A Reformulation of Classical Mechanics

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This paper presents a reformulation of classical mechanics which is invariant under transformations between inertial and non-inertial reference frames and which can be applied in any reference frame without introducing fictitious forces.

Introduction

The reformulation of classical mechanics presented in this paper is obtained starting from an auxiliary system of particles (called free-system) that is used to obtain kinematic magnitudes (such as inertial position, inertial velocity, etc.) that are invariant under transformations between inertial and non-inertial reference frames.

The inertial position $\mathbf{r}_i$, the inertial velocity $\mathbf{v}_i$ and the inertial acceleration $\mathbf{a}_i$ of a particle i are given by:

$$\mathbf{r}_i \doteq (\vec{r}_i - \vec{R})$$

$$\mathbf{v}_i \doteq (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{a}_i \doteq (\vec{a}_i - \vec{A}) - 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R})$$

($\mathbf{v}_i \doteq d(\mathbf{r}_i)/dt$) and ($\mathbf{a}_i \doteq d^2(\mathbf{r}_i)/dt^2$) where $\vec{r}_i$ is the position vector of particle i, $\vec{R}$ is the position vector of the center of mass of the free-system, and $\vec{\omega}$ is the angular velocity vector of the free-system (see Appendix I)

The net force $\mathbf{F}_i$ acting on a particle i of mass m_i produces an inertial acceleration $\mathbf{a}_i$ according to the following equation:

$$\mathbf{F}_i = m_i\,\mathbf{a}_i$$

Additionally, inertial reference frames and non-inertial reference frames must not introduce fictitious forces into $\mathbf{F}_i$.

The magnitudes [m_i, $\mathbf{r}_i$, $\mathbf{v}_i$, $\mathbf{a}_i$ and $\mathbf{F}_i$] are invariant under transformations between inertial and non-inertial reference frames.

A reference frame S is non-rotating if the angular velocity $\vec{\omega}$ of the free-system relative to S is equal to zero, and the reference frame S is also inertial if the acceleration $\vec{A}$ of the center of mass of the free-system relative to S is equal to zero.

The Definitions

For a system of N particles, the following definitions are applicable:

Mass	$\mathrm{M} \doteq \sum_i^{\mathrm{N}} m_i$
Position CM 1	$\vec{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{r}_i$
Velocity CM 1	$\vec{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{v}_i$
Acceleration CM 1	$\vec{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \vec{a}_i$
Position CM 2	$\mathbf{R}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{r}_i$
Velocity CM 2	$\mathbf{V}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Acceleration CM 2	$\mathbf{A}_{cm} \doteq \mathrm{M}^{-1} \sum_i^{\mathrm{N}} m_i \, \mathbf{a}_i$
Linear Momentum 1	$\mathbf{P}_1 \doteq \sum_i^{\mathrm{N}} m_i \, \mathbf{v}_i$
Angular Momentum 1	$\mathbf{L}_1 \doteq \sum_i^{\mathrm{N}} m_i \left[\mathbf{r}_i \times \mathbf{v}_i \right]$
Angular Momentum 2	$\mathbf{L}_2 \doteq \sum_i^{\mathrm{N}} m_i \left[(\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{v}_i - \mathbf{V}_{cm}) \right]$
Work 1	$\mathrm{W}_1 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \Delta \, \mathrm{K}_1$
Kinetic Energy 1	$\Delta \, \mathrm{K}_1 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i)^2$
Potential Energy 1	$\Delta \, \mathrm{U}_1 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i$
Mechanical Energy 1	$\mathrm{E}_1 \doteq \mathrm{K}_1 + \mathrm{U}_1$
Lagrangian 1	$\mathrm{L}_1 \doteq \mathrm{K}_1 - \mathrm{U}_1$
Work 2	$\mathrm{W}_2 \doteq \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm}) = \Delta \, \mathrm{K}_2$
Kinetic Energy 2	$\Delta \, \mathrm{K}_2 \doteq \sum_i^{\mathrm{N}} \Delta \, {}^1\!/_2 \, m_i \, (\mathbf{v}_i - \mathbf{V}_{cm})^2$
Potential Energy 2	$\Delta \, \mathrm{U}_2 \doteq - \sum_i^{\mathrm{N}} \int_1^2 \mathbf{F}_i \cdot d(\mathbf{r}_i - \mathbf{R}_{cm})$
Mechanical Energy 2	$\mathrm{E}_2 \doteq \mathrm{K}_2 + \mathrm{U}_2$
Lagrangian 2	$\mathrm{L}_2 \doteq \mathrm{K}_2 - \mathrm{U}_2$

Work 3	$\mathrm{W}_3 \ \doteq\ \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i \ =\ \Delta\, \mathrm{K}_3$
Kinetic Energy 3	$\Delta\, \mathrm{K}_3 \ \doteq\ \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i\, \mathbf{a}_i \cdot \mathbf{r}_i$
Potential Energy 3	$\Delta\, \mathrm{U}_3 \ \doteq\ -\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \mathbf{r}_i$
Mechanical Energy 3	$\mathrm{E}_3 \ \doteq\ \mathrm{K}_3 + \mathrm{U}_3$
Work 4	$\mathrm{W}_4 \ \doteq\ \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) \ =\ \Delta\, \mathrm{K}_4$
Kinetic Energy 4	$\Delta\, \mathrm{K}_4 \ \doteq\ \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[\, (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm})\, \right]$
Potential Energy 4	$\Delta\, \mathrm{U}_4 \ \doteq\ -\sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$
Mechanical Energy 4	$\mathrm{E}_4 \ \doteq\ \mathrm{K}_4 + \mathrm{U}_4$
Work 5	$\mathrm{W}_5 \ \doteq\ \sum_i^{\mathrm{N}} \left[\, \int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R})\, \right] \ =\ \Delta\, \mathrm{K}_5$
Kinetic Energy 5	$\Delta\, \mathrm{K}_5 \ \doteq\ \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[\, (\vec{v}_i - \vec{V})^2 + (\vec{a}_i - \vec{A}) \cdot (\vec{r}_i - \vec{R})\, \right]$
Potential Energy 5	$\Delta\, \mathrm{U}_5 \ \doteq\ -\sum_i^{\mathrm{N}} \left[\, \int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R})\, \right]$
Mechanical Energy 5	$\mathrm{E}_5 \ \doteq\ \mathrm{K}_5 + \mathrm{U}_5$
Work 6	$\mathrm{W}_6 \ \doteq\ \sum_i^{\mathrm{N}} \left[\, \int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm})\, \right] \ =\ \Delta\, \mathrm{K}_6$
Kinetic Energy 6	$\Delta\, \mathrm{K}_6 \ \doteq\ \sum_i^{\mathrm{N}} \Delta\, {}^1\!/_2\, m_i \left[\, (\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm})\, \right]$
Potential Energy 6	$\Delta\, \mathrm{U}_6 \ \doteq\ -\sum_i^{\mathrm{N}} \left[\, \int_1^2 \mathbf{F}_i \cdot d(\vec{r}_i - \vec{R}_{cm}) + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot (\vec{r}_i - \vec{R}_{cm})\, \right]$
Mechanical Energy 6	$\mathrm{E}_6 \ \doteq\ \mathrm{K}_6 + \mathrm{U}_6$

The Relations

From the above definitions, the following relations can be obtained (see Appendix II)

$$\mathrm{K}_1 \ =\ \mathrm{K}_2 + {}^1\!/_2\, \mathrm{M}\, \mathbf{V}_{cm}^2$$

$$\mathrm{K}_3 \ =\ \mathrm{K}_4 + {}^1\!/_2\, \mathrm{M}\, \mathbf{A}_{cm} \cdot \mathbf{R}_{cm}$$

$$\mathrm{K}_5 \ =\ \mathrm{K}_6 + {}^1\!/_2\, \mathrm{M} \left[\, (\vec{V}_{cm} - \vec{V})^2 + (\vec{A}_{cm} - \vec{A}) \cdot (\vec{R}_{cm} - \vec{R})\, \right]$$

$$\mathrm{K}_5 \ =\ \mathrm{K}_1 + \mathrm{K}_3 \quad \& \quad \mathrm{U}_5 \ =\ \mathrm{U}_1 + \mathrm{U}_3 \quad \& \quad \mathrm{E}_5 \ =\ \mathrm{E}_1 + \mathrm{E}_3$$

$$\mathrm{K}_6 \ =\ \mathrm{K}_2 + \mathrm{K}_4 \quad \& \quad \mathrm{U}_6 \ =\ \mathrm{U}_2 + \mathrm{U}_4 \quad \& \quad \mathrm{E}_6 \ =\ \mathrm{E}_2 + \mathrm{E}_4$$

The Principles

The linear momentum $[\,\mathbf{P}_1\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its weak form.

$$\mathbf{P}_1 \;=\; \text{constant} \qquad \Big[\; d(\mathbf{P}_1)/dt \;=\; \textstyle\sum_i^{\mathrm{N}} m_i\,\mathbf{a}_i \;=\; \sum_i^{\mathrm{N}} \mathbf{F}_i \;=\; 0 \;\Big]$$

The angular momentum $[\,\mathbf{L}_1\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its strong form.

$$\mathbf{L}_1 \;=\; \text{constant} \qquad \Big[\; d(\mathbf{L}_1)/dt \;=\; \textstyle\sum_i^{\mathrm{N}} m_i\,\big[\,\mathbf{r}_i \times \mathbf{a}_i\,\big] \;=\; \sum_i^{\mathrm{N}} \mathbf{r}_i \times \mathbf{F}_i \;=\; 0 \;\Big]$$

The angular momentum $[\,\mathbf{L}_2\,]$ of an isolated system of N particles remains constant if the internal forces obey Newton's third law in its strong form.

$$\mathbf{L}_2 \;=\; \text{constant} \qquad \Big[\; d(\mathbf{L}_2)/dt \;=\; \textstyle\sum_i^{\mathrm{N}} m_i\,\big[\,(\mathbf{r}_i - \mathbf{R}_{cm}) \times (\mathbf{a}_i - \mathbf{A}_{cm})\,\big] \;=$$

$$\textstyle\sum_i^{\mathrm{N}} m_i\,\big[\,(\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{a}_i\,\big] \;=\; \sum_i^{\mathrm{N}} (\mathbf{r}_i - \mathbf{R}_{cm}) \times \mathbf{F}_i \;=\; 0 \;\Big]$$

The mechanical energy $[\,\mathrm{E}_1\,]$ and the mechanical energy $[\,\mathrm{E}_2\,]$ of a system of N particles remain constant if the system is only subject to conservative forces.

$$\mathrm{E}_1 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_1 \;=\; \Delta\,\mathrm{K}_1 + \Delta\,\mathrm{U}_1 \;=\; 0 \;\Big]$$

$$\mathrm{E}_2 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_2 \;=\; \Delta\,\mathrm{K}_2 + \Delta\,\mathrm{U}_2 \;=\; 0 \;\Big]$$

The mechanical energy $[\,\mathrm{E}_3\,]$ and the mechanical energy $[\,\mathrm{E}_4\,]$ of a system of N particles are always zero (and therefore they always remain constant)

$$\mathrm{E}_3 \;=\; \text{constant} \qquad \Big[\; \mathrm{E}_3 \;=\; \textstyle\sum_i^{\mathrm{N}} 1/2\,\big[\,m_i\,\mathbf{a}_i \cdot \mathbf{r}_i - \mathbf{F}_i \cdot \mathbf{r}_i\,\big] \;=\; 0 \;\Big]$$

$$\mathrm{E}_4 \;=\; \text{constant} \qquad \Big[\; \mathrm{E}_4 \;=\; \textstyle\sum_i^{\mathrm{N}} 1/2\,\big[\,m_i\,\mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) - \mathbf{F}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})\,\big] \;=\; 0 \;\Big]$$

$$\textstyle\sum_i^{\mathrm{N}} 1/2\,m_i\,\big[\,(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm})\,\big] \;=\; \sum_i^{\mathrm{N}} 1/2\,m_i\,\mathbf{a}_i \cdot (\mathbf{r}_i - \mathbf{R}_{cm})$$

The mechanical energy $[\,\mathrm{E}_5\,]$ and the mechanical energy $[\,\mathrm{E}_6\,]$ of a system of N particles remain constant if the system is only subject to conservative forces.

$$\mathrm{E}_5 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_5 \;=\; \Delta\,\mathrm{K}_5 + \Delta\,\mathrm{U}_5 \;=\; 0 \;\Big]$$

$$\mathrm{E}_6 \;=\; \text{constant} \qquad \Big[\; \Delta\,\mathrm{E}_6 \;=\; \Delta\,\mathrm{K}_6 + \Delta\,\mathrm{U}_6 \;=\; 0 \;\Big]$$

Observations

All equations of this paper can be applied in any inertial reference frame and also in any non-inertial reference frame.

Additionally, inertial reference frames and non-inertial reference frames must not introduce fictitious forces into $\mathbf{F}_i$.

In this paper, the magnitudes [m, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, M, $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\mathbf{F}$, $\mathbf{P}_1$, $\mathbf{L}_1$, $\mathbf{L}_2$, W_1, K_1, U_1, E_1, L_1, W_2, K_2, U_2, E_2, L_2, W_3, K_3, U_3, E_3, W_4, K_4, U_4, E_4, W_5, K_5, U_5, E_5, W_6, K_6, U_6 and E_6] are invariant under transformations between inertial and non-inertial reference frames.

The mechanical energy E_3 of a system of particles is always zero [$E_3 = K_3 + U_3 = 0$]

Therefore, the mechanical energy E_5 of a system of particles is always equal to the mechanical energy E_1 of the system of particles [$E_5 = E_1$]

The mechanical energy E_4 of a system of particles is always zero [$E_4 = K_4 + U_4 = 0$]

Therefore, the mechanical energy E_6 of a system of particles is always equal to the mechanical energy E_2 of the system of particles [$E_6 = E_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k then the potential energy U_3 and the potential energy U_5 of the system of particles are given by: [$U_3 = (\frac{k}{2})\, U_1$] and [$U_5 = (1+\frac{k}{2})\, U_1$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k then the potential energy U_4 and the potential energy U_6 of the system of particles are given by: [$U_4 = (\frac{k}{2})\, U_2$] and [$U_6 = (1+\frac{k}{2})\, U_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_5 of the system of particles is equal to zero, then we obtain: [$K_1 = -\, K_3 = U_3 = (\frac{k}{2})\, U_1 = (\frac{k}{2+k})\, E_1$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the kinetic energy K_6 of the system of particles is equal to zero, then we obtain: [$K_2 = -\, K_4 = U_4 = (\frac{k}{2})\, U_2 = (\frac{k}{2+k})\, E_2$]

If the potential energy U_1 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle K_5 \rangle$ of the system of particles is equal to zero, then we obtain: [$\langle K_1 \rangle = -\, \langle K_3 \rangle = \langle U_3 \rangle = (\frac{k}{2})\, \langle U_1 \rangle = (\frac{k}{2+k})\, \langle E_1 \rangle$]

If the potential energy U_2 of a system of particles is a homogeneous function of degree k and if the average kinetic energy $\langle K_6 \rangle$ of the system of particles is equal to zero, then we obtain: [$\langle K_2 \rangle = -\, \langle K_4 \rangle = \langle U_4 \rangle = (\frac{k}{2})\, \langle U_2 \rangle = (\frac{k}{2+k})\, \langle E_2 \rangle$]

The average kinetic energy $\langle K_5 \rangle$ and the average kinetic energy $\langle K_6 \rangle$ of a system of particles with bounded motion (in $\langle K_5 \rangle$ relative to $\vec{R}$ and in $\langle K_6 \rangle$ relative to $\vec{R}_{cm}$) are always zero.

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : [$K_5 = \sum_i^N 1/2 \; m_i \, (\dot{r}_i \, \dot{r}_i + \ddot{r}_i \, r_i)$] where $r_i \doteq | \vec{r}_i - \vec{R} |$ and [$K_6 = \sum_{j>i}^N 1/2 \, m_i \, m_j \, M^{-1} (\dot{r}_{ij} \, \dot{r}_{ij} + \ddot{r}_{ij} \, r_{ij})$] where $r_{ij} \doteq | \vec{r}_i - \vec{r}_j |$

The kinetic energy K_5 and the kinetic energy K_6 of a system of N particles can also be expressed as follows : [$K_5 = \sum_i^N 1/2 \; m_i \, (\ddot{\tau}_i)$] where $\tau_i \doteq 1/2 \; (\vec{r}_i - \vec{R}) \cdot (\vec{r}_i - \vec{R})$ and [$K_6 = \sum_{j>i}^N 1/2 \, m_i \, m_j \, M^{-1} (\ddot{\tau}_{ij})$] where $\tau_{ij} \doteq 1/2 \, (\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j)$

The kinetic energy K_6 is the only kinetic energy that can be expressed without the necessity of introducing any magnitude that is related to the free-system [such as: $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, $\vec{\omega}$, $\vec{R}$, etc.]

In an isolated system of particles, the potential energy U_2 is equal to the potential energy U_1 if the internal forces obey Newton's third law in its weak form [$U_2 = U_1$]

In an isolated system of particles, the potential energy U_4 is equal to the potential energy U_3 if the internal forces obey Newton's third law in its weak form [$U_4 = U_3$]

In an isolated system of particles, the potential energy U_6 is equal to the potential energy U_5 if the internal forces obey Newton's third law in its weak form [$U_6 = U_5$]

A reference frame S is non-rotating if the angular velocity $\vec{\omega}$ of the free-system relative to S is equal to zero, and the reference frame S is also inertial if the acceleration $\vec{A}$ of the center of mass of the free-system relative to S is equal to zero.

If the origin of a non-rotating reference frame S [$\vec{\omega} = 0$] always coincides with the center of mass of the free-system [$\vec{R} = \vec{V} = \vec{A} = 0$] then relative to S: [$\mathbf{r}_i = \vec{r}_i$, $\mathbf{v}_i = \vec{v}_i$ and $\mathbf{a}_i = \vec{a}_i$] Therefore, it is easy to see that always: [$\mathbf{v}_i = d(\mathbf{r}_i)/dt$ and $\mathbf{a}_i = d^2(\mathbf{r}_i)/dt^2$]

This paper does not contradict Newton's first and second laws since these two laws are valid in all inertial reference frames. The equation [$\mathbf{F}_i = m_i \, \mathbf{a}_i$] is a simple reformulation of Newton's second law.

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Appendix I

The Free-System

The free-system is a system of N particles that must always be free of internal and external forces, that must be three-dimensional, and that the relative distances between the N particles must be constant.

The position $\vec{R}$, the velocity $\vec{V}$ and the acceleration $\vec{A}$ of the center of mass of the free-system relative to a reference frame S (and the angular velocity $\vec{\omega}$ and the angular acceleration $\vec{\alpha}$ of the free-system relative to the reference frame S) are given by:

$$\mathrm{M} \doteq \textstyle\sum_i^{\mathrm{N}} m_i$$

$$\vec{R} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i \, \vec{r}_i$$

$$\vec{V} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i \, \vec{v}_i$$

$$\vec{A} \doteq \mathrm{M}^{-1} \textstyle\sum_i^{\mathrm{N}} m_i \, \vec{a}_i$$

$$\vec{\omega} \doteq \overleftrightarrow{I}^{-1} \cdot \vec{L}$$

$$\vec{\alpha} \doteq d(\vec{\omega})/dt$$

$$\overleftrightarrow{I} \doteq \textstyle\sum_i^{\mathrm{N}} m_i \, [\, |\, \vec{r}_i - \vec{R}\,|^2 \, \overleftrightarrow{1} - (\vec{r}_i - \vec{R}) \otimes (\vec{r}_i - \vec{R})\,]$$

$$\vec{L} \doteq \textstyle\sum_i^{\mathrm{N}} m_i \, (\vec{r}_i - \vec{R}) \times (\vec{v}_i - \vec{V})$$

where M is the mass of the free-system, $\overleftrightarrow{I}$ is the inertia tensor of the free-system (relative to $\vec{R}$) and $\vec{L}$ is the angular momentum of the free-system relative to the reference frame S.

The Transformations

$$(\vec{r}_i - \vec{R}) \doteq \mathbf{r}_i = \mathbf{r}'_i$$

$$(\vec{r}\,'_i - \vec{R}') \doteq \mathbf{r}'_i = \mathbf{r}_i$$

$$(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{v}_i = \mathbf{v}'_i$$

$$(\vec{v}\,'_i - \vec{V}') - \vec{\omega}\,' \times (\vec{r}\,'_i - \vec{R}') \doteq \mathbf{v}'_i = \mathbf{v}_i$$

$$(\vec{a}_i - \vec{A}) - 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\, \vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{a}_i = \mathbf{a}'_i$$

$$(\vec{a}\,'_i - \vec{A}') - 2\, \vec{\omega}\,' \times (\vec{v}\,'_i - \vec{V}') + \vec{\omega}\,' \times [\, \vec{\omega}\,' \times (\vec{r}\,'_i - \vec{R}')\,] - \vec{\alpha}\,' \times (\vec{r}\,'_i - \vec{R}') \doteq \mathbf{a}'_i = \mathbf{a}_i$$

Appendix II

The Relations

In a system of particles, these relations can be obtained (The magnitudes $\mathbf{R}_{cm}$, $\mathbf{V}_{cm}$, $\mathbf{A}_{cm}$, $\vec{R}_{cm}$, $\vec{V}_{cm}$ and $\vec{A}_{cm}$ can be replaced by the magnitudes $\mathbf{R}$, $\mathbf{V}$, $\mathbf{A}$, $\vec{R}$, $\vec{V}$ and $\vec{A}$, or by the magnitudes $\mathbf{r}_j$, $\mathbf{v}_j$, $\mathbf{a}_j$, $\vec{r}_j$, $\vec{v}_j$ and $\vec{a}_j$, respectively. On the other hand, $\mathbf{R} = \mathbf{V} = \mathbf{A} = 0$)

$$\mathbf{r}_i \doteq (\vec{r}_i - \vec{R})$$

$$\mathbf{R}_{cm} \doteq (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{r}_i - \vec{R}_{cm})$$

$$\mathbf{v}_i \doteq (\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R})$$

$$\mathbf{V}_{cm} \doteq (\vec{V}_{cm} - \vec{V}) - \vec{\omega} \times (\vec{R}_{cm} - \vec{R})$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm}) = (\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})$$

$$(\mathbf{v}_i - \mathbf{V}_{cm}) \cdot (\mathbf{v}_i - \mathbf{V}_{cm}) = \big[(\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[(\vec{v}_i - \vec{V}_{cm}) - \vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) - 2\, (\vec{v}_i - \vec{V}_{cm}) \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + 2\, (\vec{r}_i - \vec{R}_{cm}) \cdot \big[\vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm}) \cdot (\vec{v}_i - \vec{V}_{cm}) + \big[2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big] =$$

$$(\vec{v}_i - \vec{V}_{cm})^2 + \big[2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm}) \big]^2$$

$$(\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = \big\{ (\vec{a}_i - \vec{A}_{cm}) - 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) + \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})] -$$

$$\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) +$$

$$\big\{ \vec{\omega} \times [\vec{\omega} \times (\vec{r}_i - \vec{R}_{cm})] \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[\vec{\alpha} \times (\vec{r}_i - \vec{R}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) = (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) -$$

$$\big[2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big\{ \big[\vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \big] \vec{\omega} - (\vec{\omega} \cdot \vec{\omega})\, (\vec{r}_i - \vec{R}_{cm}) \big\} \cdot (\vec{r}_i - \vec{R}_{cm}) =$$

$$(\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm}) - \big[2\, \vec{\omega} \times (\vec{v}_i - \vec{V}_{cm}) \big] \cdot (\vec{r}_i - \vec{R}_{cm}) + \big[\vec{\omega} \cdot (\vec{r}_i - \vec{R}_{cm}) \big]^2 - (\vec{\omega})^2 (\vec{r}_i - \vec{R}_{cm})^2$$

$$\longrightarrow \quad (\mathbf{v}_i - \mathbf{V}_{cm})^2 + (\mathbf{a}_i - \mathbf{A}_{cm}) \cdot (\mathbf{r}_i - \mathbf{R}_{cm}) = (\vec{v}_i - \vec{V}_{cm})^2 + (\vec{a}_i - \vec{A}_{cm}) \cdot (\vec{r}_i - \vec{R}_{cm})$$

Appendix III

The Magnitudes

The magnitudes $\mathbf{L}_2$, W_2, K_2, U_2, W_4, K_4, U_4, W_6, K_6 and U_6 of a system of N particles can also be expressed as follows:

$$\mathbf{L}_2 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, (\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{v}_i - \mathbf{v}_j) \,\big]$$

$$W_2 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$\Delta K_2 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} (\mathbf{v}_i - \mathbf{v}_j)^2 \;=\; W_2$$

$$\Delta U_2 \;=\; -\textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$W_4 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$\Delta K_4 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{a}_i - \mathbf{a}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big] \;=\; W_4$$

$$\Delta U_4 \;=\; -\textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) \,\big]$$

$$W_6 \;=\; \textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big]$$

$$\Delta K_6 \;=\; \textstyle\sum_{j>i}^{N} \Delta\, {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\vec{v}_i - \vec{v}_j)^2 + (\vec{a}_i - \vec{a}_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big] \;=\; W_6$$

$$\Delta U_6 \;=\; -\textstyle\sum_{j>i}^{N} m_i\, m_j\, M^{-1} \big[\, \int_1^2 (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot d(\vec{r}_i - \vec{r}_j) + \Delta\, {}^1\!/_2\, (\mathbf{F}_i/m_i - \mathbf{F}_j/m_j) \cdot (\vec{r}_i - \vec{r}_j) \,\big]$$

The magnitudes $W_{(1\ \text{to}\ 6)}$ and $U_{(1\ \text{to}\ 6)}$ of an isolated system of N particles, whose internal forces obey Newton's third law in its weak form, can be reduced to:

$$W_1 \;=\; W_2 \;=\; \textstyle\sum_{i}^{N} \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$\Delta U_1 \;=\; \Delta U_2 \;=\; -\textstyle\sum_{i}^{N} \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i$$

$$W_3 \;=\; W_4 \;=\; \textstyle\sum_{i}^{N} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$\Delta U_3 \;=\; \Delta U_4 \;=\; -\textstyle\sum_{i}^{N} \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i$$

$$W_5 \;=\; W_6 \;=\; \textstyle\sum_{i}^{N} \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i \,\big]$$

$$\Delta U_5 \;=\; \Delta U_6 \;=\; -\textstyle\sum_{i}^{N} \big[\, \int_1^2 \mathbf{F}_i \cdot d\vec{r}_i + \Delta\, {}^1\!/_2\, \mathbf{F}_i \cdot \vec{r}_i \,\big]$$