Investigations on the Chebyshev functions II: Discrete Formulas for first and Second Functions

By Edigles Guedes

March 9, 2015

"I am that bread of life." - John 6:48.

Abstract. In present article, we create discrete formulas for first and second Chebyshev functions.

1. Introduction

The first Chebyshev function, $\vartheta(x)$ or $\theta(x)$, is defined by

$$\vartheta(x) \stackrel{\text{def}}{=} \sum_{n \leqslant x} \ln(p_n),$$

If $x = m \in \mathbb{N}$, then, we can write the first Chebyshev function as follows

$$\vartheta(m) = \sum_{n=1}^{m} \ln(p_n). \tag{1}$$

The second Chebyshev function, $\psi(x)$, is defined by

$$\psi(x) \stackrel{\text{def}}{=} \sum_{p_n^k \leqslant x} \ln(p_n) = \sum_{n \leqslant x} \Lambda(n) = \sum_{n \leqslant x} \lfloor \log_{p_n}(x) \rfloor \ln(p_n),$$

where $\Lambda(n)$ denotes the von Mangoldt function, which is defined as

$$\Lambda(n) \stackrel{\text{def}}{=} \begin{cases} \ln(p_n), \text{if } n = p_n^k \text{ for some } p_n \text{ and integer } k \geqslant 1; \\ 0, \text{otherwise.} \end{cases}$$

An direct relationship between them is given by

$$\psi(x) = \sum_{n=1}^{\infty} \vartheta(x^{1/n}).$$

If $x = m \in \mathbb{N}$, then, we can write the second Chebyshev function as follows

$$\psi(m) = \sum_{n=1}^{m} \lfloor \log_{p_n}(m) \rfloor \ln(p_n). \tag{2}$$

In this present paper, we prove that, for $x \in \mathbb{R}_{\geq 5}$, then

$$\vartheta(x) = \ln 6 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right] \ln k$$

$$= \ln 6 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right] \ln k$$

$$= \ln 6 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{\left(e^{-\frac{2\pi i \Gamma(k)}{k}} - 1\right)\left(e^{-\frac{2\pi i}{k}} - 1\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right] \ln k$$

$$= \ln 6 - \sum_{k=5}^{\lfloor x \rfloor} \left(\csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\} \right) \ln k,$$

and

$$
\begin{aligned}
\psi(x) &= \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right] \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k \\
&= \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3 \\
&+ \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right] \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k \\
&= \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{\left(e^{-\frac{2\pi i \Gamma(k)}{k}} - 1\right)\left(e^{-\frac{2\pi i}{k}} - 1\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right] \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k \\
&= \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3 - \sum_{k=5}^{\lfloor x \rfloor} \left(\csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\} \right) \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k.
\end{aligned}
$$

2. PRELIMINARIES

In previous articles [1 and 2], we demonstrate that

$$
\begin{aligned}
\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} &= \frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \\
&= \frac{\left(e^{-\frac{2\pi i \Gamma(k)}{k}} - 1\right)\left(e^{-\frac{2\pi i}{k}} - 1\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} = -\csc\left(\frac{\pi}{k}\right) \sin\left(\frac{\pi\Gamma(k)}{k}\right) \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\} \\
&= \begin{cases} 1, \text{if } k \text{ is prime;} \\ 0, \text{if } k \text{ is composite;} \end{cases}
\end{aligned}
\tag{3}
$$

for $k \in \mathbb{N}_{\geq 5}$.

3. THEOREMS

Theorem 1. *If* $n \in \mathbb{N}_{\geq 5}$, *then*

$$
\begin{aligned}
\vartheta(n) &= \ln 6 + \sum_{k=5}^{n} \left[\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right] \ln k \\
&= \ln 6 + \sum_{k=5}^{n} \left[\frac{1 - \cos\left(\frac{2\pi}{k}\right) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right] \ln k \\
&= \ln 6 + \sum_{k=5}^{n} \left[\frac{\left(e^{-\frac{2\pi i \Gamma(k)}{k}} - 1\right)\left(e^{-\frac{2\pi i}{k}} - 1\right)}{2 - 2\cos\left(\frac{2\pi}{k}\right)} \right] \ln k \\
&= \ln 6 - \sum_{k=5}^{n} \left(\csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\} \right) \ln k,
\end{aligned}
$$

where $\ln k$ *denotes the natural logarithm function,* e^x *denotes the exponential logarithm function,* $\csc x$ *denotes the cosecant function,* $\sin x$ *denotes the sine function and* $\cos x$ *denotes the cosine function.*

Proof. From (1) and (3), it follows that

$$\begin{aligned}
\vartheta(n) &= \ln 2+\ln 3+\sum_{k=5}^{n}\left[\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right]\ln k=\ln 6+\sum_{k=5}^{n}\left[\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right]\ln k\\
&=\ln 6+\sum_{k=5}^{n}\left[\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left(\frac{2\pi\Gamma(k)}{k}\right)+\cos\left(\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right]\ln k\\
&=\ln 6+\sum_{k=5}^{n}\left[\frac{\left(e^{-\frac{2\pi i\Gamma(k)}{k}}-1\right)\left(e^{-\frac{2\pi i}{k}}-1\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right]\ln k\\
&=\ln 6-\sum_{k=5}^{n}\left(\csc\left(\frac{\pi}{k}\right)\sin\left[\frac{\pi\Gamma(k)}{k}\right]\cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\}\right)\ln k,
\end{aligned}$$

which are the desired results. □

Theorem 2. *If $n\in\mathbb{N}_{\geq 5}$, then*

$$\begin{aligned}
\psi(n) &= \lfloor\log_2(n)\rfloor\ln 2+\lfloor\log_3(n)\rfloor\ln 3+\sum_{k=5}^{n}\left[\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right]\lfloor\log_k(n)\rfloor\ln k\\
&=\lfloor\log_2(n)\rfloor\ln 2+\lfloor\log_3(n)\rfloor\ln 3\\
&+\sum_{k=5}^{n}\left[\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left(\frac{2\pi\Gamma(k)}{k}\right)+\cos\left(\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right]\lfloor\log_k(n)\rfloor\ln k\\
&=\lfloor\log_2(n)\rfloor\ln 2+\lfloor\log_3(n)\rfloor\ln 3+\sum_{k=5}^{n}\left[\frac{\left(e^{-\frac{2\pi i\Gamma(k)}{k}}-1\right)\left(e^{-\frac{2\pi i}{k}}-1\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right]\lfloor\log_k(n)\rfloor\ln k\\
&=\lfloor\log_2(n)\rfloor\ln 2+\lfloor\log_3(n)\rfloor\ln 3-\sum_{k=5}^{n}\left(\csc\left(\frac{\pi}{k}\right)\sin\left[\frac{\pi\Gamma(k)}{k}\right]\cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\}\right)\lfloor\log_k(n)\rfloor\ln k,
\end{aligned}$$

where $\ln k$ *denotes the natural logarithm function,* e^x *denotes the exponential logarithm function,* $\csc x$ *denotes the cosecant function,* $\sin x$ *denotes the sine function and* $\cos x$ *denotes the cosine function.*

Proof. From (2) and (3), it follows that

$$\begin{aligned}
\psi(n) &= \lfloor\log_2(n)\rfloor\ln 2+\lfloor\log_3(n)\rfloor\ln 3+\sum_{k=5}^{n}\left[\frac{e^{\frac{2\pi i\Gamma(k)}{k}}-1}{e^{-\frac{2\pi i}{k}}-1}\right]\lfloor\log_k(n)\rfloor\ln k\\
&=\lfloor\log_2(n)\rfloor\ln 2+\lfloor\log_3(n)\rfloor\ln 3\\
&+\sum_{k=5}^{n}\left[\frac{1-\cos\left(\frac{2\pi}{k}\right)-\cos\left(\frac{2\pi\Gamma(k)}{k}\right)+\cos\left(\frac{2\pi\Gamma(k)}{k}+\frac{2\pi}{k}\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right]\lfloor\log_k(n)\rfloor\ln k\\
&=\lfloor\log_2(n)\rfloor\ln 2+\lfloor\log_3(n)\rfloor\ln 3+\sum_{k=5}^{n}\left[\frac{\left(e^{-\frac{2\pi i\Gamma(k)}{k}}-1\right)\left(e^{-\frac{2\pi i}{k}}-1\right)}{2-2\cos\left(\frac{2\pi}{k}\right)}\right]\lfloor\log_k(n)\rfloor\ln k\\
&=\lfloor\log_2(n)\rfloor\ln 2+\lfloor\log_3(n)\rfloor\ln 3-\sum_{k=5}^{n}\left(\csc\left(\frac{\pi}{k}\right)\sin\left[\frac{\pi\Gamma(k)}{k}\right]\cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\}\right)\lfloor\log_k(n)\rfloor\ln k,
\end{aligned}$$

which are the desired results. □

Theorem 3. *If* $x \in \mathbb{R}_{\geq 5}$, *then*

$$\vartheta(x) = \ln 6 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right] \ln k$$

$$= \ln 6 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{1 - \cos(\frac{2\pi}{k}) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos(\frac{2\pi}{k})} \right] \ln k$$

$$= \ln 6 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{\left(e^{-\frac{2\pi i \Gamma(k)}{k}} - 1\right)\left(e^{-\frac{2\pi i}{k}} - 1\right)}{2 - 2\cos(\frac{2\pi}{k})} \right] \ln k$$

$$= \ln 6 - \sum_{k=5}^{\lfloor x \rfloor} \left(\csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\} \right) \ln k,$$

where $\ln k$ *denotes the natural logarithm function,* e^x *denotes the exponential logarithm function,* $\csc x$ *denotes the cosecant function,* $\sin x$ *denotes the sine function,* $\cos x$ *denotes the cosine function and* $\lfloor x \rfloor$ *denotes the floor function.*

Proof. From (1) and (3), it follows that

$$\vartheta(x) = \ln 2 + \ln 3 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right] \ln k = \ln 6 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right] \ln k$$

$$= \ln 6 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{1 - \cos(\frac{2\pi}{k}) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos(\frac{2\pi}{k})} \right] \ln k$$

$$= \ln 6 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{\left(e^{-\frac{2\pi i \Gamma(k)}{k}} - 1\right)\left(e^{-\frac{2\pi i}{k}} - 1\right)}{2 - 2\cos(\frac{2\pi}{k})} \right] \ln k$$

$$= \ln 6 - \sum_{k=5}^{\lfloor x \rfloor} \left(\csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\} \right) \ln k,$$

which are the desired results. □

Theorem 4. *If* $x \in \mathbb{R}_{\geq 5}$, *then*

$$\psi(x) = \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right] \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k$$

$$= \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3$$

$$+ \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{1 - \cos(\frac{2\pi}{k}) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos(\frac{2\pi}{k})} \right] \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k$$

$$= \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{\left(e^{-\frac{2\pi i \Gamma(k)}{k}} - 1\right)\left(e^{-\frac{2\pi i}{k}} - 1\right)}{2 - 2\cos(\frac{2\pi}{k})} \right] \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k$$

$$= \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3 - \sum_{k=5}^{\lfloor x \rfloor} \left(\csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\} \right) \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k,$$

where $\ln k$ *denotes the natural logarithm function,* e^x *denotes the exponential logarithm function,* $\csc x$ *denotes the cosecant function,* $\sin x$ *denotes the sine function,* $\cos x$ *denotes the cosine function and* $\lfloor x \rfloor$ *denotes the floor function.*

Proof. From (2) and (3), it follows that

$$\psi(x) = \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{e^{\frac{2\pi i \Gamma(k)}{k}} - 1}{e^{-\frac{2\pi i}{k}} - 1} \right] \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k$$

$$= \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3$$

$$+ \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{1 - \cos(\frac{2\pi}{k}) - \cos\left(\frac{2\pi\Gamma(k)}{k}\right) + \cos\left(\frac{2\pi\Gamma(k)}{k} + \frac{2\pi}{k}\right)}{2 - 2\cos(\frac{2\pi}{k})} \right] \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k$$

$$= \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3 + \sum_{k=5}^{\lfloor x \rfloor} \left[\frac{\left(e^{-\frac{2\pi i \Gamma(k)}{k}} - 1\right)\left(e^{-\frac{2\pi i}{k}} - 1\right)}{2 - 2\cos(\frac{2\pi}{k})} \right] \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k$$

$$= \lfloor \log_2(\lfloor x \rfloor) \rfloor \ln 2 + \lfloor \log_3(\lfloor x \rfloor) \rfloor \ln 3 - \sum_{k=5}^{\lfloor x \rfloor} \left(\csc\left(\frac{\pi}{k}\right) \sin\left[\frac{\pi\Gamma(k)}{k}\right] \cos\left\{\frac{\pi[\Gamma(k)+1]}{k}\right\} \right) \lfloor \log_k(\lfloor x \rfloor) \rfloor \ln k,$$

which are the desired results. □

References

[1] Guedes, Edigles and Gandhi, Raja Rama, *An Elementary proof of Legendre's Conjecture*, viXra:1307.0142, available in March 9, 2015.

[2] Gandhi, Raja Rama and Guedes, Edigles, *An Elementary proof of Legendre's Conjecture*, Asia Journal of Mathematics and Physics, Volume 2013, Article ID amp0041, pp. 1-7, ISSN 2308-3131, http://scienceasia.asia.