Tail properties and asymptotic distribution for maximum of LGMD [*]

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Abstract. We introduce logarithmic generalized Maxwell distribution which is an extension of the generalized Maxwell distribution. Some interesting properties of this distribution are studied and the asymptotic distribution of the partial maximum of an independent and identically distributed sequence from the logarithmic generalized Maxwell distribution is gained.

Keywords. Extreme value distribution; Logarithmic generalized Maxwell distribution; Mills' ratio; Partial Maximum; Tail Properties.

Mathematics Subject Classification: Primary 62E20, 60E05; Secondary 60F15, 60G15.

1 Introduction

The generalized Maxwell distribution (GMD for short), a generalization of ordinary Maxwell distribution, is proposed by Vodǎ (2009). The GMD has a variety of applications in statistics, physics and chemistry. Its most common application is in statistical mechanics. Some recent examples of this have: constructing fractional rheological constitutive equations (Schiessel et al., 1995); be friction model suitable for quick simulation and control (Farid et al., 2005); forecasting the temporal change of opening angle in multiple time scales and electroscalar wave (Zhang et al., 2008; Arbab and Satti, 2009); project of the time related to behavior of viscoelastic materials (Monsia, 2011).

The probability density function (pdf) and the cumulative distribution function (cdf) of the GMD with the parameter $k > 0$ are respectively,

$$g_k(x) = \frac{k}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)}x^{2k}\exp\left(-\frac{x^{2k}}{2\sigma^2}\right)$$

and

$$G_k(x) = \int_{-\infty}^{x} g_k(t)\,\mathrm{d}t$$

for $x > 0$, where σ is a positive constant and $\Gamma(\cdot)$ is the Gamma function.

Mills (1926) gave a well-known inequality and Mills' ratio result for the standard normal cdf $\Phi(x)$ with pdf $\phi(x)$ as follows:

$$x^{-1}(1 + x^{-2})^{-1}\phi(x) < \Phi(-x) < x^{-1}\phi(x), \tag{1.1}$$

for $x > 0$ and

$$\frac{\Phi(-x)}{\phi(x)} \sim \frac{1}{x}, \tag{1.2}$$

as $x \to \infty$.

Peng et al. (2009) extended the Mills' results to the case of the general error distribution:

$$\frac{2\lambda^v}{v}x^{1-v}\left(1 + \frac{2(v-1)\lambda^v}{v}x^{-v}\right)^{-1} < \frac{T_v(-x)}{t_v(x)} < \frac{2\lambda^v}{v}x^{1-v}, \tag{1.3}$$

for $v > 1$ and $x > 0$, and for $v > 0$

$$\frac{T_v(-x)}{t_v(x)} \sim \frac{2\lambda^v}{v}x^{1-v}, \tag{1.4}$$

as $x \to \infty$, where $\lambda = \left[\frac{2^{-2/v}\Gamma(1/v)}{\Gamma(3/v)}\right]^{1/2}$, and $T_v(x)$ is the cdf with pdf $t_v(x)$. Huang and Chen (2014) investigated the similar results of GMD, i.e.,

$$\frac{\sigma^2}{k}x^{1-2k} < \frac{G_k(-x)}{g_k(x)} < \frac{\sigma^2}{k}x^{1-2k}\left(1 + \left(\frac{\sigma^2}{k}x^{2k} - 1\right)^{-1}\right), \tag{1.5}$$

for $k > 1/2$, $\sigma > 0$ and $x > 0$, and for $k > 0$,

$$\frac{G_k(-x)}{g_k(x)} \sim \frac{\sigma^2}{k}x^{1-2k}, \tag{1.6}$$

as $x \to \infty$. The above-mentioned inequalities such as (1.1), (1.3), and (1.5) and Mills' type-ratios such as (1.2), (1.4), and (1.6) play an important role in considering some tail behavior and extremes of economic and financial data.

In this paper, we define the logarithmic generalized Maxwell distribution (denoted by LGMD), which is a natural extension of the generalized Maxwell distribution. One motivation of considering LGMD is to obtain more efficient results as parameter estimators when random models were supposed with the LGMD error terms instead of normal ones. Meanwhile, the LGMD can be expected to be a better model for certain modern areas.

The present paper is to derive the Mills-type inequality, Mills-type ratio, and the tail distributional representation for the LGMD. As an important application, the asymptotic distribution of the partial maximum of independent and identically distributed (i.i.d.) with common LGMD is investigated.

First we provide the definition of LGMD.

Definition 1.1. *Let X denote a random variable which obeys the GMD. Set $Y = \exp(X)$. Then we call that Y obeys the LGMD, denoted by $Y \sim LGMD(k)$ with parameter $k > 0$.*

Easily check that the pdf of $Y \sim LGMD(k)$ is

$$f_k(x) = \frac{kx^{-1}}{2^{k/2}\sigma^{2+1/k}\Gamma(1+k/2)}(\log x)^{2k}\exp\left(-\frac{(\log x)^{2k}}{2\sigma^2}\right)$$

for $x > 0$, where parameter $k > 0$, and σ is a positive constant. Let $F_k(x)$ denote the cdf of Y, i.e.,

$$F_k(x) = \int_0^x f_k(t)\,\mathrm{d}t$$

for $x > 0$. Note that the LGMD reduces to the logarithmic Maxwell distribution when $k = 1$.

The rest of the article is organized as follows. In Sec. 2, we derive some results concerning Mills-type ratios and tail behavior of LGMD. Sec. 3, we consider the limit distribution of the partial maximum of i.i.d random variables following the LGMD and the suitable norming constants needed. The result is also extended to the case of a finite mixture of LGMDs.

2 Mills' Ratio and Tail Properties of LGMD

In this section, we derive some results including Mills' inequality, Mills' ratio of LGMD.

For LGMD and GMD, note that $1 - G_k(\log x) = 1 - F_k(x)$ and

$$\frac{1 - G_k(\log x)}{x^{-1}g_k(\log x)} = \frac{1 - F_k(x)}{f_k(x)}.$$

So, by Lemma 2.2 and Theorem 2.1 in Huang and Chen (2014), we have the following two results.

Theorem 2.1. *Let F_k and f_k respectively denote the cdf and pdf of LGMD with parameter $k > 1/2$. We have the inequality below for all $x > 1$,*

$$\frac{\sigma^2}{k}x(\log x)^{1-2k} < \frac{1 - F_k(x)}{f_k(x)} < \frac{\sigma^2}{k}x(\log x)^{1-2k}\left(1 + \left(\frac{\sigma^2}{k}(\log x)^{2k} - 1\right)^{-1}\right), \qquad (2.1)$$

where σ is a positive constant.

Corollary 2.1. *For fixed $k > 0$, as $x \to \infty$, we have*

$$\frac{1 - F_k(x)}{f_k(x)} \sim \frac{\sigma^2}{k}x(\log x)^{1-2k}. \qquad (2.2)$$

Remark 2.1. *Since the LGMD(k) are reduced to the logarithmic Maxwell distribution as $k = 1$, so by Theorem 2.1 and Corollary 2.1, we derive the inequality and Mills' ratio of the logarithmic Maxwell distribution, i.e.,*

$$\sigma^2 x(\log x)^{-1}f_1(x) < 1 - F_1(x) < \sigma^2 x(\log x)^{-1}(1 + (\sigma^2(\log x)^2 - 1)^{-1})f_1(x),$$

for $x > 1$ and

$$\frac{1 - F_1(x)}{f_1(x)} \sim \frac{\sigma^2 x}{\log x},$$

as $x \to \infty$.

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Remark 2.2. *For $k > 1/2$, Corollary 2.1 gives $F_k \in D(\Lambda)$, i.e., there exist norming constants $\alpha_n > 0$ and $\beta_n \in \mathbb{R}$ which make sure $F_k^n(\alpha_n x + \beta_n)$ converges to $\exp(-\exp(-x))$. Since*

$$\frac{(d/dx)f_k(x)}{f_k(x)} = -\frac{1}{x}\left(1 - \frac{2k}{\log x} + \frac{k}{\sigma^2}(\log x)^{2k-1}\right),$$

by Corollary 2.1, we have

$$\frac{1 - F_k(x)}{f_k(x)}\frac{(d/dx)f_k(x)}{f_k(x)} \to -1$$

as $x \to \infty$. Hence, it follows by Proposition 1.18 in Resnick (1987) that $F_k \in D(\Lambda)$. The choice of norming constants α_n and β_n is discussed by Theorem 3.2.

Finner et al. (2008) investigated the asymptotic behavior of the ratio of the Student's t and normal distributions as the degrees of freedom $u = u(x)$ satisfies

$$\lim_{x \to \infty} \frac{x^4}{u} = \beta \in [0, \infty). \tag{2.3}$$

The main motivation of the work is to consider the false discovery rate in multiple testing problems with large numbers of hypotheses and extremely small critical values for the smallest ordered p value; in detail, see Finner et al. (2007). In this section, we study the asymptotic behavior of the ratio of pdfs and the ratio of the tails of the LGMD and the logarithmic Maxwell distribution. Firstly, we consider the case of $k \to 1$. Secondly, we consider the case of $x \to \infty$ for fixed k.

Theorem 2.2. *For $k > 0$, let $x = x(k)$ be such that*

$$k - 1 = \frac{\gamma}{2(\log x)^2 \log \log x}$$

for some $\gamma \in \mathbb{R}$. Then

$$\lim_{k \to 1} \frac{f_1(x)}{f_k(x)} = \exp\left(\frac{\gamma}{2\sigma^2}\right) \tag{2.4}$$

and

$$\lim_{k \to 1} \frac{1 - F_1(x)}{1 - F_k(x)} = \exp\left(\frac{\gamma}{2\sigma^2}\right). \tag{2.5}$$

Proof. Note that $\frac{2^{(k+1)/2}\sigma^{2+1/k}\Gamma(1+k/2)}{k\sigma^3\pi^{1/2}} \to 1$ as $k \to 1$, so

$$\begin{aligned}
\lim_{k \to 1} \frac{f_1(x)}{f_k(x)} &= \lim_{k \to 1}(\log x)^{2-2k}\exp\left(\frac{(\log x)^{2k}}{2\sigma^2} - \frac{(\log x)^2}{2\sigma^2}\right) \\
&= \lim_{k \to 1}\exp\left(\frac{(\log x)^2}{2\sigma^2}\left((\log x)^{2k-2} - 1\right)\right) \\
&= \lim_{k \to 1}\exp\left(\frac{(\log x)^2}{2\sigma^2}(\exp((2k-2)\log\log x) - 1)\right) \\
&= \lim_{k \to 1}\exp\left(\frac{(\log x)^2}{2\sigma^2}\left(\exp\left(\frac{\gamma}{(\log x)^2}\right) - 1\right)\right) \\
&= \exp\left(\frac{\gamma}{2\sigma^2}\right).
\end{aligned}$$

The condition of the theorem deduces $(\log x)^{2-2k} \to 1$ as $k \to 1$. According to Corollary 2.1, Remark 2.1 and (2.4), (2.5) can be deduced. $\square$

Theorem 2.3. *For fixed k, we have*

$$\frac{f_1(x)}{f_k(\exp((\log x)^{1/k}))} = \frac{2^{(k+1)/2}\Gamma(1+k/2)\exp((\log x)^{1/k})}{\pi^{1/2}k\sigma^{1-1/k}x} \tag{2.6}$$

and

$$\lim_{x\to\infty}\frac{(\log x)^{1/k-1}(1-F_1(x))}{1-F_k(\exp((\log x)^{1/k}))} = \frac{2^{(k+1)/2}\Gamma(1+k/2)}{\pi^{1/2}\sigma^{1-1/k}}. \tag{2.7}$$

Proof. Note that (2.6) follows from fundamental calculation. By Corollary 2.1, Remark 2.1 and (2.6), we have

$$\lim_{x\to\infty}\frac{(\log x)^{1/k-1}(1-F_1(x))}{1-F_k(\exp((\log x)^{1/k}))}$$
$$= \lim_{x\to\infty}\frac{kx}{\exp((\log x)^{1/k})}\frac{f_1(x)}{f_k(\exp((\log x)^{1/k}))}$$
$$= \frac{2^{(k+1)/2}\Gamma(1+k/2)}{\pi^{1/2}\sigma^{1-1/k}},$$

so (2.7) follows. □

3 Asymptotic Distribution of the Maximum

By applying Corollary 2.1, we could establish the distributional tail representation for the LGMD.

Theorem 3.1. *Under the conditions of Theorem* 2.1*, we have*

$$1 - F_k(x) = c(x)\exp\left(-\int_e^x \frac{g(t)}{f(t)}\,\mathrm{d}t\right)$$

for large enough x, where

$$c(x) = \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\left(-1/(2\sigma^2)\right)(1+\theta_1(x))$$

and

$$f(t) = \frac{\sigma^2}{k}t(\log t)^{1-2k},\ \ g(t) = 1 - \frac{\sigma^2}{k}(\log t)^{-2k},$$

where $\theta_1(x)\to 0$ as $x\to\infty$.

Proof. For large enough x, by Corollary 2.1, we have

$$1 - F_k(x) = \frac{\sigma^2}{k}(\log x)^{1-2k}xf_k(x)(1+\theta_1(x))$$

$$= \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\left(\log\log x - \frac{(\log x)^{2k}}{2\sigma^2}\right)(1+\theta_1(x))$$

$$= \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)}\exp\left(-\frac{1}{2\sigma^2}\right)\exp\left(-\int_e^x\left(\frac{k(\log t)^{2k-1}}{\sigma^2 t} - \frac{1}{t\log t}\right)\mathrm{d}t\right)(1+\theta_1(x))$$

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$$= \frac{1}{2^{k/2}\sigma^{1/k}\Gamma(1+k/2)} \exp\left(-\frac{1}{2\sigma^2}\right) \exp\left(-\int_e^x \frac{1-k^{-1}\sigma^2(\log t)^{-2k}}{k^{-1}\sigma^2 t(\log t)^{1-2k}}\, dt\right)(1+\theta_1(x))$$

$$= c(x)\exp\left(-\int_e^x \frac{g(t)}{f(t)}\, dt\right),$$

where $\theta_1(x) \to 0$ as $x \to \infty$. The desired result follows. $\qquad\square$

Remark 3.1. *As $\lim_{t\to\infty} g(t) = 1$, $f(t) > 0$ on $[1,\infty)$ is absolutely continuous function and $\lim_{t\to\infty} f'(t) = 0$ in Theorem 3.1, an application of Theorem 3.1 and Corollary 1.7 in Resnick (1987) shows $F_k \in D(\Lambda)$, where $D(\Lambda)$ denotes the domain of attraction $\Lambda(x) = \exp(-\exp(-x))$.*

In this we consider the asymptotic distribution of the normalized maximum of a sequence of i.i.d. random variables following LGMD. Remark 2.2 and Theorem 3.1 showed that the distribution of partial maximum converges to $\Lambda(x)$. So, the following work is to find the suitable norming constants.

Theorem 3.2. *Let $\{X_n, n \geq 1\}$ be an i.i.d. sequence from the LGMD with $k > 1/2$. Let $M_n = \max\{X_k, 1 \leq k \leq n\}$. We have*

$$\lim_{n\to\infty} P(M_n \leq \alpha_n x + \beta_n) = \exp(-\exp(-x)),$$

where

α_n
$$= \frac{\sigma^2 \exp\left(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)}\right)\left(1 + \frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2))\right)}{k\left(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)} + \log\left(1 + \frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2))\right)\right)^{2k-1}}$$

and

β_n
$$= \exp\left(2^{1/(2k)}\sigma^{1/k}(\log n)^{1/(2k)}\right)\left(1 + \frac{\sigma^{1/k}(\log n)^{1/(2k)-1}}{2^{2-1/(2k)}k^2}\left(\log\log n - (k^2-1)\log 2 - 2k\log\Gamma(1+k/2)\right)\right).$$

Proof. Since $F_k \in D(\Lambda)$, there must be norming constants $a_n > 0$ and $b_n \in \mathbb{R}$ which make sure that $\lim_{n\to\infty} P((M_n - b_n)/a_n \leq x) = \exp(-\exp(-x))$. By Proposition 1.1 in Resnick (1987) and Theorem 3.1, the norming constants can be chosen that a_n and b_n satisfy the equations: $b_n = (1/(1-F_k))^{\leftarrow}(n)$ and $a_n = f(b_n)$. Note that $F_k(x)$ is continuous, then $1 - F_k(b_n) = n^{-1}$. By Corollary 2.1, we have
$$nk^{-1}\sigma^2(\log b_n)^{1-2k}b_n f_k(b_n) \to 1,$$

as $n \to \infty$, i.e.,
$$n2^{-\frac{k}{2}}\sigma^{-\frac{1}{k}}\Gamma^{-1}\left(1+\frac{k}{2}\right)\log b_n \exp\left(-\frac{(\log b_n)^{2k}}{2\sigma^2}\right) \to 1,$$

as $n \to \infty$, and so
$$\log n - \frac{k}{2}\log 2 - \frac{1}{k}\log\sigma - \log\Gamma\left(1+\frac{k}{2}\right) + \log\log b_n - \frac{(\log b_n)^{2k}}{2\sigma^2} \to 0, \qquad (3.1)$$

as $n \to \infty$, which deduces
$$\frac{(\log b_n)^{2k}}{2\sigma^2\log n} \to 1,$$

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as $n \to \infty$, thus
$$2k \log \log b_n - \log 2 - 2 \log \sigma - \log \log n \to 0,$$

as $n \to \infty$, hence
$$\log \log b_n = \frac{1}{2k}(\log 2 + 2 \log \sigma + \log \log n) + o(1).$$

Putting the above equality into (3.1), we have
$$(\log b_n)^{2k} = 2\sigma^2 \left(\log n + \frac{1}{2k} \log \log n - \frac{k^2 - 1}{2k} \log 2 - \log \Gamma \left(1 + \frac{k}{2} \right) \right) + o(1),$$

which deduces that
$$\log b_n = 2^{\frac{1}{2k}} \sigma^{\frac{1}{k}} (\log n)^{\frac{1}{2k}} \left(1 + \frac{\log \log n - (k^2 - 1) \log 2 - 2k \log \Gamma(1 + \frac{k}{2})}{2^2 k^2 \log n} + o \left((\log n)^{-1} \right) \right),$$

therefore
$$
\begin{aligned}
b_n &= \exp \left(2^{\frac{1}{2k}} \sigma^{\frac{1}{k}} (\log n)^{\frac{1}{2k}} \right) \left(1 + \frac{\sigma^{\frac{1}{k}} (\log n)^{\frac{1}{2k} - 1}}{2^{2 - \frac{1}{2k}} k^2} \left(\log \log n - (k^2 - 1) \log 2 - 2k \log \Gamma(1 + \frac{k}{2}) \right) \right. \\
&\quad \left. + o \left((\log n)^{\frac{1}{2k} - 1} \right) \right) \\
&= \beta_n + o \left((\log n)^{\frac{1}{2k} - 1} \exp \left(2^{\frac{1}{2k}} \sigma^{\frac{1}{k}} (\log n)^{\frac{1}{2k}} \right) \right),
\end{aligned}
$$

where
$$\beta_n = \exp \left(2^{1/(2k)} \sigma^{1/k} (\log n)^{1/(2k)} \right) \left(1 + \frac{\sigma^{1/k} (\log n)^{1/(2k) - 1}}{2^{2 - 1/(2k)} k^2} (\log \log n - (k^2 - 1) \log 2 - 2k \log \Gamma(1 + k/2)) \right).$$

Hence, we have
$$\alpha_n = f(\beta_n)$$
$$= \frac{\sigma^2 \exp(2^{1/(2k)} \sigma^{1/k} (\log n)^{1/(2k)}) \left(1 + \frac{\sigma^{1/k} (\log n)^{1/(2k) - 1}}{2^{2 - 1/(2k)} k^2} (\log \log n - (k^2 - 1) \log 2 - 2k \log \Gamma(1 + k/2)) \right)}{k \left(2^{1/(2k)} \sigma^{1/k} (\log n)^{1/(2k)} + \log \left(1 + \frac{\sigma^{1/k} (\log n)^{1/(2k) - 1}}{2^{2 - 1/(2k)} k^2} (\log \log n - (k^2 - 1) \log 2 - 2k \log \Gamma(1 + k/2)) \right) \right)^{2k-1}}.$$

It is easy to check that $\lim_{n \to \infty} \alpha_n / a_n = 1$ and $\lim_{n \to \infty} (b_n - \beta_n) / \alpha_n = 0$. Hence, by Theorem 1.2.3 in Leadbetter et al. (1983), the result follows. □

Remark 3.2. *Theorem 3.2 shows that the limit distribution of the normalized maximum from the logarithmic Maxwell distribution is the Gumbel extreme value distribution with norming constants*
$$\alpha_n = \frac{\sigma^2 \exp \left(2^{1/2} \sigma (\log n)^{1/2} \right) \left(1 + \frac{\sigma}{2^{3/2} (\log n)^{1/2}} \left(\log \log n - 2 \log(\pi^{1/2}/2) \right) \right)}{2^{1/2} \sigma (\log n)^{1/2} + \log \left(1 + \frac{\sigma}{2^{3/2} (\log n)^{1/2}} \left(\log \log n - 2 \log(\pi^{1/2}/2) \right) \right)}$$

and
$$\beta_n = \exp \left(2^{1/2} \sigma (\log n)^{1/2} \right) \left(1 + \frac{\sigma}{2^{3/2} (\log n)^{1/2}} \left(\log \log n - 2 \log(\pi^{1/2}/2) \right) \right).$$

At the end of this section, we extend the result of Theorem 3.2 to the case of a finite mixture of LGMDs.

Finite mixture distributions or models have been widely applied in various areas like Chemistry (Roeder, 1994) and image and video databases (Yang and Ahuja, 1998). Recently, some extreme statistical scholars have also studied them. Mladenović (1999) have considered extreme values of the sequences of independent random variables with common mixed distributions containing normal, Cauchy and uniform distributions. Peng et al. (2010) have investigated the limit distribution and its corresponding uniform convergence rate for a finite mixed of exponential distribution.

If the distribution function (df) F of a random variable ξ have

$$F(x) = p_1 F_1(x) + p_2 F_2(x) + \cdots + p_r F_r(x),$$

we say that ξ obeys a finite mixture distribution F, where $F_i, 1 \leq i \leq r$ denote different dfs of the mixture components. The weight coefficients have the condition that $p_i > 0, i = 1, 2, \cdots, r$ and $\sum_{j=1}^{r} p_j = 1$.

Next, we consider the extreme value distribution from a finite mixture with component dfs F_{k_i} obeying LGMD(k_i), where the parameter $k_i > 1$ for $1 \leq i \leq r$ and $k_i \neq k_j$ for $i \neq j$. Denote the df of the finite mixture by

$$F(x) = p_1 F_{k_1}(x) + p_2 F_{k_2}(x) + \cdots + p_r F_{k_r}(x) \tag{3.2}$$

for $x > 0$.

Theorem 3.3. *Let $\{Z_n, n \geq 1\}$ be a sequence of i.i.d. random variables following the common df F given by (3.2). Let $M_n = \max\{Z_1, Z_2, \cdots, Z_n\}$. Then*

$$\lim_{n \to \infty} P\left(\frac{M_n - \beta_n}{\alpha_n} \leq x\right) = \exp(-\exp(-x))$$

holds with the norming constants

$$\alpha_n = \frac{\sigma^{1/k} \exp\left(2^{1/(2k)} \sigma^{1/k} (\log n)^{1/(2k)}\right)}{2^{1-1/(2k)} k (\log n)^{1-1/(2k)}}$$

and

$$\beta_n = \exp\left(2^{1/(2k)} \sigma^{1/k} (\log n)^{1/(2k)}\right) \left(1 + \frac{\sigma^{1/k} (\log n)^{1/(2k)-1}}{2^{2-1/(2k)} k^2} \left(\log \log n + 2k \log p - (k^2 - 1) \log 2\right.\right.$$

$$\left.\left. - 2k \log \Gamma(1 + k/2)\right)\right),$$

where $\sigma = \max\{\sigma_1, \cdots, \sigma_r\}$, and $p = p_{i_1} + \cdots + p_{i_j}$, where $i_s \in \{i, \sigma_i = \sigma \text{ and } k = k_i\}, 1 \leq s \leq j \leq r$ and $k = \min\{k_1, \cdots, k_r\}$.

Proof. By (3.2), we have

$$1 - F(x) = \sum_{i=1}^{r} p_i (1 - F_{k_i}(x)).$$

By Theorem 2.1, we have

$$\sum_{i=1}^{r} \frac{p_i \sigma_i^2}{k_i} (\log x)^{1-2k_i} x f_{k_i}(x) < 1 - F(x)$$

$$< \sum_{i=1}^{r} \frac{p_i \sigma_i^2}{k_i} (\log x)^{1-2k_i} x \left(1 + (\frac{\sigma_i^2}{k_i} (\log x)^{2k_i} - 1)^{-1} \right) f_{k_i}(x)$$

for all $x > 1$, according to the definition of f_k, which implies

$$\frac{p \log x}{2^{\frac{k}{2}} \sigma^{\frac{1}{k}} \Gamma(1 + \frac{k}{2})} \exp \left(-\frac{(\log x)^{2k}}{2\sigma^2} \right) (1 + A_k(x)) < 1 - F(x)$$

$$< \frac{p \log x}{2^{\frac{k}{2}} \sigma^{\frac{1}{k}} \Gamma(1 + \frac{k}{2})} \left(1 + (\frac{\sigma^2}{k} (\log x)^{2k} - 1)^{-1} \right) \exp \left(-\frac{(\log x)^{2k}}{2\sigma^2} \right) (1 + B_k(x)), \qquad (3.3)$$

where

$$A_k(x) = \sum_{k_i \neq k} \frac{2^{\frac{k}{2}} p_i \sigma^{\frac{1}{k}} \Gamma(1 + \frac{k}{2})}{2^{\frac{k_i}{2}} p \sigma^{\frac{1}{k_i}} \Gamma(1 + \frac{k_i}{2})} \exp \left(\frac{(\log x)^{2k}}{2\sigma^2} - \frac{(\log x)^{2k_i}}{2\sigma_i^2} \right) \to 0 \qquad (3.4)$$

and

$$B_k(x) = \sum_{k_i \neq k} \frac{2^{\frac{k}{2}} p_i \sigma^{\frac{1}{k}} \Gamma(1 + \frac{k}{2})}{2^{\frac{k_i}{2}} p \sigma^{\frac{1}{k_i}} \Gamma(1 + \frac{k_i}{2})} \frac{1 + (\frac{\sigma_i^2}{k_i} (\log x)^{2k_i} - 1)^{-1}}{1 + (\frac{\sigma^2}{k} (\log x)^{2k} - 1)^{-1}} \exp \left(\frac{(\log x)^{2k}}{2\sigma^2} - \frac{(\log x)^{2k_i}}{2\sigma_i^2} \right) \to 0 \quad (3.5)$$

as $x \to \infty$ since $k = \min\{k_1, k_2, \cdots, k_r\}$. Combining (3.3)-(3.5) and (2.2) together, for large enough x, we obtain

$$1 - F(x) \sim p(1 - F_k(x)) \qquad (3.6)$$

as $x \to \infty$, where F_k denotes the cdf of the LGMD(k), and σ and p are defined by Theorem 3.3. By Proposition 1.19 in Resnick (1987), we can derive $F \in D(\Lambda)$. The norming constants can be obtained by Theorem 3.2 and (3.6). The proof is complete. $\qquad \square$

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