A note on quantum entanglement in Dempster-Shafer evidence theory

Xinyang Deng[a,b], Yong Deng[a,c,*]

[a] *School of Computer and Information Science, Southwest University, Chongqing, 400715, China*
[b] *Center for Quantitative Sciences, Vanderbilt University School of Medicine, Nashville, TN, 37232, USA*
[c] *Department of Civil & Environmental Engineering, School of Engineering, Vanderbilt University, Nashville, TN, 37235, USA*

Abstract

Dempster-Shafer evidence theory is an efficient mathematical tool to deal with uncertain information. In this theory, basic probability assignment (BPA) is the basic structure for the expression and inference of uncertainty. In this paper, quantum entanglement involved in Dempster-Shafer evidence theory is studied. A criterion is given to determine whether a BPA is in an entangled state or not. Based on that, the information volume involved in a BPA is discussed. The discussion shows that a non-quantum strategy (or observation) can not obtained all information contained in a BPA which is in an entangled state.

Keywords: Dempster-Shafer evidence theory, Belief function, Quantum entanglement, Entangled state, Information volume, Uncertainty

*Corresponding author: Yong Deng, School of Computer and Information Science, Southwest University, Chongqing, 400715, China.
Email address: prof.deng@hotmail.com (Yong Deng)

1. Introduction

Uncertainty is ubiquitous in nature. Several uncertainty theories have been developed, such as probability theory [1], fuzzy set theory [2], possibility theory [3], Dempster-Shafer evidence theory [4, 5], generalized evidence theory [6] and D numbers [7].

Among these theories, Dempster-Shafer evidence theory [4, 5] has attracted increasing interest from scientific communities because of its inherent advantages in representing and handling uncertain information. Recently, this theory has been used to alleviate the difficulties that appear when we use Kolmogorov's theory for quantum probabilities [8, 9]. In this paper, we study the quantum entanglement involved in Dempster-Shafer evidence theory. This work is inspired by our previous work [10] and based on a new proposed uncertainty measure of BPA, Deng entropy [11]. The main contribution of this work is that a criterion is proposed to judge if a BPA is in the entangled state. And by means of that criterion, we reconsider the information volume contained in a BPA, which corrects our previous conclusion and helps us to understand that why the upper bound of uncertainty contained in a BPA should be larger than $log_2 2^{|X|}$ [10], where $|X|$ is the cardinality of the frame of discernment X.

The paper is organized as follows. Knowledge background about Dempster-Shafer evidence theory is briefly introduced in Section 2. Section 3 presents the idea of quantum entanglement in Dempster-Shafer evidence theory. The issue of information volume of a BPA is discussed in Section 4. Finally, this paper is concluded in Section 5.

2. Dempster-Shafer evidence theory

Dempster-Shafer evidence theory (short for D-S theory), also called belief function theory, as introduced by Dempster[4] and then developed by Shafer[5], has emerged from their works on statistical inference and uncertain reasoning.

Let X be a set of mutually exclusive and collectively exhaustive events, indicated by

$$X = \{\theta_1, \theta_2, \cdots, \theta_i, \cdots, \theta_n\} \tag{1}$$

where set X is called a frame of discernment (FOD). The power set of X is indicated by 2^X, namely

$$2^X = \{\emptyset, \{\theta_1\}, \cdots, \{\theta_{|X|}\}, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_2, \cdots, \theta_i\}, \cdots, X\} \tag{2}$$

For a FOD $X = \{\theta_1, \theta_2, \cdots, \theta_{|X|}\}$, a mass function is a mapping m from 2^X to $[0, 1]$, formally defined by:

$$m: \quad 2^X \to [0, 1] \tag{3}$$

which satisfies the following condition:

$$m(\emptyset) = 0 \quad and \quad \sum_{A \in 2^X} m(A) = 1 \tag{4}$$

In D-S theory, a mass function is also called a basic probability assignment (BPA). Assume there are two BPAs indicated by m_1 and m_2, Dempster's rule of combination is used to combine them as follows:

$$m(A) = \begin{cases} \frac{1}{1-K} \sum\limits_{B \cap C = A} m_1(B) m_2(C) \ , & A \neq \emptyset; \\ 0 \ , & A = \emptyset. \end{cases} \tag{5}$$

with

$$K = \sum_{B \cap C = \emptyset} m_1(B) m_2(C) \tag{6}$$

Note that the Dempster's rule of combination is only applicable to such two BPAs which satisfy the condition $K < 1$.

D-S theory has more advantages in in handling uncertainty compared with classical probability theory. When information is adequate, probability theory is effective to handle that situation. However, when information is not adequate, probability theory is invalid to such uncertain situation. Here is an example.

As shown in Figure 1, assume there are two boxes. There are red balls the left box, and green balls in the right box. The number of balls in each box is unknown. Now, a person is assigned to pick a boll from these two boxes. We know that he chooses the the left box with a probability $P1 = 0.6$, and chooses the right box with a probability $P2 = 0.4$. Based on probability theory, it can be obtained that the probability of picking a red ball is 0.6, the probability of picking a green ball is 0.4, namely $p(R) = 0.6$, $p(G) = 0.4$.

Now, let us change the configuration, as shown in Figure 2. In the left box, there are still only red balls. But in the right box, there are not only red balls but also green balls. In accord with above, the exact number of balls in each box is still unknown, and the ratio of them are completely unknown. This

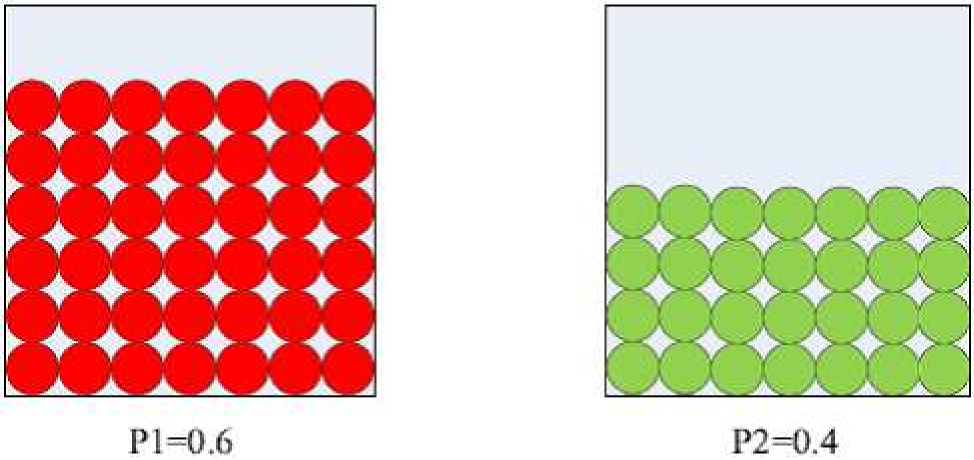

Figure 1: A game of picking ball which can be handled by probability theory

person also has 0.6 probability to choose the left box and 0.4 probability to choose the right box. The question is how possible that a red ball is picked. Obviously, in this case due to lack of adequate information, $p(R)$ and $p(G)$ cannot be obtained. Facing the situation of inadequate information, probability theory is incapable. However, if using D-S theory to analyze this problem, we can obtain a BPA that $m(R) = 0.6$ and $m(R, G) = 0.4$, which means the probability of red ball being picked is 0.6 and the probability of red ball or green ball being picked is 0.4. In the framework of D-S theory, the uncertainty has been expressed more effective. D-S theory has more ability to express uncertain information than probability theory.

3. Quantum entanglement in D-S theory

As we found, in D-S theory a BPA can assign its mass to subsets of FOD. For example, given a FOD $X = \{A, B\}$ which includes all possible

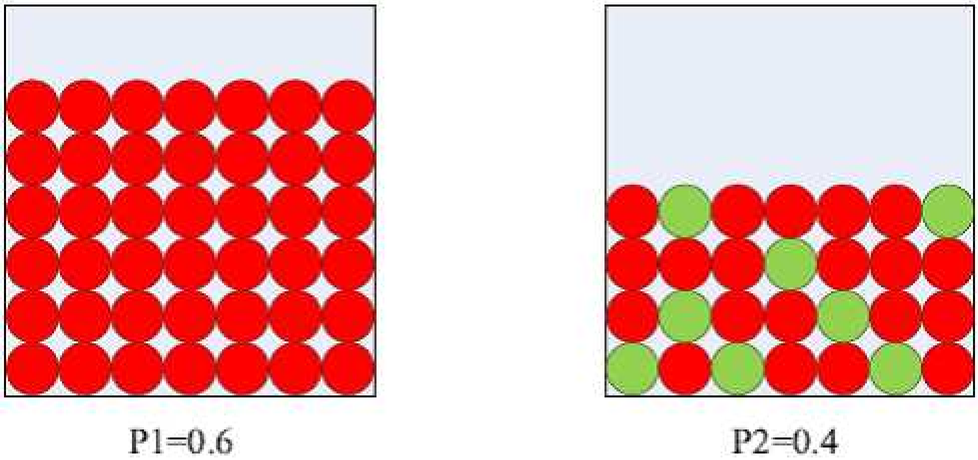

Figure 2: A game of picking ball where probability theory is unable but D-S theory is able to handle

categories that a target T belongs to, a BPA can be $m(A) = a$, $m(B) = b$, $m(A, B) = 1 - a - b$, where $a, b \in [0, 1]$. Herein, $m(A, B)$ indicates that at that moment the target T can be either A or B. This is very similar with a famous example in quantum mechanism, Schrödinger's cat. The example of Schrödinger's cat, shows that a cat would be either alive or dead before the box is opened by a conscious observer [12]. According to the idea of quantum mechanism, this cat is in an entangled state of state "ALIVE" and state "DEAD". Back to D-S theory, therefore we suspect that there is also quantum entanglement in D-S theory. In this paper, we give a criterion to determine whether a BPA is in an entangled state or not. At first, an approach is given to translate a BPA into a Hilbert space.

Given a FOD $X = \{\theta_1, \theta_2, \cdots, \theta_i, \cdots, \theta_n\}$, it generates a high dimensional Hilbert space S_H of a multi-state system. In S_H, $|\theta_i >$ indicates a vector that θ_i happens, and $|\bar{\theta}_i>$ indicates θ_i does not happen. A BPA m can be translated to a state $|\psi>$ in S_H, formally given by:

$$|\psi>=\sum_j \sqrt{x_j}|f(\theta_1)f(\theta_2)\cdots f(\theta_n)> \tag{7}$$

where x_j is the mass value of subset F_j in m, i.e., $m(F_j)=x_j$, and

$$f(\theta_i)=\begin{cases}\theta_i, & \theta_i \in F_j \\ \bar{\theta}_i, & \theta_i \notin F_j\end{cases} \tag{8}$$

Here is an example.

Example 1. *Given a FOD $X=\{\theta_1,\theta_2\}$, there is a BPA: $m(\theta_1)=x_1$, $m(\theta_2)=x_2$, $m(\theta_1,\theta_2)=x_3$, where $x_1+x_2+x_3=1$ and $x_1,x_2,x_3\geq 0$. The BPA m can be translated as*

$$|\psi>=\sqrt{x_1}|\theta_1\bar{\theta}_2>+\sqrt{x_2}|\bar{\theta}_1\theta_2>+\sqrt{x_3}|\theta_1\theta_2>+0|\bar{\theta}_1\bar{\theta}_2> \tag{9}$$

Then, the criterion to determine whether a BPA is in an entangled state or not is given as follows.

Criterion 1. *Given a FOD $X=\{\theta_1,\theta_2,\cdots,\theta_i,\cdots,\theta_n\}$, a BPA m, which corresponds to $|\psi>=\sum_j\sqrt{x_j}|f(\theta_1)f(\theta_2)\cdots f(\theta_n)>$, is in an entangled state if there does not exist $|\psi_1>,|\psi_2>,\cdots,|\psi_n>$, such that*

$$|\psi>=|\psi_1>\otimes|\psi_2>\otimes\cdots\otimes|\psi_n> \tag{10}$$

where $|\psi_i>=\alpha_i|\theta_i>+\beta_i|\bar{\theta}_i>$, and $|\alpha_i|^2+|\beta_i|^2=1$, for $i=1,2,\cdots,n$, and $\otimes$ is a tensor product operator.

Criterion 1 shows that a BPA is in an entangled state if it can not be factored in Hilbert space S_H by using tensor product operator. Two examples are given as below.

Example 2. *Given a FOD* $X = \{\theta_1, \theta_2\}$*, there is a BPA* $m(\theta_1, \theta_2) = 1$*.*

At first, we translates m *to the state in Hilbert space,*

$$|\psi >= 1|\theta_1\theta_2 > +0|\theta_1\bar{\theta}_2 > +0|\bar{\theta}_1\theta_2 > +0|\bar{\theta}_1\bar{\theta}_2 > \tag{11}$$

Then, we assume that $|\psi >$ *is not entangled, which means that there exist* $|\psi_1 >= \alpha_1|\theta_1 > +\beta_1|\bar{\theta}_1 >$ *and* $|\psi_2 >= \alpha_2|\theta_2 > +\beta_2|\bar{\theta}_2 >$*, where* $|\alpha_1|^2 + |\beta_1|^2 = 1$ *and* $|\alpha_2|^2 + |\beta_2|^2 = 1$*, so that* $|\psi >= |\psi_1 > \otimes|\psi_2 >$*. Because,*

$$\begin{aligned}|\phi_1 > \otimes|\phi_2 > &= (\alpha_1|\theta_1 > +\beta_1|\bar{\theta}_1 >) \otimes (\alpha_2|\theta_2 > +\beta_2|\bar{\theta}_2 >)\\ &= \alpha_1\alpha_2|\theta_1\theta_2 > +\alpha_1\beta_2|\theta_1\bar{\theta}_2 > +\alpha_2\beta_1|\bar{\theta}_1\theta_2 > +\beta_1\beta_2|\bar{\theta}_1\bar{\theta}_2 >\end{aligned}$$

The following condition must be meet,

$$\begin{cases}\alpha_1\alpha_2 = 1\\ \alpha_1\beta_2 = 0\\ \alpha_2\beta_1 = 0\\ \beta_1\beta_2 = 0\\ |\alpha_1|^2 + |\beta_1|^2 = 1\\ |\alpha_2|^2 + |\beta_2|^2 = 1\end{cases} \tag{12}$$

It is readily to find that $\alpha_1 = \alpha_2 = \pm 1$*,* $\beta_1 = \beta_2 = 0$*, which means that* $|\psi_1 >$ *and* $|\psi_2 >$ *are existing so that* $|\psi >= |\psi_1 > \otimes|\psi_2 >$*. Therefore, the BPA* $m(\theta_1, \theta_2) = 1$ *is not in an entangled state.*

Example 3. *Given a FOD* $X = \{\theta_1, \theta_2\}$*, there is a BPA* $m(\theta_1) = m(\theta_2) = m(\theta_1, \theta_2) = 1/3$*.*

Similarly, at first, a state $|\psi >$ *is generated,*

$$|\psi >= \sqrt{1/3}|\theta_1\theta_2 > +\sqrt{1/3}|\theta_1\bar{\theta}_2 > +\sqrt{1/3}|\bar{\theta}_1\theta_2 > +0|\bar{\theta}_1\bar{\theta}_2 > \tag{13}$$

Then, assume $|\psi >$ *is not an entangled state, so there exist* $|\psi_1 >= \alpha_1|\theta_1 > +\beta_1|\bar{\theta}_1 >$ *and* $|\psi_2 >= \alpha_2|\theta_2 > +\beta_2|\bar{\theta}_2 >$*, where* $|\alpha_1|^2 + |\beta_1|^2 = 1$ *and* $|\alpha_2|^2 + |\beta_2|^2 = 1$*, so that* $|\psi >= |\psi_1 > \otimes|\psi_2 >$*. The following condition must be meet if* $|\psi >$ *is not entangled,*

$$\begin{cases}\alpha_1\alpha_2 = \sqrt{1/3}\\ \alpha_1\beta_2 = \sqrt{1/3}\\ \alpha_2\beta_1 = \sqrt{1/3}\\ \beta_1\beta_2 = 0\\ |\alpha_1|^2 + |\beta_1|^2 = 1\\ |\alpha_2|^2 + |\beta_2|^2 = 1\end{cases} \tag{14}$$

Obviously, there is not solution for Eq.(14) so that $|\psi>=|\psi_1>\otimes|\psi_2>$. *Therefore, the BPA* $m(\theta_1)=m(\theta_2)=m(\theta_1,\theta_2)=1/3$ *is in an entangled state.*

4. Understanding the information volume contained in a BPA

In order to measure the uncertainty or entropy contained in a BPA, an index, called Deng entropy, has been proposed in [11], which is shown as follows,

$$E_d = -\sum_i m(F_i)\log_2 \frac{m(F_i)}{2^{|F_i|}-1} \tag{15}$$

where, F_i is a proposition in mass function m, and $|F_i|$ is the cardinality of F_i. Specially, Deng entropy definitely degenerate to Shannon entropy if the mass values of m are only assigned to single elements, namely,

$$E_d = -\sum_i m(\theta_i)\log_2 \frac{m(\theta_i)}{2^{|\theta_i|}-1} = -\sum_i m(\theta_i)\log_2 m(\theta_i) \tag{16}$$

According to the definition of Deng entropy, the upper bound of uncertainty, also called information volume, contained in a BPA can be calculated.

Example 4. *Given a FOD* $X=\{\theta_1,\theta_2,\cdots,\theta_N\}$, *let us consider three special cases of mass functions as follows.*

- $m_1(F_i)=m_1(F_j)$ *and* $\sum_i m_1(F_i)=1, \quad \forall F_i,F_j\subseteq X,\ F_i,F_j\neq\emptyset.$
- $m_2(X)=1.$
- $m_3(\theta_1)=m_3(\theta_2)=\cdots=m_3(\theta_N)=1/N.$

Obviously, in terms of Deng entropy, the uncertainty contained in each BPA changes with size of FOD, N*. Figure 3 gives these results. As found in Figure 3,* m_1 *has the maximum uncertainty. In other words, in order to eliminate all uncertainty contained in* m_1*, the information volume we need is the maximum.*

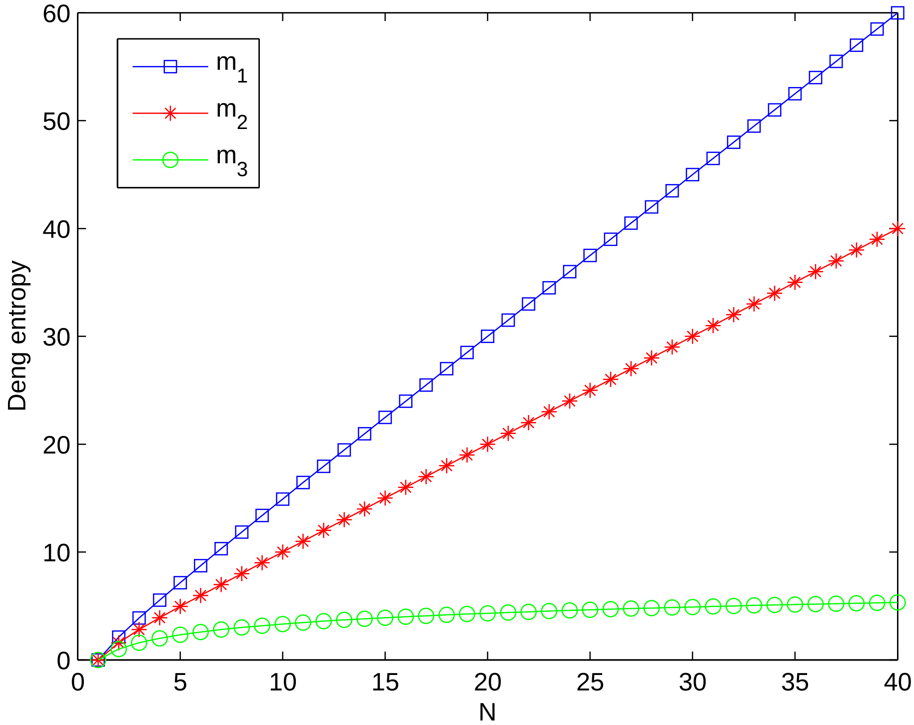

Figure 3: Deng entropy as a function of the size of frame of discernment in three types of mass functions

For example, if $N = 32$, we can obtained that $E_d(m_1) = 48$, $E_d(m_2) = 32$, $E_d(m_3) = 5$. Moreover, it is easy to verify that the maximum Deng entropy is 48 for an evidential system whose size of FOD is 32. Namely, given FOD $X = \{a_1, a_2, \cdots, a_{32}\}$, for any BPA m on X, we have $E_d(m) \leq 48$. The maximum information volume is 48.

In previous study [10], the information volume of a BPA has been investigated. In [10], our conclusion is that the range of uncertainty contained in a BPA is $[0, \log_2 2^{|X|}]$, where $|X|$ is the size of FOD. However, based on Deng entropy, the previous conclusion is not correct, and we find that the upper bound of uncertainty is larger than $\log_2 2^{|X|}$. For example, if $|X| = 32$, the upper bound of uncertainty is 48, but not 32. How to explain the gap between 32 and 48? In [13], we conjecture that it is caused by quantum entanglement. Now, let us reconsider this problem from the perspective of quantum entanglement.

Given a FOD $X = \{\theta_1, \theta_2, \cdots, \theta_{32}\}$, the information volume 32 is obtained from BPA m_{FOD}:

$m_{FOD}(\theta_1, \theta_2, \cdots, \theta_{32}) = 1.$

The information volume 48 is obtained from BPA m_{AVG}:

$m_{AVG}(F_i) = 1/(2^{32} - 1), \quad \forall F_i, F_i \in 2^X \ \ and \ \ F_i \neq \emptyset.$

At first, these two BPAs are translated to vectors in Hilbert space, ψ_{FOD} and ψ_{AVG}. Based on Criterion 1, we find that $|\psi_{FOD}>=|\psi_1> \otimes \cdots \otimes |\psi_i> \otimes \cdots \otimes |\psi_{32}>$, where $|\psi_i>= 1|\theta_i> +0|\bar{\theta}_i>$ for all $i = 1, 2, \cdots, 32$. So, m_{FOD} is not entangled. However, it is easy to demonstrate that there does not exist $|\psi_i>= \alpha_i|\theta_i> +\beta_i|\bar{\theta}_i>$, where $|\alpha_i|^2+|\beta_i|^2 = 1$ and $i = 1, 2, \cdots, 32$, so that $|\psi_{AVG}>=|\psi_1> \otimes \cdots \otimes |\psi_i> \otimes \cdots \otimes |\psi_{32}>$. Therefore, m_{AVG} is in an entangled state.

Here, we found that the information volume of m_{AVG} is larger than that of m_{FOD} since the impact of quantum entanglement. In our view, the BPA m_{FOD} represents a non-quantum strategy (or observation) to acquire information of a system. For example, in order to find the top 1 student (or students), we can ask all students one by one [10, 13]. "One by one" implies that the observation is decomposable, such observation or strategy is classical or non-quantum. However, all information contained in m_{AVG} which is entangled, can not be completely by using such a non-quantum strategy or observation.

5. Conclusion

In this paper, we studied the quantum entanglement in D-S theory. At first, an approach was proposed to convert a BPA to a state defined in the Hilbert space. Then, we gave a criterion to decide whether a BPA is in the entangled state or not. And examples are given to show that criterion. Finally, based on the proposed criterion, we reconsidered the uncertainty bound of BPA. We found that quantum entanglement helps to explain the information volume of a BPA calculated by using Deng entropy. In the future research, the degree of entanglement in a BPA will be studied.

Acknowledgments

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