On the Natural Logarithm Function and its Applications

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"We love him, because he first loved us." - I John 4:19

Abstract. In present article, we create new integral representations for the natural logarithm function, the Euler-Mascheroni constant, the natural logarithm of Riemann zeta function and the first derivative of Riemann zeta function.

1. Introduction

In this paper, we prove the amazing integral representations for the natural logarithm function, the Euler-Mascheroni constant, the natural logarithm of Riemann zeta function and the first derivative of Riemann zeta function, as follows:

$$\ln x = (x-1)\int_0^1 \frac{du}{u(x-\ln u)(1-\ln u)},$$

$$\ln x = \int_0^\infty \int_0^\infty \left[\sin u - \frac{\sin(u\sqrt{x})}{\sqrt{x}}\right] e^{-u\sqrt{t}} du dt,$$

$$\frac{1}{x(x-1)} - \frac{\ln x}{(x-1)^2} = \int_0^1 \frac{du}{u(\ln u - 1)(\ln u - x)^2},$$

$$\gamma = 1 - \int_0^1 \frac{1}{u(1-\ln u)^2}\left[\frac{1}{\ln u} + \frac{1}{\ln(1-\ln u)}\right] du,$$

$$\gamma = -\int_0^\infty \frac{t - e^{\frac{1}{t}}(1+t)\Gamma\left(0, \frac{1}{t}\right)}{(1+t)t^2} dt,$$

$$\gamma = -2\int_0^\infty \frac{\sin u - 2F\left(\frac{u}{2}\right)}{u^2} du,$$

$$\ln \zeta(x) = \int_0^1 \frac{1}{u(\ln u - 1)}\left[\frac{(x-1)\zeta(x) - 1}{\ln u - (x-1)\zeta(x)} + \frac{2-x}{(x-1)\ln u - 1}\right] du,$$

$$\zeta'(2) = \frac{1}{6}\int_0^\infty \frac{\pi^2 t - 6(1+t)[\gamma + \psi(1+t)]}{(1+t)t^2} dt$$

and

$$\zeta'(3) = \frac{1}{6}\int_0^\infty \frac{6(1+t)[\gamma + \psi(1+t)] - t[\pi^2(1+t) - 6t\zeta(3)]}{(1+t)t^3} dt.$$

2. Definition and Lemma

Definition 1. *The natural logarithm function can be defined by the Frullani integral as follows*

$$\ln x \equiv \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt, \tag{1}$$

for $\mathrm{Re}(x) > 0$.

Lemma 2. *If* $\mathrm{Re}(x) > 0$, *then*

$$\frac{1}{x} = \frac{1}{\Gamma(x+1)}\int_0^\infty t^{x-1} e^{-t} dt. \tag{2}$$

Proof. We note that the left hand side of (2) can be written as

$$\frac{1}{x}=\int_0^1 t^{x-1}dt=B(x,1)=\frac{\Gamma(x)\Gamma(1)}{\Gamma(x+1)}=\frac{\Gamma(x)}{\Gamma(x+1)}=\frac{1}{\Gamma(x+1)}\int_0^\infty t^{x-1}e^{-t}dt, \tag{3}$$

which is the desired result. □

3. Theorems

Theorem 3. *If* $\mathrm{Im}(x)\neq 0$ *or* $\mathrm{Re}(x)\geq 0$*, then*

$$\ln x=(x-1)\int_0^1 \frac{du}{u(x-\ln u)(1-\ln u)}, \tag{4}$$

where $\ln x$ *denotes the natural logarithm function.*

Proof. Substituting (3) in (1), we encounter

$$\begin{aligned}\ln x&=\int_0^\infty\left(e^{-t}\int_0^1 u^{t-1}du-e^{-xt}\int_0^1 u^{t-1}du\right)dt\\&=\int_0^\infty e^{-t}\int_0^1 u^{t-1}du\,dt-\int_0^\infty e^{-xt}\int_0^1 u^{t-1}du\,dt\\&=\int_0^1\int_0^\infty u^{t-1}e^{-t}dt\,du-\int_0^1\int_0^\infty u^{t-1}e^{-xt}dt\,du.\end{aligned} \tag{5}$$

Observe that

$$e^{-t}=\sum_{k=0}^\infty(-1)^k\frac{t^k}{k!} \tag{6}$$

and

$$e^{-xt}=\sum_{k=0}^\infty(-1)^k\frac{x^kt^k}{k!}. \tag{7}$$

We take (6) and (7) into (5) and obtain

$$\begin{aligned}\ln x&=\int_0^1\int_0^\infty u^{t-1}\sum_{k=0}^\infty(-1)^k\frac{t^k}{k!}dt\,du-\int_0^1\int_0^\infty u^{t-1}\sum_{k=0}^\infty(-1)^k\frac{x^kt^k}{k!}dt\,du\\&=\sum_{k=0}^\infty(-1)^k\frac{1}{k!}\int_0^1\int_0^\infty u^{t-1}t^k\,dt\,du-\sum_{k=0}^\infty(-1)^k\frac{x^k}{k!}\int_0^1\int_0^\infty u^{t-1}t^k\,dt\,du\\&=\sum_{k=0}^\infty(-1)^k\frac{1}{k!}\int_0^1\frac{\Gamma(k+1)(-\ln u)^{-(k+1)}}{u}du-\sum_{k=0}^\infty(-1)^k\frac{x^k}{k!}\int_0^1\frac{\Gamma(k+1)(-\ln u)^{-(k+1)}}{u}du\\&=\sum_{k=0}^\infty(-1)^k\int_0^1\frac{(-\ln u)^{-(k+1)}}{u}du-\sum_{k=0}^\infty(-1)^kx^k\int_0^1\frac{(-\ln u)^{-(k+1)}}{u}du\\&=\sum_{k=0}^\infty x^k\int_0^1\frac{(\ln u)^{-(k+1)}}{u}du-\sum_{k=0}^\infty\int_0^1\frac{(\ln u)^{-(k+1)}}{u}du\\&=\int_0^1\frac{1}{u}\sum_{k=0}^\infty x^k(\ln u)^{-(k+1)}du-\int_0^1\frac{1}{u}\sum_{k=0}^\infty(\ln u)^{-(k+1)}du\\&=\int_0^1\frac{1}{u(\ln u-x)}du-\int_0^1\frac{1}{u(\ln u-1)}du\\&=(x-1)\int_0^1\frac{du}{u(x-\ln u)(1-\ln u)},\end{aligned}$$

which is the desired result. □

Theorem 4. *If* $\mathrm{Re}(z)>0$*, then*

$$\frac{1}{z}=\int_0^1\frac{du}{u(1-\ln u)^{z+1}}, \tag{8}$$

where $\ln u$ *denotes the natural logarithm function.*

Proof. We well-know that

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt, \tag{9}$$

for $\mathrm{Re}(z)>0$.

We substitute (3) into (9), and find

$$\begin{aligned}\Gamma(z)&=\int_0^\infty e^{-t}t^z\int_0^1 u^{t-1}du\,dt\\&=\int_0^\infty\int_0^1 e^{-t}t^zu^{t-1}du\,dt\\&=\int_0^1\int_0^\infty e^{-t}t^zu^{t-1}dt\,du.\end{aligned} \tag{10}$$

We take (6) into (10), and encounter

$$\begin{aligned}\Gamma(z)&=\int_0^1\int_0^\infty\sum_{k=0}^\infty(-1)^k\frac{t^k}{k!}t^zu^{t-1}dt\,du\\&=\int_0^1\sum_{k=0}^\infty(-1)^k\frac{1}{k!}\int_0^\infty t^{z+k}u^{t-1}dt\,du\\&=\int_0^1\sum_{k=0}^\infty(-1)^k\frac{(-\ln u)^{-(k+z+1)}\Gamma(k+z+1)}{k!u}du\\&=\int_0^1\frac{1}{u}\sum_{k=0}^\infty(-1)^k\frac{(-\ln u)^{-(k+z+1)}\Gamma(k+z+1)}{k!}du\\&=\Gamma(z+1)\int_0^1\frac{du}{u(1-\ln u)^{z+1}},\end{aligned}$$

it implies that

$$\frac{\Gamma(z)}{\Gamma(z+1)}=\int_0^1\frac{du}{u(1-\ln u)^{z+1}}\Leftrightarrow\frac{1}{z}=\int_0^1\frac{du}{u(1-\ln u)^{z+1}},$$

which is the desired result. □

4. Applications

4.1. The Natural Logarithm Function.

Corollary 5. *If* $\mathrm{Re}(x)>0$*, then*

$$\frac{1}{x(x-1)}-\frac{\ln x}{(x-1)^2}=\int_0^1\frac{du}{u(\ln u-1)(\ln u-x)^2}, \tag{11}$$

where $\ln x$ *denotes the natural logarithm function.*

Proof. Differentiating both sides in (4) with respect to x, we obtain the desired result. □

4.2. The Euler-Mascheroni constant.

Corollary 6. *We have*

$$\gamma=1-\int_0^1\frac{1}{u(1-\ln u)^2}\left[\frac{1}{\ln u}+\frac{1}{\ln(1-\ln u)}\right]du,$$

where γ *denotes the Euler-Mascheroni constant and* $\ln u$ *denotes the natural logarithm function.*

Proof. We well-know that

$$\gamma=1+\sum_{k=2}^\infty\left(\frac{1}{k}-\int_{k-1}^k\frac{dx}{x}\right). \tag{12}$$

From (8) and (12), it follows that

$$\begin{aligned}\gamma =& 1+\sum_{k=2}^{\infty}\left(\int_0^1 \frac{du}{u(1-\ln u)^{k+1}}-\int_{k-1}^{k}\int_0^1 \frac{du\,dx}{u(1-\ln u)^{x+1}}\right)\\ =&1+\int_0^1\sum_{k=2}^{\infty}\frac{du}{u(1-\ln u)^{k+1}}-\sum_{k=2}^{\infty}\int_0^1\int_{k-1}^{k}\frac{dx\,du}{u(1-\ln u)^{x+1}}\\ =&1-\int_0^1\frac{du}{u(1-\ln u)^2\ln u}-\sum_{k=2}^{\infty}\int_0^1\frac{\ln u\,du}{u(1-\ln u)^{k+1}\ln(1-\ln u)}\\ =&1-\int_0^1\frac{du}{u(1-\ln u)^2\ln u}-\int_0^1\frac{\ln u}{u\ln(1-\ln u)}\sum_{k=2}^{\infty}\frac{du}{(1-\ln u)^{k+1}}\\ =&1-\left(\int_0^1\frac{du}{u(1-\ln u)^2\ln u}+\int_0^1\frac{\ln u\,du}{u(1-\ln u)^2\ln(1-\ln u)}\right)\\ =&1-\int_0^1\frac{1}{u(1-\ln u)^2}\left[\frac{1}{\ln u}+\frac{1}{\ln(1-\ln u)}\right]du,\end{aligned}$$

which is the desired result. □

4.3. The Logarithm of the Riemann Zeta Function.

Corollary 7. *If* $\mathrm{Re}(x)>1$*, then*

$$\ln\zeta(x)=\int_0^1\frac{1}{u(\ln u-1)}\left[\frac{(x-1)\zeta(x)-1}{\ln u-(x-1)\zeta(x)}+\frac{2-x}{(x-1)\ln u-1}\right]du,$$

where $\ln u$ *denotes the natural logarithm function and* $\zeta(x)$ *denotes the Riemman zeta function.*

Proof. We knew [1, page 64] that

$$\ln\zeta(x)=\ln[(x-1)\zeta(x)]+\ln\frac{1}{x-1}.$$

We use the Theorem 3 and encounter

$$\begin{aligned}\ln\zeta(x)=&\int_0^1\frac{(x-1)\zeta(x)-1}{u[(x-1)\zeta(x)-\ln u](1-\ln u)}du\\ &+\left(\frac{1}{x-1}-1\right)\int_0^1\frac{du}{u\left(\frac{1}{x-1}-\ln u\right)(1-\ln u)}\\ =&\int_0^1\frac{(x-1)\zeta(x)-1}{u[(x-1)\zeta(x)-\ln u](1-\ln u)}du+\int_0^1\frac{(2-x)du}{u[1-(x-1)\ln u](1-\ln u)}\\ =&\int_0^1\frac{1}{u(1-\ln u)}\left[\frac{(x-1)\zeta(x)-1}{(x-1)\zeta(x)-\ln u}+\frac{2-x}{1-(x-1)\ln u}\right]du\\ =&\int_0^1\frac{1}{u(\ln u-1)}\left[\frac{(x-1)\zeta(x)-1}{\ln u-(x-1)\zeta(x)}+\frac{2-x}{(x-1)\ln u-1}\right]du,\end{aligned}$$

which is the desired result. □

4.4. Again the Natural Logarithm Function.

Corollary 8. *If* $\mathrm{Re}(x)>0$*, then*

$$\ln x=\int_0^{\infty}\frac{(x-1)dt}{(x+t)(1+t)}, \tag{13}$$

where $\ln x$ *denotes the natural logarithm function.*

Proof. We take $u=e^{-t}$ in Theorem 3, and this completes the proof. □

4.5. The Riemann Zeta Function.

Corollary 9. *If* $\mathrm{Re}(s)>1$, *then*

$$\frac{\Gamma(s)\zeta(s)}{\Gamma(s-1)}=-\int_0^\infty \frac{\mathrm{Li}_{s-1}(-t)}{(1+t)t}\,dt,$$

where $\zeta(s)$ *denotes the Riemann zeta function and* $\mathrm{Li}_a(t)$ *denotes the polylogarithm function.*

Proof. Changing of members $x-1$ and $\ln x$ in (13), it follows that

$$\frac{1}{x-1}=\frac{1}{\ln x}\int_0^\infty \frac{dt}{(x+t)(1+t)}. \tag{14}$$

Let $x=e^u$ into (14) and obtain

$$\frac{1}{e^u-1}=\frac{1}{u}\int_0^\infty \frac{dt}{(e^u+t)(1+t)}. \tag{15}$$

On the other hand, we well-know that

$$\Gamma(s)\zeta(s)=\int_0^\infty \frac{u^{s-1}}{e^u-1}du, \tag{16}$$

for $\mathrm{Re}(s)>1$.

Substituting (15) into (16), we find

$$\begin{aligned}\Gamma(s)\zeta(s)=&\int_0^\infty u^{s-2}\int_0^\infty \frac{dt}{(e^u+t)(1+t)}du\\ =&\int_0^\infty \frac{1}{1+t}\int_0^\infty \frac{u^{s-2}}{e^u+t}du\,dt\\ =&-\Gamma(s-1)\int_0^\infty \frac{\mathrm{Li}_{s-1}(-t)}{(1+t)t}\,dt,\end{aligned}$$

therefore, we deduce easily that

$$\frac{\Gamma(s)\zeta(s)}{\Gamma(s-1)}=-\int_0^\infty \frac{\mathrm{Li}_{s-1}(-t)}{(1+t)t}\,dt,$$

which is the desired result. □

4.6. Once Again the Natural Logarithm Function.

Corollary 10. *If* $\mathrm{Re}(z)>0$, *then*

$$\ln z=\int_0^\infty\int_0^\infty \left[\sin u-\frac{\sin(u\sqrt{z})}{\sqrt{z}}\right]e^{-u\sqrt{t}}du\,dt, \tag{17}$$

where $\ln z$ *denotes the natural logarithm function and* $\sin z$ *denotes the sine function.*

Proof. In [2], we find

$$\frac{1}{(s^2+w^2)(s^2+a^2)}=\frac{1}{w^2-a^2}\int_0^\infty \left[\frac{\sin(au)}{a}-\frac{\sin(wu)}{w}\right]e^{-su}du, \tag{18}$$

for $a\neq w$.

We get $s=\sqrt{t}$, $w=\sqrt{z}$ and $a=1$ in (18) and obtain

$$\frac{1}{(z+t)(1+t)}=\frac{1}{z-1}\int_0^\infty \left[\sin u-\frac{\sin(u\sqrt{z})}{\sqrt{z}}\right]e^{-u\sqrt{t}}du. \tag{19}$$

Integrating (19) from 0 at infinity with respect to t, we encounter

$$(z-1)\int_0^\infty \frac{dt}{(z+t)(1+t)}=\int_0^\infty\int_0^\infty \left[\sin u-\frac{\sin(u\sqrt{z})}{\sqrt{z}}\right]e^{-u\sqrt{t}}du\,dt. \tag{20}$$

We substitute the left hand side of (13) in the left hand side of (20), and this conclude the desired proof. □

Question 11. *Prove that, if* $\mathrm{Re}(x) > 0$, *then*

$$\ln x = \int_0^\infty \frac{(x-1)dt}{(1+xt)(1+t)}, \tag{21}$$

where $\ln x$ *denotes the natural logarithm function.*

4.7. Again the Euler-Mascheroni Constant.

Corollary 12. *We have*

$$\gamma = -\int_0^\infty \frac{t - e^{\frac{1}{t}}(1+t)\Gamma\left(0, \frac{1}{t}\right)}{(1+t)t^2}\, dt,$$

where γ *denotes the Euler-Mascheroni constant,* e^x *denotes the exponential function and* $\Gamma(a, x)$ *denotes the upper incomplete gamma function.*

Proof. We well-know that

$$\gamma = -\int_0^\infty e^{-v} \ln v\, dv. \tag{22}$$

We use (21) into (22) and have

$$\begin{aligned}
\gamma &= -\int_0^\infty e^{-v} \int_0^\infty \frac{(v-1)}{(1+vt)(1+t)} dt\, dv \\
&= -\int_0^\infty \frac{1}{1+t} \int_0^\infty \frac{(v-1)e^{-v}}{1+vt} dv\, dt \\
&= -\int_0^\infty \frac{t - e^{\frac{1}{t}}(1+t)\Gamma\left(0, \frac{1}{t}\right)}{(1+t)t^2}\, dt,
\end{aligned}$$

which is the desired result. □

Corollary 13. *We have*

$$\gamma = -2\int_0^\infty \frac{\sin u - 2F\left(\frac{u}{2}\right)}{u^2}\, du,$$

where γ *denotes the Euler-Mascheroni constant,* $\sin u$ *denotes the sine function and* $F(u)$ *denotes the Dawson's integral.*

Proof. We use (17) into (22) and have

$$\begin{aligned}
\gamma &= -\int_0^\infty e^{-v} \int_0^\infty \int_0^\infty \left[\sin u - \frac{\sin(u\sqrt{v})}{\sqrt{v}}\right] e^{-u\sqrt{t}} du\, dt\, dv \\
&= -\int_0^\infty \int_0^\infty \left[\sin u - \int_0^\infty \frac{\sin(u\sqrt{v})e^{-v}}{\sqrt{v}} dv\right] e^{-u\sqrt{t}} du\, dt \\
&= -\int_0^\infty \int_0^\infty \left[\sin u - 2F\left(\frac{u}{2}\right)\right] e^{-u\sqrt{t}} du\, dt \\
&= -\int_0^\infty e^{-u\sqrt{t}} \int_0^\infty \left[\sin u - 2F\left(\frac{u}{2}\right)\right] du\, dt \\
&= -\int_0^\infty \left[\sin u - 2F\left(\frac{u}{2}\right)\right] \int_0^\infty e^{-u\sqrt{t}} dt\, du \\
&= -2\int_0^\infty \frac{\sin u - 2F\left(\frac{u}{2}\right)}{u^2}\, du,
\end{aligned}$$

which is the desired result. □

Question 14. *Prove that*

$$\gamma = \lim_{x \to \infty} \left[\int_0^x e^t \Gamma(0, t) \mathrm{d}t - \ln x\right],$$

where γ denotes the Euler-Mascheroni constant, $\Gamma(a,\ t)$ denotes the upper incomplete gamma function and $\ln x$ denotes the natural logarithm function.

Corollary 15. *If* $\mathrm{Re}(s)>0$*, then*

$$\zeta'(s) = -\int_0^\infty \frac{1}{1+t}\left(\sum_{n=1}^\infty \frac{1}{n^s}\cdot\frac{n-1}{n+t}\right)dt, \tag{23}$$

where $\zeta'(s)$ denotes the first derivative of Riemann zeta function.

Proof. We well-know that

$$\zeta'(s) = -\sum_{n=1}^\infty \frac{\ln n}{n^s}, \tag{24}$$

for $\mathrm{Re}(s)>1$.

We substitute (13) into (23) and encounter

$$\begin{aligned}\zeta'(s) &= -\sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty \frac{(n-1)dt}{(n+t)(1+t)}\\ &= -\int_0^\infty \frac{1}{1+t}\left(\sum_{n=1}^\infty \frac{1}{n^s}\cdot\frac{n-1}{n+t}\right)dt,\end{aligned}$$

which is the desired result. □

Remark 16. If $s=2$ and 3 in (23), we encounter the integral representations:

$$\zeta'(2) = \frac{1}{6}\int_0^\infty \frac{\pi^2 t - 6(1+t)[\gamma+\psi(1+t)]}{(1+t)t^2}dt,$$

$$\zeta'(3) = \frac{1}{6}\int_0^\infty \frac{6(1+t)[\gamma+\psi(1+t)] - t[\pi^2(1+t) - 6t\zeta(3)]}{(1+t)t^3}dt.$$

References

[1] Havil, Julian, *Gamma: Exploring Euler's Constant*, First Edition, Princenton University Press, 2003.

[2] http://www.vibrationdata.com/math/Laplace_Transforms.pdf, available in March 8, 2015.