A Model to Comprehend the Dynamics of an Observable Phenomenon in the View of Newtonian Mechanics

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Abstract

This paper's purpose is to show the effectiveness of an approach to formulate and understand the dynamics of an observable phenomenon through Newtonian mechanics. The approach starts by taking a simpler analogue of the problem we need to understand, and solving both the simpler and the real case simultaneously. Then, I have split the real case into two independent cases. The major part of the paper will be concentrated on demonstrating that the approach I choose is effective in comprehending the dynamics of the phenomenon. The problem I have considered is, in a circus, a co-worker saves a person who falls down during a rehearsal by dashing him in the horizontal direction, from a few feet above the ground. In the rest of the paper, for ease, I will be calling the person who saved as rescuer and the person who is being saved as casualty. The analysis of the problem is based on the assumptions that the collision between the rescuer and the casualty is an inelastic collision and after the total mass of rescuer and casualty come in the contact with the ground, it remains stationary.

1 Introduction

The Position, Velocity, and Acceleration components along with Energy, Momentum, and Frictional forces at impact for this problem are analyzed in this paper. In order to give a comprehensive view of the problem, I have considered two cases. In the first case, I have considered the casualty falling to the ground with none rescuing him. In the second case, the casualty is considered to be saved by the rescuer. For the second case, I have separated the trajectory into two independent trajectories. The first independent trajectory is of the casualty till he comes in contact with the rescuer and the second independent trajectory involves the trajectory of the rescuer and the casualty after collision and till they come in contact with the ground. At the end of the analysis, for a range of rescuer's horizontal velocities, Kinetic energy after collision is compared with the Kinetic energy of casualty before collision. By substituting numerical values in the momentum equations, I will find the rescuer's horizontal velocity v_o, for which the Kinetic energy after collision is minimal.

2 Position, Velocity, and Acceleration Components

2.1 Equations of 1D motion with constant acceleration

Position function, $x_t = x_o + v_{ox}t + \frac{1}{2}a_x t^2$
Velocity function, $v_{xt} = v_{ox} + a_x t$
Acceleration function, $a_{xt} = a_x$

Replace x with y and z for the corresponding directions. Let us consider this to be a 2D problem, so we won't be mentioning z-direction components in rest of the paper.

2.2 Case (i): If the casualty falls to the ground without being rescued

The time when he starts to fall at A be 0 and the time when he touches the ground at D be t1. For time 0 and t1, the x and y direction components of position, velocity, and acceleration are decomposed in the tables below.

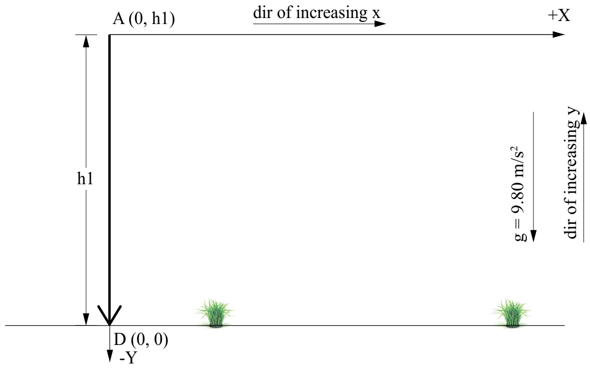

Figure 1: fig a

Position, Velocity, and Acceleration components for fig a:

at time 0 (at pt A)	at time t_1 (at pt D)
$x_o = 0$ (my free choice)	$x_{t1} = 0$
$v_{xo} = 0$	$v_{xt1} = 0$
$a_{xo} = 0$	$a_{xt1} = 0$

Table 1: x direction components for fig a

at time 0 (at pt A)	at time t_1 (at pt D)
$y_o = h_1$ (my free choice)	$y_{t1} = h_1 + 0 - \frac{1}{2}gt_1^2$
$v_{yo} = 0$ (initial velocity is 0)	$v_{yt1} = 0 - gt_1$
$a_{yo} = -g$	$a_{yt1} = -g$

Table 2: y direction components for fig a

2.3 Case (ii): If the casualty is rescued by a rescuer

Here for case(ii), we can separate the motion showed in fig b to two independent motions as shown in fig b(i) and fig b(ii). In fig b(i), the time at point A is 0 and time at point B be t2. In fig b(ii), the time at point B is 0 and the time at point C be t3. For each of the independent trajectories in case(ii), position, velocity, and acceleration components in x and y directions are decomposed in the below tables.

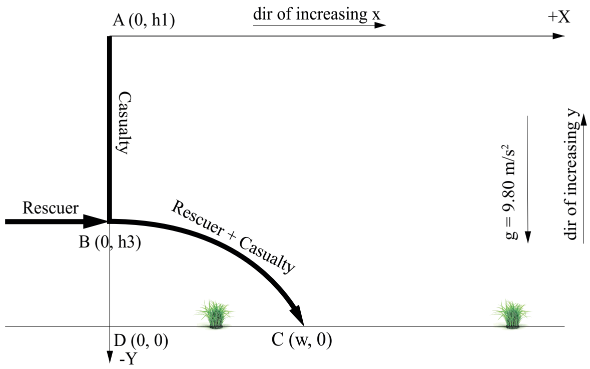

Figure 2: fig b

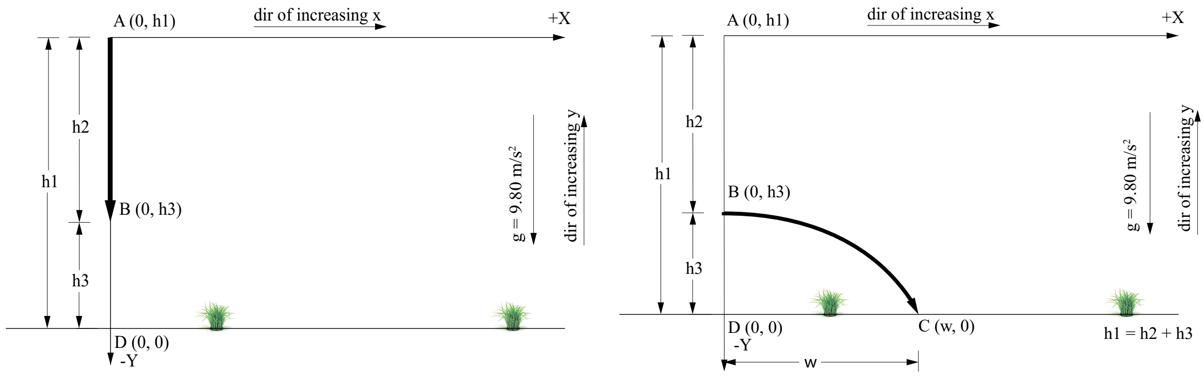

Figure 3: fig b(i) and fig b(ii)

Position, Velocity, and Acceleration components for fig b (i):

at time 0 (at pt A)	at time t_2 (at pt B)
$x_o = 0$ (my free choice)	$x_{t2} = 0$
$v_{xo} = 0$	$v_{xt2} = 0$
$a_{xo} = 0$	$a_{xt2} = 0$

Table 3: x direction components for fig b(i)

at time 0 (at pt A)	at time t_2 (at pt B)
$y_o = h_1$ (my free choice)	$y_{t2} = h_1 + 0 - \frac{1}{2}gt_2^2$
$v_{yo} = 0$ (intial velocity is 0)	$v_{yt2} = 0 - gt_2$
$a_{yo} = -g$	$a_{yt2} = -g$

Table 4: y direction components for fig b(i)

Position, Velocity, and Acceleration components for fig b (ii):

at time 0 (at pt B)	at time t_3 (at pt C)
$x_o = 0$	$x_{t3} = 0 + v_x t3 + 0 = w$
$v_{xo} = v_x$ (+ve value)	$v_{xt3} = v_x + 0$
$a_{xo} = 0$ (v_x is const)	$a_{xt3} = 0$

Table 5: x direction components for fig b(ii)

at time 0 (at pt B)	at time t_3 (at pt C)
$y_o = h_3$	$y_{t3} = h_3 + v_{y1}t + \frac{1}{2}\frac{dv_{y1}}{dt}t^2$
$v_{yo} = v_{y1}$ (-ve value)	$v_{yt3} = v_{y1} + t\frac{dv_{y1}}{dt}$ (-ve value)
$a_{yo} = \frac{dv_{y1}}{dt} = \frac{(v_{y1}+\Delta v_{y1})-v_{y1}}{\Delta t}$ (-ve value)	$a_{yt3} = \frac{dv_{y1}}{dt}$ (-ve value)

Table 6: y direction components for fig b(ii)

2.4 Velocity components at point B

Velocity components of casualty just before the collision: $v_x = 0$ and $v_y = v_{by}$
Velocity components of rescuer just before the collision: $v_x = v_o$ and $v_y = 0$
Velocity components of total mass (rescuer and casualty) just after the collision: $v_x = v_x$ and $v_y = v_{y1}$

3 Energy equations

Let us consider the mass of recuer and casualty to be equal. The energy equations for all the three trajectories are formulated below.

3.1 Case (i): Work done by gravity to move the object from A to D

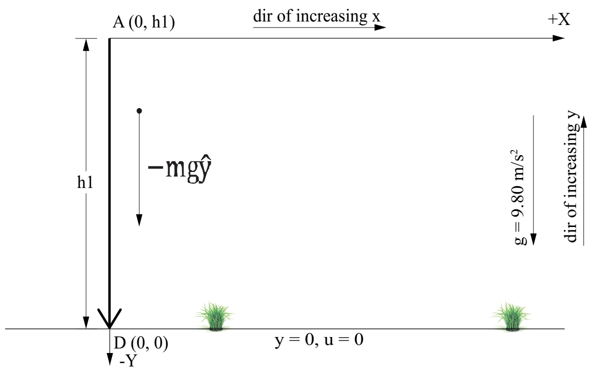

Figure 4: fig a

$\mathrm{WD_{AD}} = \mathrm{KE_D} - \mathrm{KE_A}$ (No initial velocity at A, so $\mathrm{KE_A}$ is 0)
$\mathrm{WD_{AD}} = \frac{1}{2}mv_D^2$
At point A, $\mathrm{gPE_A} = mgh_1$ and $\mathrm{KE_A} = 0$
At point D, $\mathrm{gPE_D} = 0$ (height at pt D is 0) and $\mathrm{KE_D} = \frac{1}{2}mv_D^2$
By Conservation of Total Mechanical Energy, $\mathrm{gPE_A} + \mathrm{KE_A} = \mathrm{gPE_D} + \mathrm{KE_D}$
Substituting values we get, $mgh_1 + 0 = 0 + \frac{1}{2}mv_D^2$

$$h_1 = \frac{v_D^2}{2g} \tag{1}$$

3.2 Case (ii): Work done by gravity to move the object from A to B

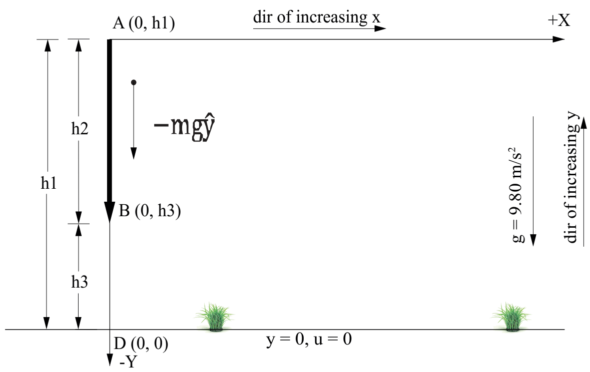

Figure 5: fig b(i)

$\mathrm{WD_{AB}} = \mathrm{KE_B} - \mathrm{KE_A}$ (No initial velocity at A, so $\mathrm{KE_A}$ is 0)
$\mathrm{WD_{AB}} = \frac{1}{2}mv_B^2$
At point A, $\mathrm{gPE_A} = mgh_1$ and $\mathrm{KE_A} = 0$
At point B, $\mathrm{gPE_B} = mgh_3$ and $\mathrm{KE_B} = \frac{1}{2}mv_B^2$
By Conservation of Total Mechanical Energy, $\mathrm{gPE_A} + \mathrm{KE_A} = \mathrm{gPE_B} + \mathrm{KE_B}$
Substituting values we get, $mgh_1 + 0 = mgh_3 + \frac{1}{2}mv_B^2$

$$h_1 - h_3 = h_2 = \frac{v_B^2}{2g} \tag{2}$$

3.3 Case (ii): Work done by gravity to move the object from B to C

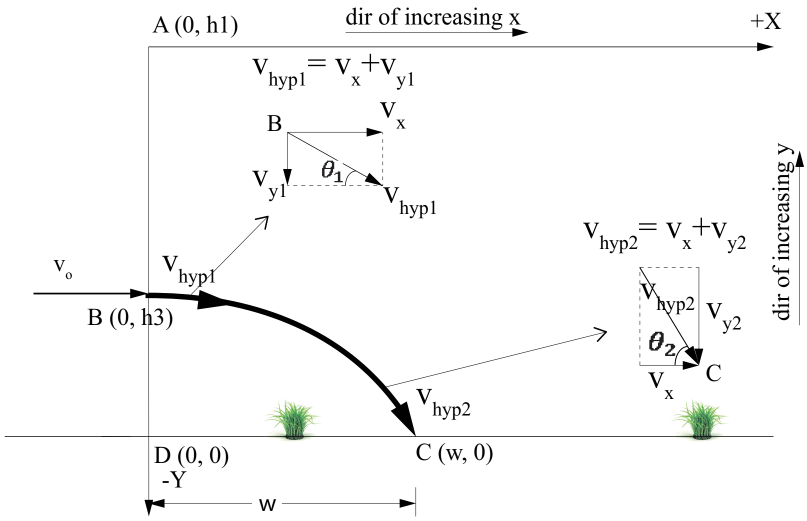

Figure 6: fig b(ii)

$v_{hyp1} = \frac{v_{y1}}{sin\theta_1}$ and $v_{hyp2} = \frac{v_{y2}}{sin\theta_2}$
$\text{WD}_{\text{BC}} = \text{KE}_{\text{C}} - \text{KE}_{\text{B}} = (\frac{1}{2} \times 2mv_C^2) - (\frac{1}{2} \times 2mv_B^2) = mv_C^2 - mv_B^2$
$\text{WD}_{\text{BC}} = m(v_{Cx}^2 - v_{Bx}^2) + m(v_{Cy}^2 - v_{By}^2) = m(v_x^2 - v_x^2) + m(v_{Cy}^2 - v_{By}^2)$
$\text{WD}_{\text{BC}} = mv_{Cy}^2 - mv_{By}^2$
At point B, $\text{PE}_{\text{B}} = mgh_3$ and $\text{KE}_{\text{B}} = \frac{1}{2} \times 2m \times v_{hyp1}^2 = \frac{mv_{y1}^2}{(sin\theta_1)^2}$
At point C, $\text{PE}_{\text{C}} = 0$ and $\text{KE}_{\text{C}} = \frac{1}{2} \times 2m \times v_{hyp2}^2 = \frac{mv_{y2}^2}{(sin\theta_2)^2}$
By Conservation of Total Mechanical Energy, $\text{gPE}_{\text{B}} + \text{KE}_{\text{B}} = \text{gPE}_{\text{C}} + \text{KE}_{\text{C}}$

$$mgh_3 + \frac{mv_{y1}^2}{(sin\theta_1)^2} = 0 + \frac{mv_{y2}^2}{(sin\theta_2)^2}$$

$$h_3 = \frac{1}{g}(\frac{v_{y2}^2}{(sin\theta_2)^2} - \frac{v_{y1}^2}{(sin\theta_1)^2}) \tag{3}$$

Now we will take the sum of total mechanical energy of casualty and total mechanical energy of rescuer just before the collision and equate it with the sum of total mechanical energy of total mass (rescuer and casualty) and energy lost to other forms just after collision.

$$(\frac{1}{2}mv_{By}^2 + mgh_3) + (\frac{1}{2}mv_o^2 + mgh_3) = (\frac{mv_{y1}^2}{(sin\theta_1)^2} + 2mgh_3) + \text{E}_{\text{LostToOtherForms}} \tag{4}$$

From equation (4), we can note that

$$mgh_3 + mgh_3 = 2mgh_3 \tag{5}$$

Equation (5) clearly shows that Potential energy in Equation (4) is conserved.

$$\frac{1}{2}mv_{By}^2 + \frac{1}{2}mv_o^2 = \frac{mv_{y1}^2}{(sin\theta_1)^2} + \text{E}_{\text{LostToOtherForms}} \tag{6}$$

4 Momentum equations

For ease of calculation, we will consider the mass of rescuer and the casualty to be 1 kg each. Let us take the velocity of casualty just before collision, $v_{By} = -6y$ m/s. So, the Kinetic energy of the casualty before collision is 18 J. Then for a range of rescuer's horizontal velocities, $|v_o| = 2|v_{by}|, |v_{by}|, |\frac{v_{by}}{2}|, |\frac{v_{by}}{3}|, |\frac{v_{by}}{6}|$, and $|\frac{v_{by}}{12}|$, we will be finding the Kinetic energy of total mass (rescuer and casualty) just after collision.

I will use $\text{KE}_{\text{CasBefCollision}}$ for the Kinetic energy of casualty just before collision and $\text{KE}_{\text{AftCollision}}$ for the Kinetic energy of total mass (rescuer and casualty) just after collision. I will be using $\text{KE}_{\text{TotBefCollision}}$ for the sum of the kinetic energies of both rescuer and casualty before collision.

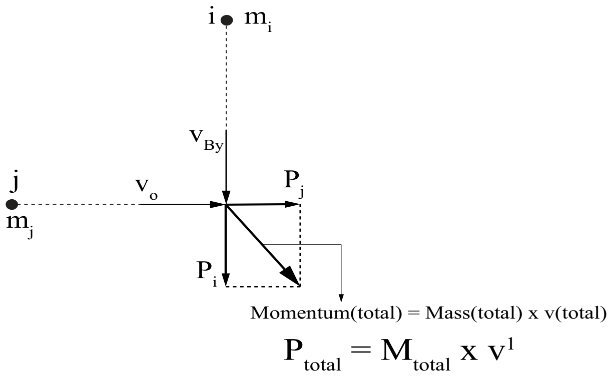

Figure 7: fig c

Consider $|v_o| > |v_{By}|$. Lets say $|v_o| = 2|v_{By}|$
So, $m_i = 1, v_{By} = -6y$ and $m_j = 1, v_o = 12x$
If $|v_o| = 2|v_{By}|$, then $\text{KE}_{\text{AftCollision}}$ (45J)> $\text{KE}_{\text{CasBefCollision}}$ (18J)

Consider $|v_o| = |v_{By}|$
So, $m_i = 1, v_{By} = -6y$ and $m_j = 1, v_o = 6x$
If $|v_o| = |v_{By}|$, then $\text{KE}_{\text{AftCollision}}$ (18J)= $\text{KE}_{\text{CasBefCollision}}$ (18J)

Consider $|v_o| < |v_{By}|$. Lets say $|v_o| = \frac{|v_{By}|}{2}$
So, $m_i = 1, v_{By} = -6y$ and $m_j = 1, v_o = 3x$
If $|v_o| = \frac{|v_{By}|}{2}$, then $\text{KE}_{\text{AftCollision}}$ (11.25J)< $\text{KE}_{\text{CasBefCollision}}$ (18J)

Consider $|v_o| < |v_{By}|$. Lets say $|v_o| = \frac{|v_{By}|}{3}$
So, $m_i = 1, v_{By} = -6y$ and $m_j = 1, v_o = 2x$
If $|v_o| = \frac{|v_{By}|}{3}$, then $\text{KE}_{\text{AftCollision}}$ (10J)< $\text{KE}_{\text{CasBefCollision}}$ (18J)

Consider $|v_o| < |v_{By}|$. Lets say $|v_o| = \frac{|v_{By}|}{6}$
So, $m_i = 1, v_{By} = -6y$ and $m_j = 1, v_o = x$
If $|v_o| = \frac{|v_{By}|}{6}$, then $\text{KE}_{\text{AftCollision}}$ (9.25J)< $\text{KE}_{\text{CasBefCollision}}$ (18J)

Consider $|v_o| < |v_{By}|$. Lets say $|v_o| = \frac{|v_{By}|}{12}$
So, $m_i = 1, v_{By} = -6y$ and $m_j = 1, v_o = 0.5x$
If $|v_o| = \frac{|v_{By}|}{12}$, then $\text{KE}_{\text{AftCollision}}$ (9.0625J)< $\text{KE}_{\text{CasBefCollision}}$ (18J)

4.1 Improbable Case

Let us consider the case, $|v_o| = 0$. Imagine the rescuer and the casualty to be in outer space. Let us assume that the force experienced by them due to gravity of any celestial body is zero. Now let us give the casualty an initial velocity v_{By} and the rescuer is stationary. In this case, after the casualty hits the rescuer, their masses stick together and continue to move with a velocity $\frac{v_{By}}{2}$. But arranging such a collision in the presence of earth's gravitational field between the rescuer and the casualty is improbable. Let the velocity of total mass after collision be v^1.

Consider the value of $m_i = 1, v_{By} = -6y$ and $m_j = 1, v_o = 0$
$\mathrm{P_{TotBefCollision}} = m_i v_i + m_j v_j = 1 \times -6y + 1 \times 0 = -6y$
$2v^1 = -6y \Rightarrow v^1 = -3y$
$|v^1| = \sqrt{(-3)^2} = \sqrt{9} = 3$
$\mathrm{KE_{TotBefCollision}} = \frac{1}{2} \times 1 \times 36 + \frac{1}{2} \times 1 \times 0 = 18\mathrm{J}$
$\mathrm{KE_{AftCollision}} = \frac{1}{2} \times 2 \times 9 = 9\mathrm{J}$
$\mathrm{KE_{CasBefCollision}} = \frac{1}{2} \times 1 \times 36 = 18\mathrm{J}$

$$\mathrm{If}|v_o| = 0, \mathrm{KE_{AftCollision}} = \frac{\mathrm{KE_{CasBefCollision}}}{2} \tag{7}$$

In this improbable case, Kinetic energy of the total mass after inelastic collision is reduced to half the Kinetic energy of the casualty before collision.

5 Frictional Forces

Assumption:

The total mass of rescuer and casualty (2m), after they come in contact with the ground at point C, remains stationary. In other words, the total mass (2m) after impact at point C doesn't moves along the direction of increasing x, due the presence of frictional forces between the total mass and ground.

The maximum frictional force (F_{frmax}) and Normal force (N) experienced by the mass due to the ground, at the instant of impact at point C, are found based on the above assumption.

5.1 Just before the impact

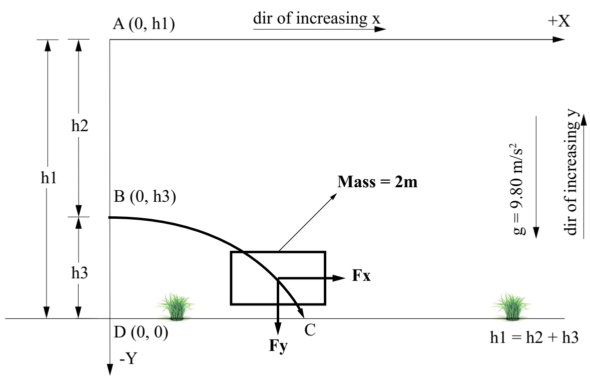

Figure 8: fig d

Just before the mass hits the ground, velocity in the x-direction is constant. So, acceleration in the x-direction is 0.

$$a_x = 0 \Rightarrow f_x = ma_x = 0$$

$$f_y = -m\frac{dv_{y2}}{dt} = \frac{-m(v_{y2} - (v_{y2} - \Delta v_{y2}))}{\Delta t_3}$$

For a particular fall, however down we move the ground, just v_{y2} increases its magnitude in the negative y direction. The horizontal velocity, v_x remains constant. See the images below.

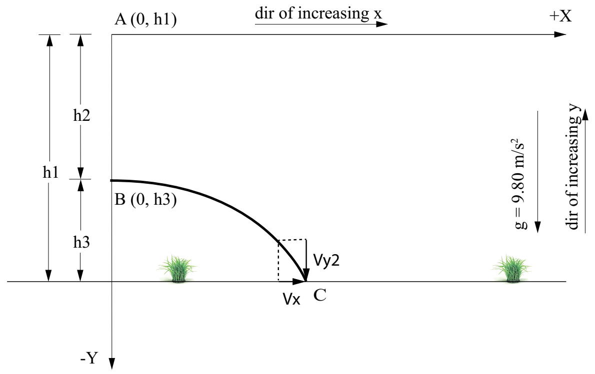

Figure 9: fig d(i)

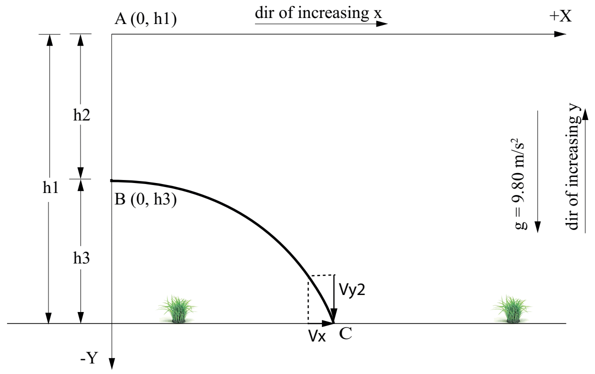

Figure 10: fig d(ii)

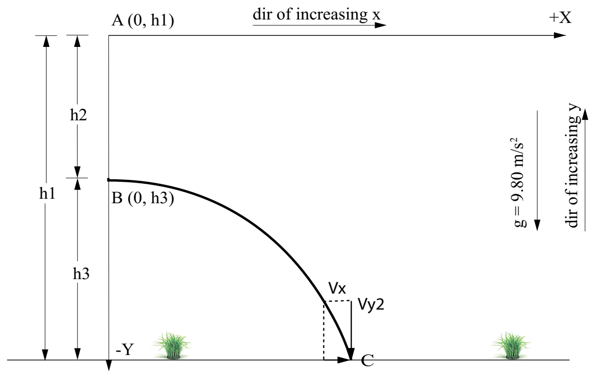

Figure 11: fig d(iii)

As this case is just before the impact, there is a small gap between the total mass and the ground. So, v_{y2} continues to increase its magnitude in the negative y direction and v_x continues to remain constant until the mass comes in contact with the ground. So, due to the manifest lack of contact between the mass and ground at this instant, there is neither a Normal force nor frictional force acting on the mass.

5.2 At the instant of impact

At the instant of impact, based on the assumption that the total mass (2m) after the impact is stationary, the values of F_x, F_y, a_x, and a_y are found.

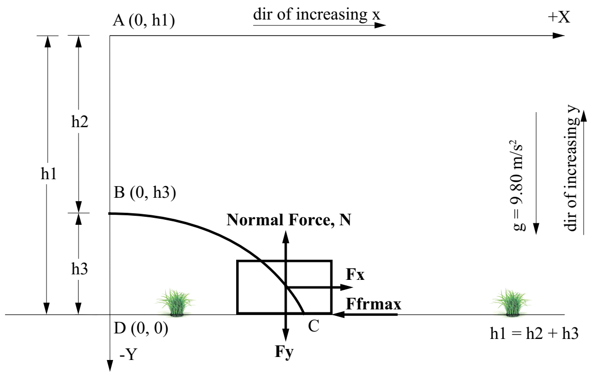

Figure 12: fig e

X-direction components:

At the instant of impact, velocity component in x direction changes from v_x to 0. So, the corresponding force and acceleration component are

$$F_x = 2m \times \frac{(0 - v_x)}{\Delta t_3} = -\frac{2mv_x}{\Delta t_3} \tag{8}$$

$$a_x = -\frac{v_x}{\Delta t_3} \tag{9}$$

Sign Interpretation for equation (9): v_x just before the impact is a +ve value. At the instant of impact, acceleration a_x is -ve value. The -ve acceleration decreases v_x from +ve value to 0.

Y-direction components:

At the instant of impact, velocity component in y direction changes from v_{y2} to 0. So, the corresponding force and acceleration component are

$$F_y = 2m \times \frac{(0 - v_{y2})}{\Delta t_3} = -\frac{2mv_{y2}}{\Delta t_3} \tag{10}$$

$$a_y = -\frac{v_{y2}}{\Delta t_3} \tag{11}$$

Sign Interpretation for equation (11): v_{y2} just before the impact is a -ve value. At the instant of impact, acceleration a_y= -(-value) =+value. Positive acceleration in y direction increases v_{y2} from -ve value to 0.

5.3 Requirements of the ground to make the total mass (2m) stationary at impact

The maximum frictional force exerted by the ground, F_{frmax} should be atleast $\frac{-2mv_x}{\Delta t_3}$ N.
The ground should be able to provide a normal force of atleast $\frac{-2mv_{y2}}{\Delta t_3}$ N.

6 Effectiveness of the approach in comprehending the phenomenon

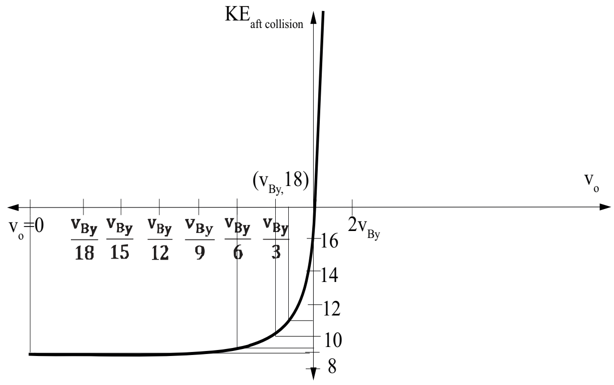

Figure 13: fig f

As we can see from the above plot, the horizontal velocity of the rescuer for which the Kinetic energy after inelastic collision is reduced to the maximum possible relative to the Kinetic energy of casualty just before the collision, is $v_o = 0$. If $v_o = 0$, then the KE of total mass (rescuer and casualty) after collision is just half the KE of casualty before collision. To assert this statement, refer to equation (7) in the improbable case explained in momentum section.We can also prove this by substituting the mass, v_{y1}, and θ_1 values used in improbable case under momentum equations section to equation (6) derived in the energy equations section.

Equation (6): $\frac{1}{2}mv_{by}^2 + \frac{1}{2}mv_o^2 = \frac{mv_{y1}^2}{(sin\theta_1)^2} + \mathrm{E_{KELostToOtherForms}}$
The first term in LHS is 18J and the second term in LHS is 0. So, it is enough to prove
$\frac{mv_{y1}^2}{(sin\theta_1)^2} = \frac{1}{2}(\frac{1}{2}mv_{by}^2)$=9J
While deriving equation (4), for the RHS terms we have considered the total mass to be 2m. So, we can here substitute m=1 kg, and $v^1 = |v_{hyp1}| = 3$ m/s
For this case, horizontal velocity of rescuer, v_o is 0. So, θ_1 value is 90^o.
$v_{y1} = sin\theta_1 \times 3 = sin90^o \times 3 = -3y$ (-ve for its direction)
$|v_{y1}| = \sqrt{(-3)^2} = \sqrt{9} = 3$
If $v_o = 0$, then $\frac{mv_{y1}^2}{(sin\theta_1)^2} = \frac{1\times 9}{1} = 9$J, which is the value of $\frac{1}{2}(\frac{1}{2}mv_{by}^2)$
Substituting $\frac{mv_{y1}^2}{(sin\theta_1)^2}$ value in equation (6), we get $\mathrm{E_{KELostToOtherForms}}$=9J

Kinetic energy of the total mass (rescuer and casualty) after collision, is reduced in comparison with the Kinetic Energy of casualty before collision by an amount, $\mathrm{E_{KELostToOtherForms}}$=9J. The above exercise demonstrates that we can infer and assert any further conclusions from the primary information available depending on the requirements.

7 Conclusion

When working with problems involving the dynamics of moving objects, we can consider a similar problem shedding the intricacies of the problem in which we are interested in. So in this way we can compare the findings of our problem with a simpler analogue, leading to better understanding of the problem. The simpler analogue we used in this paper for comparison is essentially a special case of our problem. For instance, the position, velocity, and acceleration components at point D in special case and Point C in the real case can be used for studying the impact at point D in real case. The paper also demonstrates how splitting a problem into smaller blocks will give us better ways to approach the problem. Here, in this example, the separation of the trajectory for the second case into two independent trajectories, proved easy to derive equation (4) under energy equations section.

8 Acknowledgment

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9 References

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