{J_m\left(\frac{x_{mn}\rho}{\rho_0}\right)}{J_{m+1}^2(x_{mn})} \int_0^{2\pi} d\varphi' \int_0^h dz' \\
& \quad \times (z + z' - |z - z'|) \cos(m(\varphi - \varphi')) \tilde{\mu}_i|_{\rho'=\rho_0}. \quad (5.6)
\end{aligned}$$

Let us notice the following fact: the Green function derived here is continuous with respect to every variable, $\rho, \rho', \varphi, \varphi', z, z'$. Moreover, it has a singularity if $\rho \rightarrow \rho', \varphi \rightarrow \varphi', z \rightarrow z'$. The function has the first and second derivative with respect to $\rho, \rho', \varphi, \varphi'$. However it is not differentiable with respect to z and z' and $G(\rho, \rho', \varphi, \varphi', z, z') = G(\rho, \rho', \varphi, \varphi', z', z)$. This means the function is a generalized solution of the differential equation and the derivatives with respect to z and z' should be considered in the distributional sense.

In this way we get Eqs (3.62)–(3.64), (3.77)–(3.78), (4.26)–(4.27) for a cylindrical sample substituting Eq. (5.6). Let us notice that formulas (5.5)–(5.6) have been derived for ∂V consisting of three pieces for $z = 0$ or h , $0 \leq \rho \leq \rho_0$, $\rho = \rho_0$, $0 \leq z \leq h$, $\tilde{C}_i|_{z'=0}$, $\tilde{C}_i|_{z'=h}$, $\tilde{C}_i|_{\rho=\rho_0}$ are boundary (Dirichlet) conditions on these pieces and should be consistent on borders between them, i.e.

$$\begin{aligned}
\tilde{C}_i|_{z'=0} &= \tilde{C}_i^1(\rho', \varphi') \\
\tilde{C}_i|_{z'=h} &= \tilde{C}_i^2(\rho', \varphi') \quad (5.7)
\end{aligned}$$

$$\begin{aligned}
\tilde{C}_i|_{\rho'=\rho_0} &= \tilde{C}_i^3(\varphi', z'). \\
\tilde{C}_i^1(\rho_0, \varphi') &= \tilde{C}_i^3(\varphi', 0), \quad \varphi' \in \langle 0, 2\pi \rangle \\
\tilde{C}_i^2(\rho_0, \varphi') &= \tilde{C}_i^3(\varphi', h), \quad \varphi' \in \langle 0, 2\pi \rangle. \quad (5.8)
\end{aligned}$$

The same we have for $\tilde{\mu}_i$:

$$\begin{aligned}
\tilde{\mu}_i|_{z'=0} &= \tilde{\mu}_i^1(\rho', \varphi') \\
\tilde{\mu}_i|_{z'=h} &= \tilde{\mu}_i^2(\rho', \varphi') \\
\tilde{\mu}_i|_{\rho'=\rho_0} &= \tilde{\mu}_i^3(\varphi', z') \quad (5.9) \\
\tilde{\mu}_i^1(\rho_0, \varphi') &= \tilde{\mu}_i^3(\varphi', 0), \quad \varphi' \in \langle 0, 2\pi \rangle \\
\tilde{\mu}_i^2(\rho_0, \varphi') &= \tilde{\mu}_i^3(\varphi', h), \quad \varphi' \in \langle 0, 2\pi \rangle,
\end{aligned}$$

where $i = p^+, d^+, t^+, {}^3\text{He}, {}^4\text{He}, n, \gamma$.

We rewrite Eqs (3.74) in cylindrical coordinates

$$\begin{aligned}
P \frac{\partial u}{\partial P} \left(\frac{\partial R}{\partial \rho} \frac{\partial P}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial R}{\partial \varphi} \frac{\partial P}{\partial \varphi} + \frac{\partial R}{\partial z} \frac{\partial P}{\partial z} \right) &= \tilde{f}(\rho, \varphi, z) \\
\frac{\partial R}{\partial \rho} \frac{\partial P}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial R}{\partial \varphi} \frac{\partial P}{\partial \varphi} + \frac{\partial R}{\partial z} \frac{\partial P}{\partial z} + \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right) R &= 0 \quad (5.10)
\end{aligned}$$

where $\tilde{f}(\rho, \varphi, z)$ is given by Eq. (3.68) in cylindrical coordinates and φ in Eq. (3.68) should be changed to φ_1 in order to avoid misunderstanding. Let us notice a change in notation. Now the density is denoted by P (capital Greek ρ).

$$\begin{aligned}
& + D_{p^+} \tilde{K}_{p^+} \left(\tilde{\mu}_{p^+}^1(\rho', \varphi') - \tilde{\mu}_{p^+}^2(\rho', \varphi') \right) + D_{\text{He}} \tilde{K}_{\text{He}} \left(\tilde{\mu}_{\text{He}}^1(\rho', \varphi') - \tilde{\mu}_{\text{He}}^2(\rho', \varphi') \right) \\
& \quad + D_n \tilde{K}_n \left(\tilde{\mu}_n^1(\rho', \varphi') - \tilde{\mu}_n^2(\rho', \varphi') \right) + D_{t^+} \tilde{K}_{t^+} \left(\tilde{\mu}_{t^+}^1(\rho', \varphi') - \tilde{\mu}_{t^+}^2(\rho', \varphi') \right) \\
& \quad + D_{\text{He}} \tilde{K}_{\text{He}} \left(\tilde{\mu}_{\text{He}}^1(\rho', \varphi') - \tilde{\mu}_{\text{He}}^2(\rho', \varphi') \right) + D_\gamma \tilde{K}_\gamma \left(\tilde{\mu}_\gamma^1(\rho', \varphi') - \tilde{\mu}_\gamma^2(\rho', \varphi') \right) \Big] \\
& + \frac{dJ_m}{dx}(x_{mn}) \frac{J_m\left(\frac{x_{mn}\rho}{\rho_0}\right)}{J_{m+1}^2(x_{mn})} \int_0^{2\pi} d\varphi' \int_0^h dz' (z+z'-|z-z'|) \cos(m(\varphi-\varphi')) \left[\frac{D_{d^+} \tilde{K}_{d^+}}{2M_{d^+}} \right. \\
& \quad \left(B_1 \left(\tilde{\mu}_{t^+}^3(\varphi', z') M_{t^+} + \tilde{\mu}_{p^+}^3(\varphi', z') M_{p^+} \right) + B_2 \left(\tilde{\mu}_{\text{He}}^3(\varphi', z') M_{\text{He}} + \tilde{\mu}_n^3(\varphi', z') M_n \right) \right. \\
& \quad \quad \left. + B_3 \left(\tilde{\mu}_{\text{He}}^3(\varphi', z') M_{\text{He}} \right) \right. \\
& \quad \left. + D_{p^+} \tilde{K}_{p^+} \tilde{\mu}_{p^+}^3(\varphi', z') + D_{\text{He}} \tilde{K}_{\text{He}} \tilde{\mu}_{\text{He}}^3(\varphi', z') + D_n \tilde{K}_n \tilde{\mu}_n^3(\varphi', z') \right. \\
& \quad \left. + D_{t^+} \tilde{K}_{t^+} \tilde{\mu}_{t^+}^3(\varphi', z') + D_{\text{He}} \tilde{K}_{\text{He}} \tilde{\mu}_{\text{He}}^3(\varphi', z') + D_\gamma \tilde{K}_\gamma \tilde{\mu}_\gamma^3(\varphi', z') \right] \Bigg\} \Bigg\} \Bigg\} \quad (5.12)
\end{aligned}$$

To complete an example we write

$$\vec{v} = \vec{\nabla} R = \left(\frac{\partial R}{\partial \rho} \vec{e}_\rho + \frac{1}{\rho} \frac{\partial R}{\partial \varphi} \vec{e}_\varphi + \frac{\partial R}{\partial z} \vec{e}_z \right). \quad (5.13)$$

(\vec{e}_ρ , \vec{e}_φ and \vec{e}_z are coordinate line versors) and boundary conditions for R and P (a density)

$$\tilde{R}|_{z=0}, \tilde{R}|_{z=h}, \tilde{R}|_{\rho=\rho_0}, \tilde{P}|_{z=0}, \tilde{P}|_{z=h}, \tilde{P}|_{\rho=\rho_0}. \quad (5.14)$$

One gets

$$\begin{aligned}
\tilde{R}|_{z=0} &= \tilde{R}^1(\rho, \varphi), \quad 0 \leq \varphi < 2\pi, \quad 0 \leq \rho \leq \rho_0, \\
\tilde{R}|_{z=h} &= \tilde{R}^2(\rho, \varphi), \\
\tilde{P}|_{z=0} &= \tilde{P}^1(\rho, \varphi), \\
\tilde{P}|_{z=h} &= \tilde{P}^2(\rho, \varphi), \\
\tilde{R}|_{\rho=\rho_0} &= \tilde{R}^3(\varphi, z), \quad 0 \leq \varphi < 2\pi, \quad 0 \leq z \leq h, \\
\tilde{P}|_{\rho=\rho_0} &= \tilde{P}^3(\varphi, z)
\end{aligned} \quad (5.15)$$

and consistency conditions

$$\begin{aligned}
\tilde{R}^1(\rho_0, \varphi) &= \tilde{R}^3(\varphi, 0), \quad 0 \leq \varphi < 2\pi, \\
\tilde{R}^2(\rho_0, \varphi) &= \tilde{R}^3(\varphi, h), \\
\tilde{P}^1(\rho_0, \varphi) &= \tilde{P}^3(\varphi, 0), \quad 0 \leq \varphi < 2\pi, \\
\tilde{P}^2(\rho_0, \varphi) &= \tilde{P}^3(\varphi, h).
\end{aligned} \quad (5.16)$$

We should also derive a function

$$\tilde{f}(\vec{r}) = \tilde{f}(\rho, \varphi, z) \quad (5.17)$$

getting the result

$$\begin{aligned}
\tilde{f}(\rho, \varphi, z) = & -\varphi_1 \frac{1}{\pi\rho_0} \sum_{m=-\infty}^{m=+\infty} \sum_{n=1}^{\infty} \\
& \left\{ \frac{J_m\left(\frac{x_{mn}\rho}{\rho_0}\right)}{J_{m+1}^2(x_{mn})} \left(\frac{z}{\rho_0}\right) \int_0^{2\pi} d\varphi' \int_0^{\rho_0} \rho' d\rho' J_m\left(\frac{x_{mn}\rho'}{\rho_0}\right) \cos(m(\varphi - \varphi')) \left[\frac{D_{d+}\tilde{K}_{d+}}{2M_{d+}} \right. \right. \\
& \times \left(B_1 \left((\tilde{\mu}_{t+}^1(\rho', \varphi') - \tilde{\mu}_{t+}^2(\rho', \varphi')) M_{t+} + (\tilde{\mu}_{p+}^1(\rho', \varphi') - \tilde{\mu}_{p+}^2(\rho', \varphi')) M_{p+} \right) \right. \\
& + B_2 \left((\tilde{\mu}_{3\text{He}}^1(\rho', \varphi') - \tilde{\mu}_{3\text{He}}^2(\rho', \varphi')) M_{3\text{He}} + (\tilde{\mu}_n^1(\rho', \varphi') - \tilde{\mu}_n^2(\rho', \varphi')) M_n \right) \\
& + B_3 \left((\tilde{\mu}_{4\text{He}}^1(\rho', \varphi') - \tilde{\mu}_{4\text{He}}^2(\rho', \varphi')) M_{4\text{He}} + (\tilde{\mu}_\gamma^1(\rho', \varphi') - \tilde{\mu}_\gamma^2(\rho', \varphi')) M_\gamma \right) \\
& + D_{p+}\tilde{K}_{p+} \left(\tilde{\mu}_{p+}^1(\rho', \varphi') - \tilde{\mu}_{p+}^2(\rho', \varphi') \right) + D_{3\text{He}}\tilde{K}_{3\text{He}} \left(\tilde{\mu}_{3\text{He}}^1(\rho', \varphi') - \tilde{\mu}_{3\text{He}}^2(\rho', \varphi') \right) \\
& + D_n\tilde{K}_n \left(\tilde{\mu}_n^1(\rho', \varphi') - \tilde{\mu}_n^2(\rho', \varphi') \right) + D_{t+}\tilde{K}_{t+} \left(\tilde{\mu}_{t+}^1(\rho', \varphi') - \tilde{\mu}_{t+}^2(\rho', \varphi') \right) \\
& \left. \left. + D_{4\text{He}}\tilde{K}_{4\text{He}} \left(\tilde{\mu}_{4\text{He}}^1(\rho', \varphi') - \tilde{\mu}_{4\text{He}}^2(\rho', \varphi') \right) + D_\gamma\tilde{K}_\gamma \left(\tilde{\mu}_\gamma^1(\rho', \varphi') - \tilde{\mu}_\gamma^2(\rho', \varphi') \right) \right] \right\} \\
& + \frac{dJ_m}{dx}(x_{mn}) \frac{J_m\left(\frac{x_{mn}\rho}{\rho_0}\right)}{J_{m+1}^2(x_{mn})} \int_0^{2\pi} d\varphi' \int_0^h dz' (z + z' - |z - z'|) \cos(m(\varphi - \varphi')) \left[\frac{D_{d+}\tilde{K}_{d+}}{2M_{d+}} \right. \\
& \left(B_1 \left(\tilde{\mu}_{t+}^3(\varphi', z') M_{t+} + \tilde{\mu}_{p+}^3(\varphi', z') M_{p+} \right) + B_2 \left(\tilde{\mu}_{3\text{He}}^3(\varphi', z') M_{3\text{He}} + \tilde{\mu}_n^3(\varphi', z') M_n \right) \right. \\
& \left. \left. + B_3 \left(\tilde{\mu}_{4\text{He}}^3(\varphi', z') M_{4\text{He}} \right) \right. \right. \\
& + D_{p+}\tilde{K}_{p+}\tilde{\mu}_{p+}^3(\varphi', z') + D_{3\text{He}}\tilde{K}_{3\text{He}}\tilde{\mu}_{3\text{He}}^3(\varphi', z') + D_n\tilde{K}_n\tilde{\mu}_n^3(\varphi', z') \\
& \left. \left. + D_{t+}\tilde{K}_{t+}\tilde{\mu}_{t+}^3(\varphi', z') + D_{4\text{He}}\tilde{K}_{4\text{He}}\tilde{\mu}_{4\text{He}}^3(\varphi', z') + D_\gamma\tilde{K}_\gamma\tilde{\mu}_g^3(\varphi', z') \right] \right\} \quad (5.18)
\end{aligned}$$

6 Conclusions and remarks

An effective interaction between hydrogen atoms (deuterium atoms) in a solid can be intuitively described as quasichemical bonds in quasimolecules. These quasimolecules involve quasiparticles and ordinary particles. In particular, heavy fermions in solids and hydrogen or deuterium nuclei. Such an approach is similar to muon catalysis in cold fusion and we can use some intuition from the approach. The heavy fermion catalysis paradigm can help us to find some materials where such a cold fusion is very probable and to check them for further investigations. This is the aim of the paper. Simultaneously it is a chemical topic and a mathematical chemistry topic.

To be honest, we add also kinetic equations of our Low Energy Nuclear Reactions. They are called reaction-diffusion equations and they are nonlinear partial differential

equations. They look as follows.

$$\frac{\partial C_{t^+}}{\partial t} = B_1 \sigma_{d+d^+} C_{d^+}^2 + D_{t^+} \vec{\nabla}^2 C_{t^+} \quad (6.1)$$

$$\frac{\partial C_{p^+}}{\partial t} = B_1 \sigma_{d+d^+} C_{d^+}^2 + D_{p^+} \vec{\nabla}^2 C_{p^+}, \quad (6.2)$$

$$\frac{\partial C_{{}^3\text{He}}}{\partial t} = B_2 \sigma_{d+d^+} C_{d^+}^2 + D_{{}^3\text{He}} \vec{\nabla}^2 C_{{}^3\text{He}} \quad (6.3)$$

$$\frac{\partial C_n}{\partial t} = B_2 \sigma_{d+d^+} C_{d^+}^2 + D_n \vec{\nabla}^2 C_n \quad (6.4)$$

$$\frac{\partial C_{{}^4\text{He}}}{\partial t} = B_3 \sigma_{d+d^+} C_{d^+}^2 + D_{{}^4\text{He}} \vec{\nabla}^2 C_{{}^4\text{He}} \quad (6.5)$$

$$\frac{\partial C_\gamma}{\partial t} = B_3 \sigma_{d+d^+} C_{d^+}^2 + D_\gamma \vec{\nabla}^2 C_\gamma \quad (6.6)$$

$C_i = C_i(x, y, z, t)$, $i = {}^4\text{He}, {}^3\text{He}, \gamma, n, t^+, p^+$, σ_{d+d^+} is given by formulas (4.28) (heavy fermion catalysis), (4.29) (one soliton catalysis), and (4.34) (n -soliton or cn-like waves catalysis). Eqs (6.1)–(6.6) are master equations for our cold fusion. Moreover, in our way to get reactor equation for an energy production it is better to use our approach from Section 4. This approach treats nuclear reactions as chemical reactions.

Summing up. In the paper we consider three types of a possible catalysis for a cold fusion ($d + d$ or $d + t$). They are: 1) heavy fermion (heavy electron) catalysis in a solid state sample filled with deuterons, 2) 1-soliton catalysis in an electron plasma of a metal filled with a deuter (KdV or KP equations, 1-soliton solutions), 3) soliton catalysis in an electron plasma of metal filled with a deuter (KdV or KP equations — n -soliton solutions or cn-like waves). The difference between 2) and 3) is such that in the case of 2) we follow an analogy of 1), i.e. we treat a soliton as a quasiparticle in a quasi-chemical bond. But in case 3) we consider n -soliton solutions or cn-waves as configurations of negative charge lowering a Coulomb barrier between two deuterons.

Let us notice the following fact. According to the S. Arrhenius theory of chemical reactions it is necessary for reacting molecules (or atoms) to get a sufficient kinetic energy to go over an energetic barrier. A catalysator (or an enzyme) can lower this barrier. In our case of a soliton catalysis a Coulomb barrier can be lowered by a soliton. Moreover, an energy of deuteron is very low and we need a tunnel effect to complete a reaction. This can be said also for a heavy fermion catalysis.

We consider also thermodynamical properties of a fusion reactor. In particular, we consider in details a cylindrical fusion reactor.

Let us give the following remark. Quasiparticles approach in condensed matter physics is now very popular. We have to do with some kind of “standard model” of quasiparticles. Except heavy electrons, light holes and heavy holes and composed quasiparticles as exciton, charged excitons, trions, Cooper pairs we have to do also with dressed and double dressed quasiparticles. We have to do with magnetic monopoles, skyrmions (see Refs [81]–[82]) and Majorana fermions in 2D systems. We have also the so-called spin-charge separation (see Refs [83]–[84]) and spin-orbital separation (see

Ref. [85]). Due to these effects we have new quasiparticles: spinos, holons, orbitons. All of them can (in principle) form bound systems with two deuterons going to additional catalysis of Low Energy Nuclear Reactions (LENR) and cold fusion. We can also consider as quasiparticles exact solutions of nonlinear PDE of unified field theory of electromagnetic and gravitational interactions (see Refs [86]–[87]). They are some kind of “interference effects” of electromagnetic and gravitational interaction being particle-like solutions. Moreover, a strict connection to a cold fusion problem is not known.

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