SMALLEST SYMMETRIC SUPERGROUPS OF THE ABSTRACT GROUPS UP TO ORDER 37

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Abstract. Each finite group is a subgroup of some symmetric group, known as the Cayley theorem. We find the symmetric group of smallest order which hosts the finite groups in that sense for most groups of order less than 37. For each of these small groups this is made concrete by providing a permutation group with a minimum number of moved elements in terms of a list of generators of the permutation group in reduced cycle notation.

1. Introduction

1.1. **Cayley's Theorem.** The finite groups are fundamentally defined by the multiplication table (Cayley table) of their elements g_j. We shall unify the notation by sticking to table and the enumeration of group elements as explicitly proposed by the Small Groups Table in the GAP library [5, 1]. The ith group of order o is denoted by G_o^i. Its elements are g_j, $1 \le j \le o$ where g_1 is reserved for the unit element.

Remark 1. *The maximum upper index i is the number of groups of order o, 1,1,2,1,2,1,5,2,...for $o \ge 1$, see Sequence A000001 in the Online Encyclopedia of Integer Sequences* [11].

Every group is a subgroup of a symmetric group, a fact known as Cayley's theorem [2, Corol. 2.4]. The simplest construction of such a symmetric supergroup is the representation of the elements by their standard representation [4, Ex. 1.3.3].

Remark 2. *The standard representation is a representation by $o\times o$ binary matrices with elements D_{ij} for $1 \le i, j \le o$, with $D_{ij}(g_k) = 1$ if $g_k g_j = g_i$, zero otherwise. It is a permutation representation because each matrix has one 1 in every row and column.*

1.2. **Aim.** The standard representation embeds G_o^i into the symmetric group S_n with an index $n = o$. The question arises, what the smallest index n could be such that a subgroup of S_n is (isomorphic to) G_o^i? Can n be made smaller than o? The manuscript answers this question quantitatively for most groups of order $o \le 37$ by constructing permutation groups (and therefore subgroups of the symmetric group S_n) with the smallest number n of moved elements.

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1.3. **Bounds on the Degree.** By Lagrange's theorem the order of a subgroup divides the order of the group. Since we are seeking symmetric super-groups of index n (which have order $n!$) for groups of order o, there is an immediate lower bound on n given a o, which can be read off Sequence A002034 [11].

If G_o^i is isomorphic to a symmetric group, we cannot find a $n!$ that is smaller than o. This occurs for one index i if the order o is a factorial, for $G_1^1 \cong S_1$, $G_2^1 \cong S_2$, $G_6^1 \cong S_3$, $G_{24}^{12} \cong S_4$ and so on.

When $G_o^i = G_{o'}^{i'} \times G_{o''}^{i''}$ is a direct product of two other groups, an upper bound of the n hosting G_o^i is given by the sum of the two individual n's of $G_{o'}^{i'}$ and $G_{o''}^{i''}$.

Proof. Matrix representations of G_o^i with the sum dimensions are constructed by the direct sum, placing the representation of the element of $G_{o'}^{i'}$ in the left upper and the representation of the element of $G_{o''}^{i''}$ in the right lower block of the product representation, and zeros at all remaining places. This is still a permutation representation because in each row and each column exactly one 1 appears, and the block structure of the matrices ensures that the products in G_o^i are represented by matrix products that preserve the products in the two subspaces. Obviously the degree of this representation is the sum of the degrees of the permutation representations in the factors. □

1.4. **Overview.** There is one type of groups for which the minimum n of the embedding can be found easily, which is discussed in Section 2. The general results have been found by a brute force matching program (written in Java where a auxiliary GAP program is used as a generator for some of the Java functions). The output of this program is reproduced in Section 3 and defines for each of the small groups G_o^i a permutation group acting on n elements which is isomorphic to G_o^i and has minimum n.

2. The Cyclic Groups

The close relation between the generating element of a cyclic group, its order and the order of a generating permutation in its cycle notation leads to an immediate solution of our minimization problem for all cyclic groups C_o. This handles the cases $G_2^1 \cong C_2$, $G_3^1 \cong C_3$, $G_4^1 \cong C_4$, $G_5^1 \cong C_5$, $G_6^2 \cong C_6$, $G_7^1 \cong C_7$, $G_8^1 \cong C_8$, $G_9^1 \cong C_9$, $G_{10}^2 \cong C_{10}$, $G_{11}^1 \cong C_{11}$, $G_{12}^2 \cong C_{12}$, $G_{13}^1 \cong C_{13}$, $G_{14}^2 \cong C_{14}$, $G_{15}^1 \cong C_{15}$, $G_{16}^1 \cong C_{16}$, $G_{17}^1 \cong C_{17}$ and so on [10]. This happens for one i for each o, in takes care of all groups of prime order.

Let the cycle length (order of the generating element g and number of vertices in the cycle be c. The cycle is a cyclic subgroup C_c of the group, comprising the unit element and g, g^2, $\ldots g^{c-1}$. A maximum order N of the embedding S_n follows for given c as follows: The permutation representing g has a mix of cycles; the lengths l_i of the cycles is some partition of n (of lengths larger than one because we want to avoid fixed elements to minimize n). The order of the element is the least common multiple of the lengths [4][12, Exercise 1.2]:

$$l_1 + l_2 + \cdots = n; \tag{1}$$

$$lcm(l_1, l_1, \ldots) = c; \tag{2}$$

Theorem 1. *The cycle structure that minimizes n for a given c with prime factorization $c = \prod_{i\geq 1} p_i^{e_i}$ is $l_i = p_i^{e_i}$ or any permutation of this list of cycle lengths.*

Proof. because $c_i = p_i^{e_i}$, the requirement on the least common multiple is obviously satisfied. We need to show that the l_i should be coprime to minimize n and should have different prime bases. We first see that if a cycle length pair (l_i, l_j) had a common prime factor p, we could divide one of both which has this prime factor with a lower or the same prime factor as the other through this prime factor, which would generate the same least common multiple but reduce one of the terms in (1) by the factor p, leading to a smaller n. Also, if one l_i would not be just a prime power but a product of prime powers with different bases, splitting these prime powers into two different l_i' and l_i'' terms would maintain the least common multiple, but lead to a smaller n because the sum of the prime powers would be smaller than the product of the prime powers:

$$p_1^{e_1} + p_2^{e_2} \le p_1^{e_1} p_2^{e_2} \tag{3}$$

is by division through the positive right hand side equivalent to

$$1/p_2^{e_2} + 1/p_1^{e_1} \le 1 \tag{4}$$

which is correct because one of the two terms on the left hand side is $\le 1/2$ and the other $\le 1/3$. □

Remark 3. *There are $\varphi(c)$ choices for the generator (i.e. element of order c) in the cyclic group C_c, where φ is Euler's totient function. The other elements have orders that are divisors of c and the order of the element g^k is in the k-th column of row c in A054531* [11]*. The frequency of elements of given order is detailed in Sequence* [11, A054522]*.*

This essentially completes the task of finding permutation representations of lowest n for the cyclic groups. The minimum n can be read off sequence [11, A008475] as a function of o in these cases.

3. Results

The results are represented as a list of generators in permutation form for most of the small groups G_o^i, $o \le 37$. The groups are separated by blank lines.

The first line in a group representation provides the order o of the group, its index i in the group library and after an n the index of the symmetric supergroup S_n. For groups where the results are not (yet) available (17 out of 161), a lower bound of n is printed followed by a line which says `missing`.

The minimum degree of this permutation group (defined as the minimum number of points moved for any non-identity element among the o permutations) is also printed after `deg` for comparison with earlier work [3, 8, 7].

This line is followed by one or more lines of generators—which are not necessarily a smallest set of generators—sufficient to generate the entire group as products of generators [9, 6]. A generator specification starts with a `g`, then the index of the element in the small group library (starting from `g1` for the unit element), and the mapping by a permutation in reduced cycle notation that represents the group element. By construction, the largest moved element in the set of generators equals n.

```
Order 2 Id 1 n 2
deg 2 at g2 (1 2)
g2 (1 2)
```

```
Order 3 Id 1 n 3
deg 3 at g2 (1 2 3)
g2 (1 2 3)

Order 4 Id 1 n 4
deg 4 at g2 (1 2 3 4)
g2 (1 2 3 4)

Order 4 Id 2 n 4
deg 2 at g2 (3 4)
g2 (3 4)
g3 (1 2)

Order 5 Id 1 n 5
deg 5 at g2 (1 2 3 4 5)
g2 (1 2 3 4 5)

Order 6 Id 1 n 3
deg 2 at g2 (2 3)
g2 (2 3)
g3 (1 2 3)

Order 6 Id 2 n 5
deg 2 at g2 (1 2)
g4 (1 2)(3 4 5)

Order 7 Id 1 n 7
deg 7 at g2 (1 2 3 4 5 6 7)
g2 (1 2 3 4 5 6 7)

Order 8 Id 1 n 8
deg 8 at g2 (1 2 3 4 5 6 7 8)
g2 (1 2 3 4 5 6 7 8)

Order 8 Id 2 n 6
deg 2 at g3 (5 6)
g3 (5 6)
g2 (1 2 3 4)

Order 8 Id 3 n 4
deg 2 at g2 (3 4)
g2 (3 4)
g3 (1 3)(2 4)

Order 8 Id 4 n 8
deg 8 at g2 (1 2 3 4)(5 6 7 8)
g2 (1 2 3 4)(5 6 7 8)
g3 (1 5 3 7)(2 8 4 6)

Order 8 Id 5 n 6
deg 2 at g2 (5 6)
g2 (5 6)
g3 (3 4)
```

```
g4 (1 2)

Order 9 Id 1 n 9
deg 9 at g2 (1 2 3 4 5 6 7 8 9)
g2 (1 2 3 4 5 6 7 8 9)

Order 9 Id 2 n 6
deg 3 at g2 (4 5 6)
g2 (4 5 6)
g3 (1 2 3)

Order 10 Id 1 n 5
deg 4 at g2 (2 3)(4 5)
g2 (2 3)(4 5)
g4 (1 4)(3 5)

Order 10 Id 2 n 7
deg 2 at g2 (1 2)
g4 (1 2)(3 4 5 6 7)

Order 11 Id 1 n 11
deg 11 at g2 (1 2 3 4 5 6 7 8 9 10 11)
g2 (1 2 3 4 5 6 7 8 9 10 11)

Order 12 Id 1 n 7
deg 3 at g4 (5 6 7)
g4 (5 6 7)
g2 (1 2 3 4)(6 7)

Order 12 Id 2 n 7
deg 3 at g3 (1 2 3)
g5 (1 2 3)(4 5 6 7)

Order 12 Id 3 n 4
deg 3 at g2 (2 3 4)
g3 (1 2)(3 4)
g2 (2 3 4)

Order 12 Id 4 n 5
deg 2 at g2 (4 5)
g2 (4 5)
g9 (1 2)(3 4)

Order 12 Id 5 n 7
deg 2 at g2 (6 7)
g2 (6 7)
g7 (1 2)(3 4 5)

Order 14 Id 1 n 7
deg 6 at g2 (2 3)(4 5)(6 7)
g2 (2 3)(4 5)(6 7)
g4 (1 4)(3 6)(5 7)
```

```
Order 14 Id 2 n 9
deg 2 at g2 (1 2)
g4 (1 2)(3 4 5 6 7 8 9)

Order 15 Id 1 n 8
deg 3 at g2 (1 2 3)
g5 (1 2 3)(4 5 6 7 8)

Order 16 Id 1 n 16
deg 16 at g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16)
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16)

Order 16 Id 2 n 8
deg 4 at g2 (5 6 7 8)
g2 (5 6 7 8)
g3 (1 2 3 4)

Order 16 Id 3 n 8
deg 2 at g3 (7 8)
g3 (7 8)
g2 (1 7)(2 8)(3 4 5 6)

Order 16 Id 4 n 8
deg 4 at g3 (1 3 2 4)
g2 (3 4)(5 6 7 8)
g3 (1 3 2 4)

Order 16 Id 5 n 10
deg 2 at g3 (9 10)
g3 (9 10)
g2 (1 2 3 4 5 6 7 8)

Order 16 Id 6 n 8
deg 4 at g3 (5 6)(7 8)
g3 (5 6)(7 8)
g2 (1 5 2 7 3 6 4 8)

Order 16 Id 7 n 8
deg 6 at g2 (3 4)(5 6)(7 8)
g2 (3 4)(5 6)(7 8)
g3 (1 3)(2 5)(4 7)(6 8)

Order 16 Id 8 n 8
deg 6 at g3 (3 4)(5 6)(7 8)
g3 (3 4)(5 6)(7 8)
g2 (1 3 2 5)(4 7 6 8)

Order 16 Id 9 (n>=15)
missing

Order 16 Id 10 n 8
deg 2 at g3 (7 8)
g3 (7 8)
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g4 (5 6)
g2 (1 2 3 4)

Order 16 Id 11 n 6
deg 2 at g2 (5 6)
g2 (5 6)
g3 (3 5)(4 6)
g4 (1 2)

Order 16 Id 12 n 10
deg 2 at g4 (1 2)
g2 (3 4 5 6)(7 8 9 10)
g3 (3 7 5 9)(4 10 6 8)
g4 (1 2)

Order 16 Id 13 n 8
deg 4 at g2 (5 6)(7 8)
g2 (5 6)(7 8)
g3 (1 5)(2 6)(3 7)(4 8)
g12 (1 7)(2 8)(3 6)(4 5)

Order 16 Id 14 n 8
deg 2 at g2 (7 8)
g2 (7 8)
g3 (5 6)
g4 (3 4)
g5 (1 2)

Order 17 Id 1 n 17
deg 17
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17)

Order 18 Id 1 n 9
deg 8 at g2 (2 3)(4 5)(6 7)(8 9)
g2 (2 3)(4 5)(6 7)(8 9)
g5 (1 4)(3 6)(5 8)(7 9)

Order 18 Id 2 n 9
deg 2 at g2 (1 2)
g5 (1 2)(3 4 5 6 7 8 9 10 11)

Order 18 Id 3 n 6
deg 2 at g2 (5 6)
g2 (5 6)
g8 (1 2 3)(4 5 6)

Order 18 Id 4 n 6
deg 3 at g3 (2 3 4)
g2 (3 4)(5 6)
g5 (2 3)(5 6)
g6 (1 5)(3 4)

Order 18 Id 5 n 8
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deg 2 at g2 (1 2)
g3 (6 7 8)
g6 (1 2)(3 4 5)

Order 19 Id 1 n 19
deg 19 at g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19)
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19)

Order 20 Id 1 n 9
deg 4 at g3 (1 3)(2 4)
g4 (5 6 7 8 9)
g2 (1 2 3 4)(6 9)(7 8)

Order 20 Id 2 n 9
deg 4 at g2 (1 2 3 4)
g5 (1 2 3 4)(5 6 7 8 9)

Order 20 Id 3 n 5
deg 4 at g2 (2 4 3 5)
g3 (2 3)(4 5)
g6 (1 4 2 3)

Order 20 Id 4 n 7
deg 2 at g3 (1 2)
g2 (4 5)(6 7)
g9 (1 2)(3 6)(5 7)

Order 20 Id 5 n 9
deg 2 at g2 (8 9)
g2 (8 9)
g7 (1 2)(3 4 5 6 7)

Order 21 Id 1 n 7
deg 6 at g2 (2 3 4)(5 6 7)
g2 (2 3 4)(5 6 7)
g5 (1 5 2)(4 7 6)

Order 21 Id 2 n 10
deg 3 at g2 (1 2 3)
g5 (1 2 3)(4 5 6 7 8 9 10)

Order 22 Id 1 n 11
deg 10 at g2 (2 3)(4 5)(6 7)(8 9)(10 11)
g2 (2 3)(4 5)(6 7)(8 9)(10 11)
g4 (1 4)(3 6)(5 8)(7 10)(9 11)

Order 22 Id 2 n 13
deg 2 at g2 (1 2)
g4 (1 2)(3 4 5 6 7 8 9 10 11 12 13)

Order 23 Id 1 n 23
deg 23
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23)
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Order 24 Id 1 n 11
deg 3 at g5 (9 10 11)
g5 (9 10 11)
g2 (1 2 3 4 5 6 7 8)(10 11)

Order 24 Id 2
deg 3 at g3 (1 2 3)
g6 (1 2 3)(4 5 6 7 8 9 10 11)

Order 24 Id 3 n 8
deg 6 at g2 (3 4 5)(6 7 8)
g2 (3 4 5)(6 7 8)
g7 (1 3 6)(2 7 5)

Order 24 Id 4 n 11
deg 3 at g5 (1 2 3)
g2 (2 3)(4 5 6 7)(8 9 10 11)
g14 (1 2)(4 8 6 10)(5 11 7 9)

Order 24 Id 5 n 7
deg 2 at g2 (6 7)
g2 (6 7)
g14 (1 6)(2 3 4 5)

Order 24 Id 6 n 7
deg 3 at g5 (3 6 7)
g2 (4 5)(6 7)
g14 (1 4)(2 5)(3 6)

Order 24 Id 7  n 9
deg 2 at g3 (1 2)
g2 (4 5)(6 7 8 9)
g14 (1 2)(3 4)(6 7 8 9)

Order 24 Id 8 n 7
deg 3 at g5 (1 4 5)
g2 (4 5)(6 7)
g14 (1 4)(2 6 3 7)

Order 24 Id 9 n 9
deg 2 at g3 (8 9)
g3 (8 9)
g7 (1 2 3)(4 5 6 7)

Order 24 Id 10 n 7
deg 2 at g2 (6 7)
g2 (6 7)
g9 (1 6)(2 7)(3 4 5)

Order 24 Id 11 n 11
deg 3 at g4 (1 2 3)
g2 (4 5 6 7)(8 9 10 11)
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g9 (1 2 3)(4 8 6 10)(5 11 7 9)

Order 24 Id 12 n 4
deg 2 at g2 (3 4)
g2 (3 4)
g10 (1 2 3)

Order 24 Id 13 n 6
deg 2 at g2 (1 2)
g4 (3 4)(5 6)
g6 (1 2)(4 5 6)

Order 24 Id 14 n 7
deg 2 at g2 (6 7)
g2 (6 7)
g3 (4 5)
g15 (1 2)(3 6)

Order 24 Id 15 n 9
deg 2 at g2 (8 9)
g2 (8 9)
g3 (6 7)
g11 (1 2)(3 4 5)

Order 25 Id 1 n 25
deg 25 at g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25)
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25)

Order 25 Id 2 n 10
deg 5 at g2 (6 7 8 9 10)
g2 (6 7 8 9 10)
g3 (1 2 3 4 5)

Order 26 Id 1 n 13
deg 12 at g2 (2 3)(4 5)(6 7)(8 9)(10 11)(12 13)
g2 (2 3)(4 5)(6 7)(8 9)(10 11)(12 13)
g4 (1 4)(3 6)(5 8)(7 10)(9 12)(11 13)

Order 26 Id 2 n 15
deg 2 at g2 (1 2)
g4 (1 2)(3 4 5 6 7 8 9 10 11 12 13 14 15)

Order 27 Id 1 n 27
deg 27 at g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27)
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27)

Order 27 Id 2 n 12
deg 3 at g3 (10 11 12)
g3 (10 11 12)
g2 (1 2 3 4 5 6 7 8 9)

Order 27 Id 3 n 9
deg 6 at g2 (4 5 6)(7 8 9)
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g2 (4 5 6)(7 8 9)
g3 (1 4 7)(2 5 9)(3 6 8)

Order 27 Id 4 n 9
deg 6 at g3 (4 5 6)(7 8 9)
g3 (4 5 6)(7 8 9)
g2 (1 4 7 2 6 8 3 5 9)

Order 27 Id 5 n 9
deg 3 at g2 (7 8 9)
g2 (7 8 9)
g3 (4 5 6)
g4 (1 2 3)

Order 28 Id 1 n 11
deg 4 at g3 (1 3)(2 4)
g4 (5 6 7 8 9 10 11)
g2 (1 2 3 4)(6 11)(7 10)(8 9)

Order 28 Id 2 n 11
deg 4 at g2 (1 2 3 4)
g5 (1 2 3 4)(5 6 7 8 9 10 11)

Order 28 Id 3 n 9
deg 2 at g3 (1 2)
g2 (4 5)(6 7)(8 9)
g9 (1 2)(3 6)(5 8)(7 9)

Order 28 Id 4 n 11
deg 2 at g2 (10 11)
g2 (10 11)
g7 (1 2)(3 4 5 6 7 8 9)

Order 29 Id 1 n 29
deg 29 at g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29)
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29)

Order 30 Id 1 n 8
deg 2 at g2 (7 8)
g2 (7 8)
g11 (1 7)(2 3 4 5 6)

Order 30 Id 2 n 8
deg 3 at g3 (2 3 4)
g2 (5 6)(7 8)
g11 (1 7)(2 3 4)(6 8)

Order 30 Id 3 n 8
deg 3 at g3 (1 3 4)
g2 (3 4)(5 6)(7 8)
g11 (1 3)(2 7)(6 8)

Order 30 Id 4 n 10
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deg 2 at g2 (1 2)
g11 (1 2)(3 4 5)(6 7 8 9 10)

Order 31 Id 1 n 31
deg 31
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31)

Order 32 Id 1 n 32
deg 32
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32)

Order 32 Id 2 n 12
deg 4 at g4 (1 2)(7 8)
g2 (7 8)(9 10 11 12)
g3 (1 7)(2 8)(3 4 5 6)

Order 32 Id 3 n 12
deg 4 at g3 (9 10 11 12)
g3 (9 10 11 12)
g2 (1 2 3 4 5 6 7 8)

Order 32 Id 4 n 12
deg 4 at g5 (9 11)(10 12)
g3 (5 6)(7 8)(9 10 11 12)
g2 (1 5 2 7 3 6 4 8)

Order 32 Id 5 n 12
deg 2 at g3 (11 12)
g3 (11 12)
g2 (1 11)(2 12)(3 4 5 6 7 8 9 10)

Order 32 Id 6 n 8
deg 4 at g3 (5 6)(7 8)
g3 (5 6)(7 8)
g2 (1 2 5 7)(3 4 6 8)

Order 32 Id 7 n 8
deg 4 at g3 (5 6)(7 8)
g3 (5 6)(7 8)
g2 (1 2 5 7 3 4 6 8)

Order 32 Id 8 n>=16
missing

Order 32 Id 9 n 12
deg 4 at g5 (3 5)(4 6)
g3 (7 8)(9 10)(11 12)
g2 (1 7)(2 9)(3 4 5 6)(8 11)(10 12)

Order 32 Id 10 n 12
deg 4 at g5 (9 11)(10 12)
g2 (3 4)(5 6)(7 8)(9 10 11 12)
g3 (1 3 2 5)(4 7 6 8)
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```
Order 32 Id 11 n 8
deg 4 at g7 (2 3 5 7)
g3 (1 2)(3 4)(5 6)(7 8)
g7 (2 3 5 7)

Order 32 Id 12 n 12
deg 4 at g3 (9 10 11 12)
g3 (9 10 11 12)
g2 (1 2 3 4 5 6 7 8)(10 12)

Order 32 Id 13 n 12
deg 4 at g5 (9 11)(10 12)
g2 (3 4)(5 6)(7 8)(9 10 11 12)
g7 (1 3 2 5)(4 7 6 8)(9 10 11 12)

Order 32 Id 14 n 12
deg 4 at g5 (9 11)(10 12)
g2 (3 4)(5 6)(7 8)(9 10 11 12)
g7 (1 3)(2 5)(4 7)(6 8)(9 10 11 12)

Order 32 Id 15 n>=15
missing

Order 32 Id 16 n 18
deg 2 at g3 (1 2)
g2 (3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18)
g3 (1 2)

Order 32 Id 17 n 16
deg 8 at g3 (2 10)(4 12)(6 14)(8 16)
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16)
g3 (2 10)(4 12)(6 14)(8 16)

Order 32 Id 18 n 16
deg 14 at g2 (3 4)(5 6)(7 8)(9 10)(11 12)(13 14)(15 16)
g2 (3 4)(5 6)(7 8)(9 10)(11 12)(13 14)(15 16)
g3 (1 3)(2 5)(4 7)(6 9)(8 11)(10 13)(12 15)(14 16)

Order 32 Id 19 n 16
deg 14
g8 (1 2 3 4)(5 6 7 8)(9 10 11 12)(13 14 15 16)
g3 (2 5)(4 7)(6 9)(8 11)(10 13)(12 15)(14 16)

Order 32 Id 20 n>=14
missing

Order 32 Id 21 n 10
deg 2 at g4 (9 10)
g4 (9 10)
g2 (5 6 7 8)
g3 (1 2 3 4)
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Order 32 Id 22 n 10
deg 2 at g3 (9 10)
g3 (9 10)
g4 (7 8)
g2 (1 9)(2 10)(3 4 5 6)

Order 32 Id 23 n 10
deg 2 at g4 (9 10)
g4 (9 10)
g2 (3 4)(5 6 7 8)
g3 (1 3 2 4)

Order 32 Id 24 n>=10
missing

Order 32 Id 25 n 8
deg 2 at g3 (7 8)
g3 (7 8)
g8 (5 7)(6 8)
g2 (1 2 3 4)(5 7)(6 8)

Order 32 Id 26 n>=12
missing

Order 32 Id 27 n 8
deg 4 at g2 (5 6)(7 8)
g2 (5 6)(7 8)
g3 (3 5)(4 6)
g4 (1 7)(2 8)

Order 32 Id 28 n 8
deg 4 at g2 (5 6)(7 8)
g2 (5 6)(7 8)
g4 (3 5)(4 6)
g7 (1 7)(2 8)(5 6)

Order 32 Id 29 n 12
deg 4 at g4 (1 3)(2 4)
g2 (3 4)(5 6 7 8)(9 10 11 12)
g3 (5 9 7 11)(6 12 8 10)
g4 (1 3)(2 4)

Order 32 Id 30 n>=10
missing

Order 32 Id 31 n>=11
missing

Order 32 Id 32 n>=11
missing

Order 32 Id 33 n>=12
missing
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```
Order 32 Id 34 n 8
deg 4 at g2 (5 6)(7 8)
g2 (5 6)(7 8)
g7 (3 5)(4 6)(7 8)
g8 (1 7)(2 8)(5 6)

Order 32 Id 35 n>=9
missing

Order 32 Id 36 n>=10
missing

Order 32 Id 37 n 10
deg 2
g8 (3 4 5 6 7 8 9 10)
g12 (3 5 7 9)(4 10 8 6)
g14 (1 2)(3 5 7 9)(4 6 8 10)

Order 32 Id 38 n>= 9
missing

Order 32 Id 39 n 10
deg 2 at g4 (1 2)
g2 (5 6)(7 8)(9 10)
g3 (3 5)(4 7)(6 9)(8 10)
g4 (1 2)

Order 32 Id 40 n 10
deg 2 at g4 (3 4)
g3 (5 6)(7 8)(9 10)
g4 (3 4)
g2 (1 5 2 7)(6 9 8 10)

Order 32 Id 41 n>=10
missing

Order 32 Id 42 n>=13
missing

Order 32 Id 43 n 8
deg 4 at g4 (2 4)(5 7)
g2 (1 2)(3 4)(5 6)(7 8)
g3 (2 5)(4 7)(6 8)
g4 (2 4)(5 7)

Order 32 Id 44 n>=14
missing

Order 32 Id 45 n 10
deg 2
g8 (7 8 9 10)
g18 (5 6)(7 8 9 10)
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g20 (3 4)(7 8 9 10)
g3 (1 2)

Order 32 Id 46 n 8
deg 2 at g2 (7 8)
g2 (7 8)
g3 (5 7)(6 8)
g4 (3 4)
g5 (1 2)

Order 32 Id 47 n 12
deg 2
g3 (5 6 7 8)(9 10 11 12)
g8 (5 9 7 11)(6 12 8 10)
g12 (3 4)(5 6 7 8)(9 10 11 12)
g18 (1 2)(5 10 7 12)(6 9 8 11)

Order 32 Id 48 n 10
g4 (3 4 5 6)(7 8 9 10)
g8 (3 7 5 9)(4 8 6 10)
g14 (1 2)(3 4 5 6)(7 8 9 10)
g18 (3 8 5 10)(4 9 6 7)

Order 32 Id 49 n 8
deg 4
g20 (1 2 3 4)(5 6 7 8)
g3 (5 7)(6 8)
g4 (1 5)(2 6)(3 7)(4 8)
g8 (2 4)(6 8)

Order 32 Id 50 n>=12
missing

Order 32 Id 51 n 10
deg 2
g3 (9 10)
g4 (7 8)
g6 (5 6)
g8 (3 4)
g12 (1 2)

Order 33 Id 1 n 14
deg 3 at g2 (1 2 3)
g5 (1 2 3)(4 5 6 7 8 9 10 11 12 13 14)

Order 34 Id 1 n 17
deg 2
g3 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17)
g4 (2 17)(3 16)(4 15)(5 14)(6 13)(7 12)(8 11)(9 10)

Order 34 Id 2 n 19
deg 2 at g2 (1 2)
g4 (1 2)(3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19)
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Order 35 Id 1 n 12
deg 5 at g2 (1 2 3 4 5)
g5 (1 2 3 4 5)(6 7 8 9 10 11 12)

Order 36 Id 1 n 13
deg 4 at g3 (10 12)(11 13)
g2 (2 3)(4 5)(6 7)(8 9)(10 11 12 13)
g7 (1 4)(3 6)(5 8)(7 9)(10 11 12 13)

Order 36 Id 2 n 13
deg 4 at g2 (1 2 3 4)
g6 (1 2 3 4)(5 6 7 8 9 10 11 12 13)

Order 36 Id 3 n 13
g2 (2 3 4)(5 6 7 8 9 10 11 12 13)
g8 (1 2 4)(5 6 7 8 9 10 11 12 13)

Order 36 Id 4 n 11
deg 2 at g3 (1 2)
g2 (4 5)(6 7)(8 9)(10 11)
g14 (1 2)(3 6)(5 8)(7 10)(9 11)

Order 36 Id 5 n 13
deg 2 at g2 (12 13)
g2 (12 13)
g9 (1 2)(3 4 5 6 7 8 9 10 11)

Order 36 Id 6 n 10
deg 3 at g3 (1 2 3)
g5 (8 9 10)
g6 (1 2 3)(4 5 6 7)(9 10)

Order 36 Id 7 n 10
deg 3 at g4 (8 9 10)
g4 (8 9 10)
g5 (5 6 7)
g2 (1 2 3 4)(6 7)(9 10)

Order 36 Id 8 n 10
deg 3 at g3 (8 9 10)
g3 (8 9 10)
g7 (1 2 3)(4 5 6 7)

Order 36 Id 9 n 6
deg 3 at g5 (1 6 5)
g3 (3 4)(5 6)
g7 (1 3)(2 5 4 6)

Order 36 Id 10 n 6
deg 2 at g2 (3 4)
g6 (3 4)(5 6)
g17 (1 3)(2 5 6)
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Order 36 Id 11 n 7
deg 3 at g2 (5 6 7)
g2 (5 6 7)
g18 (1 2 3)(4 6 5)

Order 36 Id 12 n 8
deg 2 at g2 (7 8)
g2 (7 8)
g20 (1 2)(3 4 5)(6 7 8)

Order 36 Id 13 n 8
deg 2 at g3 (1 2)
g2 (5 6)(7 8)
g7 (4 5)(7 8)
g15 (1 2)(3 7)(5 6)

Order 36 Id 14 n 10
deg 2 at g2 (6 7)
g7 (6 7)(8 9 10)
g10 (1 2)(3 4 5)

Order 37 Id 1 n 37
g2 (1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
```

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