Tail properties and asymptotic distribution for extreme of LGMD *

Jianwen Huang[a], Jianjun Wang[a], Guowang Luo[b]
[a]School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China
[b]School of Mathematics and Computational Science, Zunyi Normal College, Zunyi, 563002, China

Abstract. We introduce logarithmic generalized Maxwell distribution which is an extension of the generalized Maxwell distribution. Some interesting properties of this distribution are studied and the asymptotic distribution of the partial maximum of an independent and identically distributed sequence from the logarithmic generalized Maxwell distribution is gained. The expansion of the limit distribution from the normalized maxima is established under the optimal norming constants, which shows the rate of convergence of the distribution for normalized maximum tending to the extreme limit.

Keywords. Extreme value distribution; Logarithmic generalized Maxwell distribution; Mills' ratio; Maximum; Tail Properties.

Mathematics Subject Classification(2010): Primary 62E20, 60E05; Secondary 60F15, 60G15.

1 Introduction

The generalized Maxwell distribution (GMD for short), a generalization of ordinary Maxwell distribution, is proposed by Vodǎ (2009). The GMD has a variety of applications in statistics, physics and chemistry. Its most common application is in statistical mechanics. Some recent examples of this have: constructing fractional rheological constitutive equations (Schiessel et al., 1995); be friction model suitable for quick simulation and control (Farid et al., 2005); forecasting the temporal change of opening angle in multiple time scales and electroscalar wave (Zhang et al., 2008; Arbab and Satti, 2009); project of the time related to behavior of viscoelastic materials (Monsia, 2011).

The probability density function (pdf) and the cumulative distribution function (cdf) of the GMD with the parameter $k > 0$ are respectively,

$$g_k(x) = \frac{k}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)} x^{2k} \exp\left(-\frac{x^{2k}}{2\sigma^2}\right)$$

and

$$G_k(x) = \int_{-\infty}^{x} g_k(t)\,\mathrm{d}t$$

*Corresponding author. E-mail address: wjj@swu.edu.cn (J. Wang).

for $x > 0$, where σ is a positive constant and $\Gamma(\cdot)$ is the Gamma function.

Mills (1926) gave a well-known inequality and Mills' ratio result for the standard normal cdf $\Phi(x)$ with pdf $\phi(x)$ as follows:

$$x^{-1}(1+x^{-2})^{-1}\phi(x) < \Phi(-x) < x^{-1}\phi(x), \tag{1.1}$$

for $x > 0$ and

$$\frac{\Phi(-x)}{\phi(x)} \sim \frac{1}{x}, \tag{1.2}$$

as $x \to \infty$.

Peng et al. (2009) extended the Mills' results to the case of the general error distribution:

$$\frac{2\lambda^v}{v}x^{1-v}\left(1+\frac{2(v-1)\lambda^v}{v}x^{-v}\right)^{-1} < \frac{T_v(-x)}{t_v(x)} < \frac{2\lambda^v}{v}x^{1-v}, \tag{1.3}$$

for $v > 1$ and $x > 0$, and for $v > 0$

$$\frac{T_v(-x)}{t_v(x)} \sim \frac{2\lambda^v}{v}x^{1-v}, \tag{1.4}$$

as $x \to \infty$, where $\lambda = \left[\frac{2^{-2/v}\Gamma(1/v)}{\Gamma(3/v)}\right]^{1/2}$, and $T_v(x)$ is the cdf with pdf $t_v(x)$. Huang and Chen (2014) investigated the similar results of GMD, i.e.,

$$\frac{\sigma^2}{k}x^{1-2k} < \frac{G_k(-x)}{g_k(x)} < \frac{\sigma^2}{k}x^{1-2k}\left(1+\left(\frac{\sigma^2}{k}x^{2k}-1\right)^{-1}\right), \tag{1.5}$$

for $k > 1/2$, $\sigma > 0$ and $x > 0$, and for $k > 0$,

$$\frac{G_k(-x)}{g_k(x)} \sim \frac{\sigma^2}{k}x^{1-2k}, \tag{1.6}$$

as $x \to \infty$. The above-mentioned inequalities such as (1.1), (1.3), and (1.5) and Mills' type-ratios such as (1.2), (1.4), and (1.6) play an important role in considering some tail behavior and extremes of economic and financial data.

In this paper, we define the logarithmic generalized Maxwell distribution (denoted by LGMD), which is a natural extension of the generalized Maxwell distribution. One motivation of considering LGMD is to obtain more efficient results as parameter estimators when random models were supposed with the LGMD error terms instead of normal ones. Meanwhile, the LGMD can be expected to be a better model for certain modern areas.

The present paper is to derive the Mills-type inequality, Mills-type ratio, and the tail distributional representation for the LGMD. As an important application, the asymptotic distribution of the partial maximum of independent and identically distributed (i.i.d.) with common LGMD is investigated.

First we provide the definition of LGMD.

Definition 1.1. *Let X denote a random variable which obeys the GMD. Set $Y = \exp(X)$. Then we call that Y obeys the LGMD, denoted by $Y \sim LGMD(k)$ with parameter $k > 0$.*

Easily check that the pdf of $Y \sim LGMD(k)$ is

$$f_k(x) = \frac{kx^{-1}}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)}(\log x)^{2k} \exp\left(-\frac{(\log x)^{2k}}{2\sigma^2}\right)$$

for $x > 0$, where parameter $k > 0$, and σ is a positive constant. Let $F_k(x)$ denote the cdf of Y, i.e.,

$$F_k(x) = \int_0^x f_k(t)\,\mathrm{d}t$$

for $x > 0$. Note that the LGMD reduces to the logarithmic Maxwell distribution when $k = 1$.

The rest of the article is organized as follows. In Sec. 2, we derive some results concerning Mills-type ratios and tail behavior of LGMD. Sec. 3, we consider the limit distribution of the partial maximum of i.i.d random variables following the LGMD and the suitable norming constants needed. The result is also extended to the case of a finite mixture of LGMDs. Sec. 4, we establish the asymptotic expansion of the distribution of the normalized maximum from LGMD under optimal choice of norming constants. As byproduct, we obtain the convergence speed of the distribution of the normalized partial maxima to its limit.

2 Mills' Ratio and Tail Properties of LGMD

In this section, we derive some results including Mills' inequality, Mills' ratio of LGMD.

For LGMD and GMD, note that $1 - G_k(\log x) = 1 - F_k(x)$ and

$$\frac{1 - G_k(\log x)}{x^{-1}g_k(\log x)} = \frac{1 - F_k(x)}{f_k(x)}.$$

So, by Lemma 2.2 and Theorem 2.1 in Huang and Chen (2014), we have the following two results.

Theorem 2.1. *Let F_k and f_k respectively denote the cdf and pdf of LGMD with parameter $k > 1/2$. We have the inequality below for all $x > 1$,*

$$\frac{\sigma^2}{k}x(\log x)^{1-2k} < \frac{1 - F_k(x)}{f_k(x)} < \frac{\sigma^2}{k}x(\log x)^{1-2k}\left(1 + \left(\frac{\sigma^2}{k}(\log x)^{2k} - 1\right)^{-1}\right), \tag{2.1}$$

where σ is a positive constant.

Corollary 2.1. *For fixed $k > 0$, as $x \to \infty$, we have*

$$\frac{1 - F_k(x)}{f_k(x)} \sim \frac{\sigma^2}{k}x(\log x)^{1-2k}. \tag{2.2}$$

Remark 2.1. *Since the LGMD(k) are reduced to the logarithmic Maxwell distribution as $k = 1$, so by Theorem 2.1 and Corollary 2.1, we derive the inequality and Mills' ratio of the logarithmic Maxwell distribution, i.e.,*

$$\sigma^2 x(\log x)^{-1}f_1(x) < 1 - F_1(x) < \sigma^2 x(\log x)^{-1}(1 + (\sigma^2(\log x)^2 - 1)^{-1})f_1(x),$$

for $x > 1$ and

$$\frac{1 - F_1(x)}{f_1(x)} \sim \frac{\sigma^2 x}{\log x},$$

as $x \to \infty$.

Remark 2.2. *For $k > 1/2$, Corollary 2.1 gives $F_k \in D(\Lambda)$, i.e., there exist norming constants $\alpha_n > 0$ and $\beta_n \in \mathbb{R}$ which make sure $F_k^n(\alpha_n x + \beta_n)$ converges to $\exp(-\exp(-x))$. Since*

$$\frac{(d/dx)f_k(x)}{f_k(x)} = -\frac{1}{x}\left(1 - \frac{2k}{\log x} + \frac{k}{\sigma^2}(\log x)^{2k-1}\right),$$

by Corollary 2.1, we have

$$\frac{1 - F_k(x)}{f_k(x)} \frac{(d/dx)f_k(x)}{f_k(x)} \to -1$$

as $x \to \infty$. Hence, it follows by Proposition 1.18 in Resnick (1987) that $F_k \in D(\Lambda)$. The choice of norming constants α_n and β_n is discussed by Theorem 3.2.

Finner et al. (2008) investigated the asymptotic behavior of the ratio of the Student's t and normal distributions as the degrees of freedom $u = u(x)$ satisfies

$$\lim_{x\to\infty} \frac{x^4}{u} = \beta \in [0, \infty). \tag{2.3}$$

The main motivation of the work is to consider the false discovery rate in multiple testing problems with large numbers of hypotheses and extremely small critical values for the smallest ordered p value; in detail, see Finner et al. (2007). In this section, we study the asymptotic behavior of the ratio of pdfs and the ratio of the tails of the LGMD and the logarithmic Maxwell distribution. Firstly, we consider the case of $k \to 1$. Secondly, we consider the case of $x \to \infty$ for fixed k.

Theorem 2.2. *For $k > 0$, let $x = x(k)$ be such that*

$$k - 1 = \frac{\gamma}{2(\log x)^2 \log\log x}$$

for some $\gamma \in \mathbb{R}$. Then

$$\lim_{k\to 1} \frac{f_1(x)}{f_k(x)} = \exp\left(\frac{\gamma}{2\sigma^2}\right) \tag{2.4}$$

and

$$\lim_{k\to 1} \frac{1 - F_1(x)}{1 - F_k(x)} = \exp\left(\frac{\gamma}{2\sigma^2}\right). \tag{2.5}$$

Proof. Note that $\frac{2^{(k+1)/2}\sigma^{2+1/k}\Gamma(1+k/2)}{k\sigma^3\pi^{1/2}} \to 1$ as $k \to 1$, so

$$\begin{aligned}
\lim_{k\to 1} \frac{f_1(x)}{f_k(x)} &= \lim_{k\to 1} (\log x)^{2-2k} \exp\left(\frac{(\log x)^{2k}}{2\sigma^2} - \frac{(\log x)^2}{2\sigma^2}\right) \\
&= \lim_{k\to 1} \exp\left(\frac{(\log x)^2}{2\sigma^2}\left((\log x)^{2k-2} - 1\right)\right) \\
&= \lim_{k\to 1} \exp\left(\frac{(\log x)^2}{2\sigma^2}(\exp((2k-2)\log\log x) - 1)\right) \\
&= \lim_{k\to 1} \exp\left(\frac{(\log x)^2}{2\sigma^2}\left(\exp\left(\frac{\gamma}{(\log x)^2}\right) - 1\right)\right) \\
&= \exp\left(\frac{\gamma}{2\sigma^2}\right).
\end{aligned}$$

The condition of the theorem deduces $(\log x)^{2-2k} \to 1$ as $k \to 1$. According to Corollary 2.1, Remark 2.1 and (2.4), (2.5) can be deduced. □

Theorem 2.3. *For fixed* k*, we have*

$$\frac{f_1(x)}{f_k(\exp((\log x)^{1/k}))} = \frac{2^{(k+1)/2}\Gamma(1+k/2)\exp((\log x)^{1/k})}{\pi^{1/2}k\sigma^{1-1/k}x} \tag{2.6}$$

and

$$\lim_{x\to\infty}\frac{(\log x)^{1/k-1}(1-F_1(x))}{1-F_k(\exp((\log x)^{1/k}))} = \frac{2^{(k+1)/2}\Gamma(1+k/2)}{\pi^{1/2}\sigma^{1-1/k}}. \tag{2.7}$$

Proof. Note that (2.6) follows from fundamental calculation. By Corollary 2.1, Remark 2.1 and (2.6), we have

$$\begin{aligned}
&\lim_{x\to\infty}\frac{(\log x)^{1/k-1}(1-F_1(x))}{1-F_k(\exp((\log x)^{1/k}))}\\
=&\lim_{x\to\infty}\frac{kx}{\exp((\log x)^{1/k})}\frac{f_1(x)}{f_k(\exp((\log x)^{1/k}))}\\
=&\frac{2^{(k+1)/2}\Gamma(1+k/2)}{\pi^{1/2}\sigma^{1-1/k}},
\end{aligned}$$

so (2.7) follows. □

3 Asymptotic Distribution of the Maximum

By applying Corollary 2.1, we could establish the distributional tail representation for the LGMD.

Theorem 3.1. *Under the conditions of Theorem* 2.1*, we have*

$$1-F_k(x) = c(x)\exp\left(-\int_e^x \frac{g(t)}{f(t)}\,\mathrm{dt}\right)$$

for large enough x*, where*

$$c(x) = \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\left(-1/(2\sigma^2)\right)(1+\theta_1(x))$$

and

$$f(t) = \frac{\sigma^2}{k}t(\log t)^{1-2k},\ g(t) = 1-\frac{\sigma^2}{k}(\log t)^{-2k},$$

where $\theta_1(x)\to 0$ *as* $x\to\infty$*.*

Proof. For large enough x, by Corollary 2.1, we have

$$\begin{aligned}
1-F_k(x) &= \frac{\sigma^2}{k}(\log x)^{1-2k}xf_k(x)(1+\theta_1(x))\\
&= \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\left(\log\log x - \frac{(\log x)^{2k}}{2\sigma^2}\right)(1+\theta_1(x))\\
&= \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\left(-\frac{1}{2\sigma^2}\right)\exp\left(-\int_e^x\left(\frac{k(\log t)^{2k-1}}{\sigma^2 t}-\frac{1}{t\log t}\right)\mathrm{dt}\right)(1+\theta_1(x))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)} \exp\left(-\frac{1}{2\sigma^2}\right) \exp\left(-\int_e^x \frac{1-k^{-1}\sigma^2(\log t)^{-2k}}{k^{-1}\sigma^2 t(\log t)^{1-2k}}\,\mathrm{dt}\right)(1+\theta_1(x)) \\
&= c(x)\exp\left(-\int_e^x \frac{g(t)}{f(t)}\,\mathrm{dt}\right),
\end{aligned}
$$

where $\theta_1(x) \to 0$ as $x \to \infty$. The desired result follows. □

Remark 3.1. *As* $\lim_{t\to\infty} g(t) = 1$, $f(t) > 0$ *on* $[1,\infty)$ *is absolutely continuous function and* $\lim_{t\to\infty} f'(t) = 0$ *in Theorem* 3.1*, an application of Theorem* 3.1 *and Corollary* 1.7 *in Resnick (1987) shows* $F_k \in D(\Lambda)$*, and the norming constants* a_n *and* b_n *can be chosen by*

$$
\frac{1}{1-F_k(b_n)} = n,\ a_n = f(b_n) \tag{3.1}
$$

such that

$$
\lim_{n\to\infty} F_k^n(a_n x + b_n) = \Lambda(x), \tag{3.2}
$$

where $D(\Lambda)$ *denotes the domain of attraction* $\Lambda(x) = \exp(-\exp(-x))$*.*

In this we consider the asymptotic distribution of the normalized maximum of a sequence of i.i.d. random variables following LGMD. Remark 2.2 and Theorem 3.1 showed that the distribution of partial maximum converges to $\Lambda(x)$. So, the following work is to find the associated suitable norming constants.

Theorem 3.2. *Let* $\{X_n, n \geq 1\}$ *be an i.i.d. sequence from the LGMD with* $k > 1/2$*. Let* $M_n = \max\{X_k, 1 \leq k \leq n\}$*. We have*

$$
\lim_{n\to\infty} P(M_n \leq \alpha_n x + \beta_n) = \exp(-\exp(-x)),
$$

where

$$
\begin{aligned}
&\alpha_n \\
=& \frac{\sigma^2 \exp\left(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)}\right)\left(1+\frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2))\right)}{k\left(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)} + \log\left(1+\frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2))\right)\right)^{2k-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\beta_n \\
=& \exp\left(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)}\right)\left(1+\frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}\left(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2)\right)\right).
\end{aligned}
$$

Proof. Since $F_k \in D(\Lambda)$, there must be norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ which make sure that $\lim_{n\to\infty} P((M_n - b_n)/a_n \leq x) = \exp(-\exp(-x))$. By Proposition 1.1 in Resnick (1987) and Theorem 3.1, the norming constants can be chosen that a_n and b_n satisfy the equations: $b_n = (1/(1-F_k))^{\leftarrow}(n)$ and $a_n = f(b_n)$. Note that $F_k(x)$ is continuous, then $1 - F_k(b_n) = n^{-1}$. By Corollary 2.1, we have

$$
nk^{-1}\sigma^2(\log b_n)^{1-2k} b_n f_k(b_n) \to 1,
$$

as $n \to \infty$, i.e.,

$$n2^{-\frac{k}{2}}\sigma^{-\frac{1}{k}}\Gamma^{-1}\left(1+\frac{k}{2}\right)\log b_n \exp\left(-\frac{(\log b_n)^{2k}}{2\sigma^2}\right) \to 1,$$

as $n \to \infty$, and so

$$\log n - \frac{k}{2}\log 2 - \frac{1}{k}\log\sigma - \log\Gamma\left(1+\frac{k}{2}\right) + \log\log b_n - \frac{(\log b_n)^{2k}}{2\sigma^2} \to 0, \tag{3.3}$$

as $n \to \infty$, which deduces

$$\frac{(\log b_n)^{2k}}{2\sigma^2 \log n} \to 1,$$

as $n \to \infty$, thus

$$2k\log\log b_n - \log 2 - 2\log\sigma - \log\log n \to 0,$$

as $n \to \infty$, hence

$$\log\log b_n = \frac{1}{2k}(\log 2 + 2\log\sigma + \log\log n) + o(1).$$

Putting the above equality into (3.3), we have

$$(\log b_n)^{2k} = 2\sigma^2\left(\log n + \frac{1}{2k}\log\log n - \frac{k^2-1}{2k}\log 2 - \log\Gamma\left(1+\frac{k}{2}\right)\right) + o(1),$$

which deduces that

$$\log b_n = 2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}(\log n)^{\frac{1}{2k}}\left(1+\frac{\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+\frac{k}{2})}{2^2k^2\log n} + o\left((\log n)^{-1}\right)\right),$$

therefore

$$\begin{aligned} b_n =& \exp\left(2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}(\log n)^{\frac{1}{2k}}\right)\left(1+\frac{\sigma^{\frac{1}{k}}(\log n)^{\frac{1}{2k}-1}}{2^{2-\frac{1}{2k}}k^2}\left(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+\frac{k}{2})\right)\right. \\ & \left. + o\left((\log n)^{\frac{1}{2k}-1}\right)\right) \\ =& \beta_n + o\left((\log n)^{\frac{1}{2k}-1}\exp\left(2^{\frac{1}{2k}}\sigma^{\frac{1}{k}}(\log n)^{\frac{1}{2k}}\right)\right), \end{aligned}$$

where

$$\beta_n = \exp\left(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)}\right)\left(1+\frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2))\right).$$

Hence, we have

$$\begin{aligned} \alpha_n &= f(\beta_n) \\ &= \frac{\sigma^2\exp(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)})\left(1+\frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2))\right)}{k\left(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)} + \log\left(1+\frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2))\right)\right)^{2k-1}}. \end{aligned}$$

It is easy to check that $\lim_{n\to\infty}\alpha_n/a_n = 1$ and $\lim_{n\to\infty}(b_n - \beta_n)/\alpha_n = 0$. Hence, by Theorem 1.2.3 in Leadbetter et al. (1983), the result follows. □

Remark 3.2. *Theorem* 3.2 *shows that the limit distribution of the normalized maximum from the logarithmic Maxwell distribution is the Gumbel extreme value distribution with norming constants*

$$\alpha_n = \frac{\sigma^2 \exp\left(2^{1/2}\sigma(\log n)^{1/2}\right)\left(1+\frac{\sigma}{2^{3/2}(\log n)^{1/2}}\left(\log\log n - 2\log(\pi^{1/2}/2)\right)\right)}{2^{1/2}\sigma(\log n)^{1/2} + \log\left(1+\frac{\sigma}{2^{3/2}(\log n)^{1/2}}\left(\log\log n - 2\log(\pi^{1/2}/2)\right)\right)}$$

and

$$\beta_n = \exp\left(2^{1/2}\sigma(\log n)^{1/2}\right)\left(1+\frac{\sigma}{2^{3/2}(\log n)^{1/2}}\left(\log\log n - 2\log(\pi^{1/2}/2)\right)\right).$$

At the end of this section, we extend the result of Theorem 3.2 to the case of a finite mixture of LGMDs.

Finite mixture distributions or models have been widely applied in various areas like Chemistry (Roeder, 1994) and image and video databases (Ahuja 1998). Recently, some extreme statistical scholars have also studied them. Mladenović (1999) have considered extreme values of the sequences of independent random variables with common mixed distributions containing normal, Cauchy and uniform distributions. Peng et al. (2010) have investigated the limit distribution and its corresponding uniform convergence rate for a finite mixed of exponential distribution.

If the distribution function (df) F of a random variable ξ have

$$F(x) = p_1F_1(x) + p_2F_2(x) + \cdots + p_rF_r(x),$$

we say that ξ obeys a finite mixture distribution F, where $F_i, 1 \le i \le r$ denote different dfs of the mixture components. The weight coefficients have the condition that $p_i > 0, i = 1, 2, \cdots, r$ and $\sum_{j=1}^{r} p_j = 1$.

Next, we consider the extreme value distribution from a finite mixture with component dfs F_{k_i} obeying LGMD(k_i), where the parameter $k_i > 1$ for $1 \le i \le r$ and $k_i \ne k_j$ for $i \ne j$. Denote the df of the finite mixture by

$$F(x) = p_1F_{k_1}(x) + p_2F_{k_2}(x) + \cdots + p_rF_{k_r}(x) \tag{3.4}$$

for $x > 0$.

Theorem 3.3. *Let* $\{Z_n, n \ge 1\}$ *be a sequence of i.i.d. random variables following the common df* F *given by (3.4). Let* $M_n = \max\{Z_1, Z_2, \cdots, Z_n\}$. *Then*

$$\lim_{n\to\infty} P\left(\frac{M_n - \beta_n}{\alpha_n} \le x\right) = \exp(-\exp(-x))$$

holds with the norming constants

$$\alpha_n = \frac{\sigma^{1/k}\exp\left(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)}\right)}{2^{1-1/(2k)}k(\log n)^{1-1/(2k)}}$$

and

$$\begin{aligned}\beta_n =& \exp\left(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)}\right)\left(1+\frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}\Big(\log\log n + 2k\log p - (k^2-1)\log 2\right. \\ &\left. - 2k\log\Gamma(1+k/2)\Big)\right),\end{aligned}$$

where $\sigma = \max\{\sigma_1, \cdots, \sigma_r\}$, *and* $p = p_{i_1} + \cdots + p_{i_j}$, *where* $i_s \in \{i, \sigma_i = \sigma$ *and* $k = k_i\}, 1 \le s \le j \le r$ *and* $k = \min\{k_1, \cdots, k_r\}$.

Proof. By (3.4), we have

$$1 - F(x) = \sum_{i=1}^{r} p_i(1 - F_{k_i}(x)).$$

By Theorem 2.1, we have

$$\begin{aligned}
&\sum_{i=1}^{r} \frac{p_i\sigma_i^2}{k_i}(\log x)^{1-2k_i} x f_{k_i}(x) < 1 - F(x) \\
&< \sum_{i=1}^{r} \frac{p_i\sigma_i^2}{k_i}(\log x)^{1-2k_i} x \left(1 + (\frac{\sigma_i^2}{k_i}(\log x)^{2k_i} - 1)^{-1}\right) f_{k_i}(x)
\end{aligned}$$

for all $x > 1$, according to the definition of f_k, which implies

$$\begin{aligned}
&\frac{p\log x}{2^{\frac{k}{2}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})} \exp\left(-\frac{(\log x)^{2k}}{2\sigma^2}\right)(1 + A_k(x)) < 1 - F(x) \\
&< \frac{p\log x}{2^{\frac{k}{2}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})} \left(1 + (\frac{\sigma^2}{k}(\log x)^{2k} - 1)^{-1}\right) \exp\left(-\frac{(\log x)^{2k}}{2\sigma^2}\right)(1 + B_k(x)),
\end{aligned} \tag{3.5}$$

where

$$A_k(x) = \sum_{k_i \neq k} \frac{2^{\frac{k}{2}} p_i \sigma^{\frac{1}{k}} \Gamma(1+\frac{k}{2})}{2^{\frac{k_i}{2}} p \sigma^{\frac{1}{k_i}} \Gamma(1+\frac{k_i}{2})} \exp\left(\frac{(\log x)^{2k}}{2\sigma^2} - \frac{(\log x)^{2k_i}}{2\sigma_i^2}\right) \to 0 \tag{3.6}$$

and

$$B_k(x) = \sum_{k_i \neq k} \frac{2^{\frac{k}{2}} p_i \sigma^{\frac{1}{k}} \Gamma(1+\frac{k}{2})}{2^{\frac{k_i}{2}} p \sigma^{\frac{1}{k_i}} \Gamma(1+\frac{k_i}{2})} \frac{1 + (\frac{\sigma_i^2}{k_i}(\log x)^{2k_i} - 1)^{-1}}{1 + (\frac{\sigma^2}{k}(\log x)^{2k} - 1)^{-1}} \exp\left(\frac{(\log x)^{2k}}{2\sigma^2} - \frac{(\log x)^{2k_i}}{2\sigma_i^2}\right) \to 0 \tag{3.7}$$

as $x \to \infty$ since $k = \min\{k_1, k_2, \cdots, k_r\}$. Combining (3.5)-(3.7) and (2.2) together, for large enough x, we obtain

$$1 - F(x) \sim p(1 - F_k(x)) \tag{3.8}$$

as $x \to \infty$, where F_k denotes the cdf of the LGMD(k), and σ and p are defined by Theorem 3.3. By Proposition 1.19 in Resnick (1987), we can derive $F \in D(\Lambda)$. The norming constants can be obtained by Theorem 3.2 and (3.8). The proof is complete. □

4 Asymptotic expansion of Maximum

In this section, we establish an high-order expansion of the distribution of the extreme from the LGMD sample.

Theorem 4.1. *For the norming constants a_n and b_n given by (3.1), we have*

$$\lim_{n\to\infty} (\log b_n)^{\lambda} \left((\log b_n)^{2k-1}(F_k^n(a_n x + b_n) - \Lambda(x)) - I(x)\Lambda(x)\right) = l(x)\Lambda(x),$$

where

$$l(x)=\begin{cases}J_k(x)+\frac{1}{2}I^2(x), & \text{if } \frac{1}{2}<k<1,\\ J_k(x)+w(x), & \text{if } k=1,\\ J_k(x), & \text{if } k>1,\end{cases}$$

$$I(x)=\frac{1}{2}k^{-1}\sigma^2x^2e^{-x},\ \ w(x)=\frac{1}{4}\sigma^4x^4e^{-2x},$$

and

$$J_k(x)=\begin{cases}k^{-2}\sigma^4x^3\left(\frac{1}{3}-\frac{1}{4}x\right)e^{-x}, & \text{if } \frac{1}{2}<k<1,\\ \sigma^2x\left(1+\frac{1}{2}x+\frac{1}{3}\sigma^2x^2-\frac{1}{4}\sigma^2x^3\right)e^{-x}, & \text{if } k=1,\\ k^{-1}\sigma^2x\left(\frac{1}{2}(2k-1)x+1\right)e^{-x}, & \text{if } k>1.\end{cases}$$

Corollary 4.1. *Under the condition of Theorem* 4.1, *we have*

$$F_k^n(a_nx+b_n)-\Lambda(x)\sim\frac{I(x)\Lambda(x)}{(2\sigma^2\log n)^{1-1/(2k)}} \tag{4.1}$$

for large n.

In order to prove Theorem 4.1, we need several lemmas. The following lemma shows a decomposition of the distributional tail representation of the LGMD.

Lemma 4.1. *Let $F_k(x)$ denote the cdf of the LMGD. For large x, we have*

$$1-F_k(x)=\frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\left(-1/(2\sigma^2)\right)\Big[1+k^{-1}\sigma^2(\log x)^{-2k}+k^{-2}(1-2k)\sigma^4(\log x)^{-4k}$$
$$+O\left((\log x)^{-6k}\right)\Big]\exp\left(-\int_e^x\frac{g(t)}{f(t)}\,\mathrm{d}t\right)$$

with $f(t)$ and $g(t)$ given by Theorem 3.1.

Proof. By integration by parts, we have

$$\begin{aligned}1-F_k(x)&=\frac{k}{2^{\frac{1}{2}}\Gamma(1+\frac{k}{2})}\int_{\log x/\sigma^{\frac{1}{k}}}^{\infty}s^{2k}\exp\left(-\frac{1}{2}s^{2k}\right)ds\\ &=\frac{\log x}{2^{\frac{k}{2}}\sigma^{\frac{1}{k}}\Gamma(1+\frac{k}{2})}\exp\left(-\frac{(\log x)^{2k}}{2\sigma^2}\right)\Big[1+k^{-1}\sigma^2(\log x)^{-2k}+k^{-2}(1-2k)\sigma^4(\log x)^{-4k}\\ &+k^{-3}(1-2k)(1-4k)\sigma^6(\log x)^{-6k}\Big]\\ &+\frac{(1-2k)(1-4k)(1-6k)}{2^{\frac{k}{2}}k^3\Gamma(1+\frac{k}{2})}\int_{\log x/\sigma^{\frac{1}{k}}}^{\infty}s^{-6k}\exp\left(-\frac{1}{2}s^{2k}\right)ds.\end{aligned} \tag{4.2}$$

Using L'Hospital's rules yields

$$\lim_{n\to\infty}\frac{\int_{\log x/\sigma^{\frac{1}{k}}}^{\infty}s^{-6k}\exp\left(-\frac{1}{2}s^{2k}\right)ds}{(\log x)^{1-6k}\exp\left(-\frac{(\log x)^{2k}}{2\sigma^2}\right)}=0. \tag{4.3}$$

Easily check that

$$\begin{aligned}&\frac{\log x}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\left(-\frac{(\log x)^{2k}}{2\sigma^2}\right)\\&=\frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\left(-\frac{1}{2\sigma^2}\right)\exp\left(-\int_e^x\frac{g(t)}{f(t)}\,\mathrm{dt}\right)\end{aligned}\tag{4.4}$$

with $f(t)$ and $g(t)$ determined by Theorem 3.1. Combining with (4.2)-(4.4), we complete the proof. □

Lemma 4.2. *Set*

$$\begin{aligned}&B_n(x)\\=&\frac{1+k^{-1}\sigma^2(\log b_n)^{-2k}+k^{-2}(1-2k)\sigma^4(\log b_n)^{-4k}+O\left((\log b_n)^{-6k}\right)}{1+k^{-1}\sigma^2(\log(a_nx+b_n))^{-2k}+k^{-2}(1-2k)\sigma^4(\log(a_nx+b_n))^{-4k}+O\left((\log(a_nx+b_n))^{-6k}\right)}\end{aligned}$$

with the norming constants a_n and b_n given by (3.1), then

$$B_n(x)-1=2k^{-1}\sigma^4(\log b_n)^{-4k}x+O\left((\log b_n)^{1-6k}\right).$$

Proof. By (3.2), we have $n(1-F_k(a_nx+b_n))\to e^{-x}$ as $n\to\infty$, here the norming constants a_n and b_n given by (3.1). It is not difficult to verify that $\lim_{n\to\infty}B_n(x)=1$ and

$$\begin{aligned}B_n(x)-1=&\Big[k^{-1}\sigma^2\left((\log b_n)^{-2k}-(\log(a_nx+b_n))^{-2k}\right)\\&+k^{-2}(1-2k)\sigma^4\left((\log b_n)^{-4k}-(\log(a_nx+b_n))^{-4k}\right)+O\left((\log b_n)^{-6k}\right)\Big](1+o(1)).\end{aligned}\tag{4.5}$$

For large n we have

$$\begin{aligned}(\log b_n)^{-2k}-(\log(a_nx+b_n))^{-2k}=&2\sigma^2(\log b_n)^{-4k}x-k^{-1}\sigma^4(\log b_n)^{1-6k}x^2\\&+O\left((\log b_n)^{2-8k}\right)+O\left((\log b_n)^{-6k}\right)\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}(\log b_n)^{-4k}-(\log(a_nx+b_n))^{-4k}=&4\sigma^2(\log b_n)^{-6k}x-2k^{-1}\sigma^4(\log b_n)^{1-8k}x^2\\&+O\left((\log b_n)^{2-10k}\right)+O\left((\log b_n)^{-8k}\right).\end{aligned}\tag{4.7}$$

By (4.5)-(4.7), the desired result follows. □

Lemma 4.3. *Set $\lambda=1\bigwedge(2k-1)$ denote the minimum of $\{1,2k-1\}$ and $v_n(x)=n\log F_k(a_nx+b_n)+e^{-x}$ with norming constants a_n and b_n given by (3.1). Then,*

$$\lim_{n\to\infty}(\log b_n)^{\lambda}\left((\log b_n)^{2k-1}v_n(x)-I(x)\right)=J_k(x),\tag{4.8}$$

here $I(x)$, $J_k(x)$ given by Theorem 4.1.

Proof. For any positive integers m and $i > 1$, by Corollary 2.1 and the fact that $1/(1 - F_k(a_n x + b_n)) = n$, we have

$$\lim_{n\to\infty} \frac{(1 - F_k(a_n x + b_n))^i}{n^{-1}(\log b_n)^{-mk}} = 0. \tag{4.9}$$

For any $x \in \mathbb{R}$ and $a_n = k^{-1}\sigma^2 b_n (\log b_n)^{1-2k}$, we have

$$(\log b_n)^{2k-1}\left(\frac{k a_n}{\sigma^2(a_n x + b_n)(\log(a_n x + b_n))^{1-2k}} - 1\right) \to -k^{-1}\sigma^2 x \tag{4.10}$$

and

$$\frac{a_n(\log b_n)^{2k-1}}{(a_n x + b_n)\log(a_n x + b_n)} \to 0, \tag{4.11}$$

as $n \to \infty$. Here set

$$C_n(x) = \frac{k a_n}{\sigma^2(a_n x + b_n)(\log(a_n x + b_n))^{1-2k}} - \frac{a_n}{(a_n x + b_n)\log(a_n x + b_n)} - 1.$$

By Lemmas 4.1, 4.2, (4.10) and (4.11), we have

$$\begin{aligned}
&\frac{1 - F_k(b_n)}{1 - F_k(a_n x + b_n)} e^{-x} \\
=& B_n(x)\exp\left(\int_0^x \left(\frac{k a_n}{\sigma^2(a_n s + b_n)(\log(a_n s + b_n))^{1-2k}} - \frac{a_n}{(a_n s + b_n)\log(a_n s + b_n)} - 1\right) ds\right) \\
=& B_n(x)\exp\left(\int_0^x C_n(s)ds\right) \\
=& B_n(x)\left(1 + \int_0^x C_n(s)ds + \frac{1}{2}\left(\int_0^x C_n(s)ds\right)^2 (1 + o(1))\right).
\end{aligned} \tag{4.12}$$

By (4.9)-(4.12), Lemma 4.2 and dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty} (\log b_n)^{2k-1} v_n(x) \\
=& \lim_{n\to\infty} \frac{\log F_k(a_n x + b_n) + n^{-1}e^{-x}}{n^{-1}(\log b_n)^{1-2k}} \\
=& \lim_{n\to\infty} \frac{-(1 - F_k(a_n x + b_n)) - \frac{1}{2}(1 - F_k(a_n x + b_n))^2(1 + o(1)) + (1 - F_k(b_n))e^{-x}}{n^{-1}(\log b_n)^{1-2k}} \\
=& \lim_{n\to\infty} \frac{1 - F_k(a_n x + b_n)}{n^{-1}} \frac{\frac{1 - F_k(b_n)}{1 - F_k(a_n x + b_n)} e^{-x} - 1}{(\log b_n)^{1-2k}} \\
=& e^{-x} \lim_{n\to\infty} (\log b_n)^{2k-1}\left(B_n(x) + B_n(x)\int_0^x C_n(s)ds(1 + o(1)) - 1\right) \\
=& e^{-x} \lim_{n\to\infty} (\log b_n)^{2k-1} \int_0^x C_n(s)ds \\
=& -\frac{1}{2}k^{-1}\sigma^2 x^2 e^{-x} \\
=&: I(x).
\end{aligned} \tag{4.13}$$

For all $x \in \mathbb{R}$,

$$\frac{ka_n}{\sigma^2(a_ns+b_n)(\log(a_ns+b_n))^{1-2k}} - 1 + k^{-1}\sigma^2 s(\log b_n)^{1-2k}$$
$$= \left(1+k^{-1}\sigma^2(\log b_n)^{1-2k}s\right)^{-1}\left((2k-1)\left(k^{-1}\sigma^2(\log b_n)^{-2k}s - \frac{1}{2}k^{-2}\sigma^4(\log b_n)^{1-4k}s^2\right)\right.$$
$$\left. + k^{-2}\sigma^4(\log b_n)^{2-4k}s^2 + O\left((\log b_n)^{2-6k}\right)\right)$$

for large n, which implies

$$(\log b_n)^{2k-1+\lambda}\left(\frac{ka_n}{\sigma^2(a_ns+b_n)(\log(a_ns+b_n))^{1-2k}} - 1 + k^{-1}\sigma^2 s(\log b_n)^{1-2k}\right)$$
$$\to \begin{cases} k^{-2}\sigma^4 s^2, & \text{if } \frac{1}{2}<k<1, \\ \sigma^2 s(1+\sigma^2 s), & \text{if } k=1, \\ (2k-1)k^{-1}\sigma^2 s, & \text{if } k>1 \end{cases} \tag{4.14}$$

and

$$\frac{a_n(\log b_n)^{2k-1+\lambda}}{(a_ns+b_n)\log(a_ns+b_n)} \to \begin{cases} 0, & \text{if } \frac{1}{2}<k<1, \\ \sigma^2, & \text{if } k=1, \\ k^{-1}\sigma^2, & \text{if } k>1, \end{cases} \tag{4.15}$$

as $n \to \infty$.

By (4.14), (4.15) and Lemma 4.2, we have

$$\lim_{n\to\infty}(\log b_n)^{\lambda}\left((\log b_n)^{2k-1}v_n(x) - I(x)\right)$$
$$= \lim_{n\to\infty}\frac{-(1-F_k(a_nx+b_n)) + (1-F_k(b_n))e^{-x}\left(1-I(x)e^x(\log b_n)^{1-2k}\right)}{n^{-1}(\log b_n)^{1-2k-\lambda}}$$
$$= \lim_{n\to\infty}\frac{1-F_k(a_nx+b_n)}{n^{-1}}\frac{\frac{1-F_k(b_n)}{1-F_k(a_nx+b_n)}e^{-x}\left(1-I(x)e^x(\log b_n)^{1-2k}\right)-1}{(\log b_n)^{1-2k-\lambda}}$$
$$=e^{-x}\lim_{n\to\infty}\left[(\log b_n)^{2k+\lambda-1}(B_n(x)-1) + B_n(x)(\log b_n)^{2k+\lambda-1}\int_0^x\left(C_n(s)+k^{-1}\sigma^2 s(\log b_n)^{1-2k}\right)ds\right.$$
$$- B_n(x)I(x)e^x(\log b_n)^{\lambda}\int_0^x C_n(s)ds$$
$$\left.+\frac{1}{2}B_n(x)(\log b_n)^{2k+\lambda-1}\left(1-I(x)e^x(\log b_n)^{1-2k}\right)\left(\int_0^x C_n(s)ds\right)^2(1+o(1))\right]$$
$$= \begin{cases} k^{-2}\sigma^4x^3\left(\frac{1}{3}-\frac{1}{4}x\right)e^{-x}, & \text{if } \frac{1}{2}<k<1, \\ \sigma^2 x\left(1+\frac{1}{2}x+\frac{1}{3}\sigma^2x^2-\frac{1}{4}\sigma^2x^3\right)e^{-x}, & \text{if } k=1, \\ k^{-1}\sigma^2 x\left(\frac{1}{2}(2k-1)x+1\right)e^{-x}, & \text{if } k>1, \end{cases}$$
$$=: J_k(x),$$

here $\lambda = 1\bigwedge(2k-1)$. The proof is completed. □

Proof of Theorem 4.1. By Lemma 4.3, we have

$$(\log b_n)^{2k-1+\lambda}v_n^2(x) \to \begin{cases} I^2(x), & \text{if } \frac{1}{2}<k\le 1, \\ 0, & \text{if } k>1 \end{cases} \tag{4.16}$$

as $n \to \infty$. Once again by Lemma 4.3, we have

$$\begin{aligned}
&(\log b_n)^\lambda \left((\log b_n)^{2k-1}(F_k^n(a_nx+b_n)-\Lambda(x))-I(x)\Lambda(x)\right)\\
=&(\log b_n)^\lambda \left((\log b_n)^{2k-1}(\exp(u_n(x))-1)-I(x)\right)\Lambda(x)\\
=&\left((\log b_n)^\lambda \left((\log b_n)^{2k-1}v_n(x)-I(x)\right)+(\log b_n)^{2k+\lambda-1}v_n^2(x)\left(\frac{1}{2}+O(v_n(x))\right)\right)\Lambda(x)\\
\to& l(x)\Lambda(x),
\end{aligned}$$

where $l(x)$ is provided by Theorem 4.1. The proof is completed.

□

Acknowledgements

This work was supported by Natural Science Foundation of China (No. 61273020) and Fundamental Research Funds for the Central Universities(No. XDJK2015A007) and the Science and Technology Plan Project of Guizhou Province(No. LH[2015]7001, No. LH[2015]7049, No. LH[2015]70055).

References

[1] Ahuja, N. (1998). Gaussian mixture model for human skin color and its applications in image and video databases. *Proceedings of SPIE-The International Society for Optical Engineering.* **3656,** 458-466.

[2] Al-Bender, F., Lampaert, V., Swevers, J. (2005). The generalized Maxwell-slip model: a novel model for friction Simulation and compensation. *IEEE T. Automat. Contr.* **50,** 1883-1887.

[3] Arbab, AI., Satti, Z. A. (2009). On the Generalized Maxwell Equations and Their Prediction of Electroscalar Wave. *Prog. Phys.* **2,** 8-13.

[4] Finner, H., Dickhaus, T., Roters, M. (2007). Dependency and false discovery rate: asymptotics. *Ann. Statist.* **35,** 1432-1455.

[5] Finner, H., Dickhaus. T, Roters, M. (2008). Asymptotic Tail Properties of Student's t-Distribution. *Commun. Stat. Theory Methods.* **37,** 175-179.

[6] Huang. J, Chen, S. (2015). Tail behavior of the generalized Maxwell distribution. *Commun. Stat. Theory Methods.* doi:10.1080/03610926.2014.917678

[7] Leadbetter, M. R., Lindgren, G., Rootzén, H. (1983). *Extremes and Related Properties of Random Sequences and Processes.* Springer-Verlag, New York.

[8] Mills, J. P. (1926). Table of the ratio: area to bounding ordinate, for any portion of normal curve. *Biometrika.* **18,** 359-400.

[9] Mladenović, P. (1999). Extreme values of the sequences of independent random variables with mixed distributions. *Mat. Vesnik.* **51,** 29-37.

[10] Monsia, M. D. (2011). A simplified nonlinear generalized Maxwell model for predicting the time dependent behavior of viscoelastic materials. *World J. Mech.* **1,** 158-167.

[11] Peng, Z., Tong, B., Nadarajah, S. (2009). Tail behavior of the general error distribution. *Commun. Stat. Theory Methods.* **38,** 1884-1892.

[12] Peng, Z., Weng, Z., Nadarajah, S. (2010). Rates of convergence of extremes for mixed exponential distributions. *Math. Comput. Simulat.* **81,** 92-99.

[13] Resnick, S. I. (1987). *Extreme value, Regular Variation, and Point Processes.* Springer-Verlag, New York.

[14] Roederk, K. (1994). A graphical technique for determining the number of components in a mixture of normals. *J. Am. Stat. Assoc.* **89,** 487-495.

[15] Schiessel, H., Metzler, R., Blumen, A., Nonnenmacher, T. F. (1995). Generalized viscoelastic model: their fractional equations with solutions. *J. Phys. A. Math. Gen.* **28,** 6567-6584.

[16] Zhang, W., Guo, X., Gs, K. (2008). A generalized Maxwell model for creep behavior of artery opening angle. *J. Biomech. Eng.* **130,** 1-16.

[17] Vodă, V. G. (2009). A modified Weibull hazard rate as generator of a generalized Maxwell distribution. *Math. Rep.* **11,** 171-179.