The Chromatic-Complete Difference Ratio of Classes of Graphs- Domination, Asymptotes and Area

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Abstract

Much research has been done involving the chromatic number of a graph involving the least number of colors, that the vertices of a graph can be colored, so that no two adjacent vertices have the same color. The idea of how the chromatic number of a vertex cover of a graph dominates the vertex cover of the original graph, where a large number of vertices are involved, has been investigated. The difference between the energy of the complete graph,, and the energy of any other graph G, has been studied, in terms of a ratio. The complete graph, on n vertices, has chromatic number n, and is significant in terms of its easily accessible graph theoretical properties, such as its high level of connectivity and robustness. In this paper, we introduce a ratio, the chromatic-complete difference ratio, involving the difference between the chromatic number of the complete graph, and the chromatic number of any other connected graph G, on the same number n of vertices. This allowed for the investigation of the effect of the chromatic number of G, with respect to the complete graph, when a large number of vertices are involved - referred to as the chromatic-complete difference domination effect. The value of this domination effect lies on the interval [0,1], with most classes of graphs taking on the right hand end-point, while graphs with a large clique takes on the left hand end-point. When this ratio is a function f(n), of the order of a graph, we attach the average degree of G to the Riemann integral to investigate the chromatic-complete difference area aspect of classes of graphs. We applied these chromatic-complete difference aspects to complements of classes of graphs.

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1. Introduction

In this paper graphs G will be on n vertices. We shall adopt the definitions and notation of Harris, Hirst, and Mossinghoff. It is assumed that G is simple, that is, it does not contain loops or parallel edges.

Much research has been done involving the *chromatic number* of a graph, or the least number of colors, required to color the vertices of a graph, so that no two adjacent vertices are assigned the same color (see Lawler; Sopena). Applications in this area are vast; especially in scheduling problems (see, for example, Winter, Jessop and Zachariades). The chromatic-cover ratio, which allowed for the investigation of the effect, of the chromatic number of a vertex covering of graph, on the chromatic number of the same graph when a large number of vertices are involved has been researched (see Winter). The ratio, involving the difference between the energy of a graph, and the energy of the *complete graph*, on the same number of vertrices, has been investigated (see Winter and Ojako). The complete graph is well studied, in terms of its ease of accessibility in applying defined graph-theoretical properties. In this paper, we combine the chromatic number of a graph, with the importance of the complete graph, to from the chromatic-complete difference ratio. This ratio allowed for the investigation of the *domination effect* of the chromatic number of graphs on the chromatic number of the complete graph, when a large number of vertices are involved. We found that this domination effect, as a value, lies on the interval [0,1], with graphs with clique number close to n taking on the left hand end-point, and the dumbbell graph (two copies of $K_{\frac{n}{2},\frac{n}{2}}$ joined by an edge) assuming the average value of $\frac{1}{2}$.

We found the chromatic-complete different ratio of the complement of discussed classes of graphs, and showed that the relationship between the chromatic-complete difference ratio g(n), of the complete-split bipartite graph, and the chromatic-complete difference ratio f(n), of its complement, resulted in the differential equation:

$g'(n) + g(n) = 1 - [f(n)n]^{-1} + \frac{n^{-2}}{f(n)}$ with general solution:

$g(n) = 1 - [f(n)n]^{-1} + ce^{-n}$

converging to the chromatic-complete difference asymptote, of the complete-split bipartite graph.

Ratios and graphs

Ratios have been an important aspect of graph theoretical definitions. Examples of ratios of graphs are: expanders, (see Alon and Spencer), the central ratio of a graph (see Buckley), eigen-pair ratio (see Winter and Jessop), Independence and Hall ratios (see Gábor), tree-cover ratio (see Winter and Adewusi), eigen-energy formation ratio (see Winter and Sarvate), t-compete sequence ratio (see Winter, Jessop and Adewusi) the chromatic-cover ratio (see Winter) and the eigen-compete difference ratio (see Winter and Ojako)

We now introduce the idea of ratio, asymptotes and areas involving the chromatic number difference between the complete graph and G, similar to that of Winter and Adewusi; Winter and Jessop; Winter and Sarvate; Winter, Jessop and Adewusi; Winter and Ojako; and Winter.

1. Chromatic-complete difference ratio- asymptotes, domination effect and area

Let K_n be the complete graph on n vertices.

Definition 2.1

The difference between the chromatic number of K_n and the chromatic number of a graph G. on the same number of vertices n is given by:

$\langle \text{¡} D_n^G = \text{¡}(K_n) - \text{¡}(G)$

And is called the *chromatic-complete difference* associated *with G.*

If the graph G in belongs to a class $\Im$ of graphs of order n, then the *chromatic-complete-energy difference* associated with $\Im$ is defined as*:*

$\Big\langle \text{¡} D_n^{\Im} = \text{¡}(K_n) - \text{¡}(G); G \in \Im.$

Dividing the chromatic-complete difference by the chromaticnumber of K_n will give an "average" of the chromatic-complete difference with respect to G. This provides motivation for the following definition:

Definition 2.2

The *chromatic-complete difference ratio* with respect to $G(\Im)$, respectively, is defined as:

$$Rat\Big\langle \text{¡} D_n^{G} = \frac{\text{¡}(K_n) - \text{¡}(G)}{\text{¡}(K_n)} = \frac{n - \text{¡}(G)}{n}; Rat\Big\langle \text{¡} D_n^{\Im} = \frac{n - \text{¡}(G)}{n}; G \in \Im$$

Definition 2.3

If the chromatic-complete difference ratio is a function f(n) of the order of $G \in \Im$, then its horizontal asymptote results in the *chromatic-complete difference asymptote*:

$$Asymrat\Big\langle \text{¡} D_n^{\Im} = \underset{n \to \infty}{Lim}\left[\frac{n - \text{¡}(G)}{n}\right]; G \in \Im$$

This asymptote allows for the investigation of the effect of the chromatic number of a graph G on the chromatic number of the complete graph when a large number of vertices are involved, referred to as the *domination chromatic-complete difference effect.*

Definition 2.4

Attaching the average degree of graph G, with m' edges, to the Riemann integral of $Rat\langle \chi D_n^{\Im} = \frac{n-\chi(G)}{n}; G \in \Im$ we obtain the *chromatic-complete difference area:*

$$Arat\langle \chi D_n^{\Im} = \frac{2m'}{n}\int [\frac{n-\chi(G)}{n}]dn \quad \chi(G) \neq n; = \frac{2m'}{n}\int_{o}^{\chi(G)} dt = 2m'; \chi(G) = n;$$

with $Arat\langle \chi D_k^{\Im} = 0$ where k is the smallest order of $G \in \Im$.

The average degree is referred to as the *length* of the area, while the integral part is the *height* of the area.

Lemma

The chromatic-complete difference ratio of classes of graph on at least 2 vertices lies on the interval $[0,1-\frac{2}{n}]$.

Proof

$$\chi(G) \leq \chi(K_n) = n \Rightarrow Rat\langle \chi D_n^{\Im} = \frac{n-\chi(G)}{n} = 1 - \frac{\chi(G)}{n} \geq 1-1 = 0; G \in \Im$$

Also:

$$\chi(G) \geq 2 \Rightarrow Rat\langle \chi D_n^{\Im} = \frac{n-\chi(G)}{n} = 1 - \frac{\chi(G)}{n} \leq 1 - \frac{2}{n}; G \in \Im$$

Theorem

$Asymrat\langle \chi D_n^{G} \in [0,1]; G \in \Im$.

3.Examples of classes of graphs and their chromatic-complete difference aspects.

3.1The complete split-bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$.

The chromatic number of this graph is 2 and it has $\frac{n^2}{4}$ edges while the chromatic number of the complete graph is n so that:

$$Rat\langle ¡D_n^{\Im} = \left[\frac{n - ¡(K_{\frac{n}{2},\frac{n}{2}})}{n}\right] = \frac{n-2}{n}$$

$$Asymrat\langle ¡D_n^{\Im} = \lim_{n\to\infty}[\frac{n-2}{n}] = 1$$

$$Arat\langle ¡D_n^{\Im} = \frac{n}{2}\int[\frac{n-2}{n}]dn = \frac{n}{2}(n - 2\ln n + c)$$ with smallest order 2 we have :

$c = 2\ln 2 - 2$

3.2 The star graph $K_{1,n-1}$ with n-1 rays of length1.

The chromatic number of this star graph is 2 so that:

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n - \text{¡}(K_{1,n-1})}{n}\right] = \frac{n-2}{n}$$

$$Asymrat\langle \text{¡} D_n^{\Im} = \lim_{n\to\infty}[\frac{n-2}{n}] = 1$$

$$Arat\langle \text{¡} D_n^{\Im} = \frac{2(n-1)}{n}\int[\frac{n-2}{n}]dn = \frac{2(n-1)}{n}[n - 2\ln n + c]$$

With smallest star graph on 2 vertices we have:

$c = 2\ln 2 - 2$.

3.3 Star graphs $S_{r,2}$ with r rays of length 2

The chromatic nmber of this star graph with $r = n-1$ edges is 2:

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n - \text{¡}(K_{1,n-1})}{n}\right] = \frac{n-2}{n}$$

$$Asymrat\langle \text{¡} D_n^{\Im} = \lim_{n\to\infty}[\frac{n-2}{n}] = 1$$

$$Arat\langle \text{¡} D_n^{\Im} = \frac{2(n-1)}{n}\int[\frac{n-2}{n}]dn = \frac{2(n-1)}{n}[n - 2\ln n + c]$$

The smallest such star graph is non 3 vertices so that

$c = 2\ln 3 - 3$.

3.4The cycle graph C_n.

The cycle has chromatic number 2, when n is even, and 3, when n is odd. so that (k = 2 or 3:

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n - \text{¡}(C_n)}{n}\right] = \frac{n-k}{n}$$

$$Asymrat\langle \text{¡} D_n^{\Im} = \underset{n\to\infty}{Lim}[\frac{n-k}{n}] = 1$$

$$Arat\langle \text{¡} D_n^{\Im} = 2\int[\frac{n-k}{n}]dn = 2[n - k\ln n + c]$$

The smallest such cycle is on 3 or 4 vertices so that

$c = k\ln s - s,\;\; k = 2{,}3; s = 3{,}4$ for odd, even, respectively

3.5 The path graph P_n.

The path graph on at least 2 vertices has chromatic number 2 so that:

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n - \text{¡}(P_n)}{n}\right] = \frac{n-2}{n}$$

$$Asymrat\langle \text{¡} D_n^{\Im} = \underset{n\to\infty}{Lim}[\frac{n-2}{n}] = 1$$

$$Arat\langle \text{¡} D_n^{\Im} = \frac{2(n-1)}{n}\int[\frac{n-2}{n}]dn = \frac{2(n-1)}{n}[n - 2\ln n + c]$$

The smallest such path is non 2 vertices so that

$c = 2\ln 2 - 2$.

3.6 The wheel W_n with n-1 spokes.

We have the central vertex colored 1 and the vertices of the cycle colored with two or three different colors so that its chromatic number is k=3 or 4:

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n - \text{¡}(W_n)}{n}\right] = \frac{n-k}{n}$$

$$Asymrat\langle \text{¡} D_n^{\Im} = \underset{n\to\infty}{Lim}\left[\frac{n-k}{n}\right] = 1$$

$$Arat\langle \text{¡} D_n^{\Im} = \frac{4(n-1)}{n}\int\left[\frac{n-k}{n}\right]dn = \frac{4(n-1)}{n}[n - k\ln n + c]$$

The smallest such wheel is on 4 or 5 vertices so that

$c = k\ln s - s;\ \ k = 3,4; s = 4,5$; odd or even cycle, respectively.

3.7 The sun graph Su_n.

For the sun graph on an even or odd number of vertices- i.e. we have an even or odd cycle on $\frac{n}{2}$ vertices with end vertices added to each vertex of the cycle, its chromatic number is 3 or 2:

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n - \text{¡}(Su_n)}{n}\right] = \frac{n-k}{n}$$

$$Asymrat\langle \text{¡} D_n^{\Im} = \underset{n\to\infty}{Lim}\left[\frac{n-k}{n}\right] = 1$$

$$Arat\langle \text{¡} D_n^{\Im} = \frac{2n}{n}\int[\frac{n-k}{n}]dn = 2[n - k\ln n + c]$$

The smallest such sun graph is on 6 or 8 vertices so that

$c = k\ln s - s; k = 3,2; s = 6,8;$ odd or even cycle, respectively.

3.8 The fan graph F_n on n vertices

Construct the fan graph F_n on a number $n \geq 3$ of vertices, by taking a path on n-1 vertices and joining each vertex of the path to a single vertex, the *center* of the fan graph.

The chromatic number of the fan graph is 3.

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n - \text{¡}(L_n)}{n}\right] = \frac{n-3}{n}$$

$$Asymrat\langle \text{¡} D_n^{\Im} = \underset{n\to\infty}{Lim}[\frac{n-3}{n}] = 1$$

$$Arat\langle \text{¡} D_n^{\Im} = \frac{4n-6}{n}\int[\frac{n-3}{n}]dn = \frac{4n-6}{n}[n - 3\ln n + c]$$

The smallest such wheel is on 3 vertices so that

$c = 3\ln 3 - 3$.

3.9 The Ladder graph L_n on n vertices, n even.

Let the ladder on $n \geq 4$ vertices be formed by joining corresponding vertices of paths on $\frac{n}{2}$ vertices each. The chromatic number of the ladder graph is 2.

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n - \text{¡}(L_n)}{n}\right] = \frac{n-2}{n}$$

$$Asymrat\langle \text{¡} D_n^{\Im} = \lim_{n\to\infty}[\frac{n-2}{n}] = 1$$

$$Arat\langle \text{¡} D_n^{\Im} = \frac{3n-4}{n}\int[\frac{n-2}{n}]dn = \frac{3n-4}{n}[n-2\ln n + c]$$

The smallest such wheel is non 4 vertices so that

$c = 2\ln 4 - 4$.

3.10 The line graph of K_n

The line graph $L(K_n)$ of K_n has $p = \frac{n(n-1)}{2}$ vertices (see Brualdi). The number q of edges is the sum of the square of the degrees minus the number of edges of K_n:

$$q = n\frac{(n-1)^2}{2} - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}[n-1-1] = \frac{n(n-1)(n-2)}{2}$$

$$n^2 - n - 2p = 0 \Rightarrow n = \frac{1 \pm \sqrt{1+8p}}{2} = \frac{1+\sqrt{1+8p}}{2}$$

The chromatic number of the line graph of K_n is the edge chromatic number of K_n = $\text{¡}'(K_n) = n-1 = \frac{1+\sqrt{1+8p}}{2} - 1$.

$$Rat\langle \text{¡} D_p^{\Im} = \left[\frac{\text{¡}(K_p) - \text{¡}(L(K_n))}{\text{¡}(K_p)}\right] = \frac{p - \text{¡}'(K_n)}{p} = \frac{p-(n-1)}{p} = \frac{p+1-[\frac{1+\sqrt{1+8p}}{2}]}{p}$$

$$=\frac{2p+2-1-\sqrt{1+8p}}{2p}=\frac{2p+1-\sqrt{1+8p}}{2p}$$

$$Asymrat\langle \text{¡} D_p^{\Im} = \lim_{p\to\infty}[\frac{2p+1-\sqrt{1+8p}}{2p}]=1$$

$$Arat\langle \text{¡} D_p^{\Im} = \frac{2q}{p}\int\frac{2p+1-\sqrt{1+8p}}{2p}dp$$

$$=\frac{2q}{p}\int[p+\frac{1-\sqrt{1+8p}}{2p}]dp; u^2=1+8p \Rightarrow dp=\frac{udu}{4}; p=\frac{u^2-1}{8}$$

$$=\frac{2q}{p}[(p)+\int\frac{(1-u)}{(\frac{u^2-1}{4})}\frac{udu}{4}]=\frac{2q}{p}[(p)+\int\frac{(1-u)}{u^2-1}udu]=\frac{2q}{p}[(p)-\int\frac{u}{u+1}du]$$

$$=\frac{2q}{p}[(p)-[\int du-\ln(u+1)]=replacing\ \ absolute\ \ sign$$

$$\frac{2q}{p}[p-\sqrt{1+8p}+\ln(\sqrt{1+8p}+1)+c]$$

$p=3$ yields:

$c=-3+5-\ln 6$

So that the chromatic-complete area of the line graph of K_n on p vertices is:

$$\frac{2q}{p}[p-\sqrt{1+8p}+\ln(\sqrt{1+8p}+1)+2-\ln 6]$$

3.11 Lollipop graph

Complete graph joined to end vertex.

So if LP[1](G) of single extension is the complete graph on n-1vertices (the base of the lollipop graph) joined to a single end vertex v, adjacent to u, has chromatic number n-1: assign n-1 colors to the complete subgraph and color vertex v any color already assigned other than u.

To find $\sum dv$ for this graph we have:

$$\sum dv = (n-2)(n-2)+(n-1)+1 = (n-2)(n-2)+n = n^2-3n+4 = 2m'$$

Thus:

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n-E(LP^1(G))}{n}\right] = \frac{1}{n}.$$

$$Asymrat\langle \text{¡} D_n^{\Im} = \underset{n\to\infty}{Lim}[0] = 0$$

$$Arat\langle \text{¡} D_n^{\Im} = \frac{2m'}{n}[\ln n + c].$$

Smallest lollipop graph is on 3 vertices so that $c = -\ln 2$.

3.12 The q-cliqued path

Form a *q-cliqued path* P_n^q by joining q cliques, $q \geq 2$, each of size

$\frac{n}{q}; n = tq; t, q \in \aleph; q$ *fixed*,

with edges as follows:

Let Q_{i-1}, Q_i, Q_{i+1} be consecutive cliques of the path joined by edges uv and wz, where: $u \in Q_{i-1}; v \in Q_i; w \in Q_i; z \in Q_{i+1}; w \neq v$.

The number of edges of P_n^q will be: $m' = [\frac{\frac{n}{q}(\frac{n}{q}-1)}{2}]q + (q-1) = \frac{n(n-q)}{2} + (q-1)$

Lemma

The chromatic number of P_n^q is $\frac{n}{q}$.

Proof

Color each clique with $\frac{n}{q}$ colors, and for consecutive cliques as described above, color v different to u, and color z different to w.

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n - E(P_n^q)}{n}\right] = \frac{n - \frac{n}{q}}{n} = \frac{qn - n}{nq} = \frac{q-1}{q}$$; with q fixed, this is independent of n:

$$Asymrat\langle \text{¡} D_n^{\Im} = \lim_{n\to\infty}\left[\frac{q-1}{q}\right] = 1$$ and:

$$Arat\langle \text{¡} D_n^{\Im} = \frac{2m'}{n}[\int \frac{q-1}{q}]dn = \frac{2m'}{n}[n(\frac{q-1}{q}) + c].$$

Smallest q-cliqued path is on 4 vertices with q=2:

$c = -2$.

Note that the dumbbell graph, consisting of two disjoint copies of $K_{\frac{n}{2},\frac{n}{2}}$ joined by and edge, is equivalent to the $\frac{n}{2}$-*cliqued path* P_n^2 and takes on the value of $\frac{n - \frac{n}{2}}{n}$ for its chromatic-complete difference ratio, which is independent of n and has an "average" asymptote of $\frac{1}{2}$.

Also, this chromatic-complete difference ratio gives rise to the sequence:

$\frac{1}{2}, \frac{2}{3}, \ldots, \frac{q-1}{q}$ which is identical to the chromatic-cover ratio sequence, with its diagram, and similar to the famous Farey sequence and it associated diagram (see Winter).

Theorem

$Rat\langle ¡ D_n^{\Im}$; $Asymrat\langle ¡ D_n^{\Im}$ and $Arat\langle ¡ D_n^{\Im}$ for the following classes of graphs are, respectively:

$$\Im = K_{\frac{n}{2},\frac{n}{2}}: \frac{n-2}{n};\ 1; \frac{n}{2}(n-2\ln n+2\ln 2-2)$$

$$\Im = K_{1,n-1}: 1-\frac{2}{n};1;\frac{2(n-1)}{n}[n-2\ln n-2+\ln 2]$$

$$\Im = K_{2,r}: \frac{n-2}{n};1;\frac{2(n-1)}{n}[n-2\ln n-3+\ln 3].$$

$\Im = C_n:\frac{n-k}{n};1;2[n-k\ln n+c];c=k\ln s-s,\ \ k=3,2;s=3,4$ for odd, even, respectively.

$$\Im = P_n:\frac{n-2}{n};1;2[n-2\ln n+2\ln 2-2].$$

$$\Im = W_n;\frac{n-k}{n};1\ \frac{4(n-1)}{n}[n-k\ln n+c]$$

The smallest such wheel is on 4 or 5 vertices so that

$c=k\ln s-s;\ \ k=3,4;s=4,5$; odd or even cycle, respectively.

$$\Im = Su_n;\frac{n-k}{n};1;2[n-k\ln n+k\ln s-s],k=3,2;s=6,8.$$

$$\Im = F_n;\frac{n-3}{n};1;\frac{4n-6}{n}[n-3\ln n+3\ln 3-3].$$

$$\Im = L_n;\frac{n-2}{n};1;\frac{3n-4}{n}[n-2\ln n+2\ln 4-4].$$

$$\Im = L(K_n); \frac{2p+1-\sqrt{1+8p}}{2p};1; \frac{2q}{p}[p-\sqrt{1+8p}+\ln(\sqrt{1+8p}+1)+2-\ln 6]$$

Where $q = \frac{n(n-1)(n-2)}{2}$.

$$\Im = LP^1(G); \frac{1}{n};0; \frac{2m'}{n}[\ln n + -\ln 2].$$

$$\Im = P_n^q; \frac{q-1}{q}, \frac{q-1}{q}; \frac{2m'}{n}[n(\frac{q-1}{q})-2].$$

4. Eigen-complete different ratios of complements of classes of graphs

4.1 The complete-split bipartite graph

The complement of $K_{\frac{n}{2},\frac{n}{2}}$ consists of two disjoint copies of $K_{\frac{n}{2},}$. Its chromatic number is therefore $\frac{n}{2}$ so that:

$$Rat\langle ¡ D_n^{\Im} = \left[\frac{n-\frac{n}{2}}{n}\right] = \frac{1}{2}$$

$$Asymrat\langle ¡ D_n^{\Im} = \lim_{n\to\infty}[\frac{1}{2}] = \frac{1}{2}$$

$$Arat\langle ¡ D_n^{\Im} = \frac{n-2}{2}\int\left[\frac{1}{2}\right]dn = \frac{n-2}{4}[n+c]$$

Smallest such graph occurs for n=2 so that:

$c = -2$

The chromatic-complete difference ratio for the complement of the complete-split bipartite graph is $f(n) = \frac{1}{2}$

The chromatic-complete difference ratio of the original graph is:

$$g(n) = Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n-2}{n}\right] = 1 - \frac{2}{n} = 1 - \frac{1}{nf(n)} = 1 - (nf(n))^{-1}$$

$$\Rightarrow g'(n) + g(n) = 1 - \frac{2}{n} + \frac{2}{n^2} = 1 - 2n^{-1} + 2n^{-2}$$

$$IF = e^n \Rightarrow e^n g(n) = \int (1 - 2n^{-1} + 2n^{-2}) e^n dn = e^n - 2\int n^{-1} e^n dn + 2\int n^{-2} e^n dn$$

$$\int n^{-2} e^n dn = -n^{-1}.e^n + \int n^{-1} e^n dn \Rightarrow e^n g(n) = e^n - 2n^{-1} e^n + c \Rightarrow g(n) = 1 - [f(n)n]^{-1} + ce^{-n}$$

.

Check: $g'(n) + g(n) = 2n^{-2} - ce^{-n} + 1 - 2n^{-1} + ce^{-n}$

Theorem

The chromatic-complete difference ratio g(x) of the complete-split bipartite graph, and the chromatic-complete difference ratio f(x) of its complement, are related by the differential equation:

$g'(n) + g(n) = 1 - [f(n)n]^{-1} + \frac{n^{-2}}{f(n)}$ with general solution:

$$g(n) = 1 - [f(n)n]^{-1} + ce^{-n}$$

Converging to the chromatic-complete difference asymptote of the complete-split bipartite graph.

4.2 Star graphs with rays of length 1

The compliment of the star graph with rays of length one (on at least three vertices) is a complete graph on n-1 vertices together with an isolated vertex. Its chromatic number is therefore:

$$Rat\langle \text{¡}D_n^{\Im} = \left[\frac{n-n+1}{n}\right] = \frac{1}{n}.$$

$$Asymrat\langle D_n^{\Im} = \lim_{n\to\infty}\left[\frac{1}{n}\right] = 0$$

$$Arat\langle D_n^{\Im} = \frac{m'}{n}\int\left[\frac{1}{n}\right]dn = \frac{(n-1)(n-2)}{2n}[\ln(n)+c].$$

For n=3 we get $c = -\ln 3$.

The chromatic-complete difference ratio for the original graph is:

$Rat\langle \text{¡}D_n^{\Im} = \left[\frac{n-2}{n}\right] = 1-\frac{2}{n} = g(n)$ while the ratio of the complement is $f(n) = \frac{1}{n}$.

The equation of the tangent line to g(n) at n=t is :

$$y = g'(t)n + c; g(t) = 1-\frac{2}{t}; g'(t) = \frac{2}{t^2} \Rightarrow 1-\frac{2}{t} = \frac{2}{t^2}t + c \Rightarrow c = 1-\frac{4}{t}$$

$$\Rightarrow y = \frac{2}{t^2}n + 1 - \frac{4}{t}$$

Equation of tangent line to y=f(n) at n=t is:

$$y = f'(t)n + c;\ f(t) = \frac{1}{t};\ f'(t) = -\frac{1}{t^2} \Rightarrow \frac{1}{t} = -\frac{1}{t^2}t + c \Rightarrow c = \frac{2}{t}$$

$$\Rightarrow y = -\frac{1}{t^2}n + \frac{2}{t}$$

Equation of normal line to y=f(n) at n=t is:

$$y = t^2 n + c;\ f(t) = \frac{1}{t}; \Rightarrow \frac{1}{t} = t^2 t + c \Rightarrow c = \frac{1}{t} - t^3$$

$\Rightarrow y = t^2 n + \frac{1}{t} - t^3$ and tangent to y=g(x): $y = \frac{2}{t^2} n + 1 - \frac{4}{t}$

These lines intercept when:

$$t^2 n + \frac{1}{t} - t^3 = \frac{2}{t^2} n + 1 - \frac{4}{t} \Rightarrow n(t^2 - \frac{2}{t^2}) = t^3 + 1 - \frac{4}{t} \Rightarrow n(t^4 - 2)t = t^2(t^4 + t - 4)$$

$$n = \frac{t(t^4 + t - 4)}{t^4 - 2}.$$

$$t = 1 \Rightarrow n = 2; t = 2 \Rightarrow n = \frac{2(14)}{14} = 2; t = 3 \Rightarrow n = \frac{3(80)}{79}; t = 4 \Rightarrow n = \frac{4(254)}{252}$$

Theorem

The tangent line to the chromatic-complete difference ratio g(n), of the star graph with rays of length one, intercepts the normal line, to the tangent to the chromatic-complete difference ratio f(n) of its complement, when:

$$t^5 + t^2 - 4t = k(t^4 - 2); k \in \mathrm{N}.$$

4.3 The lollipop graph with complete graph on n-1 vertices as base

The compliment of the lollipop graph consists of a star graphs on n-1 vertices and an isolated vertex. Its chromatic number is therefore 2:

$$Rat\langle \text{¡} D_n^{\Im} = \left[\frac{n-2}{n}\right] = g(n) = 1 - \frac{2}{n}$$

$$Asymrat\langle \text{¡} D_n^{\Im} = \lim_{n \to \infty} \left[\frac{n-2}{n}\right] = 1.$$

$$Arat\langle\, ¡\, D_n^{\Im} = \frac{2m'}{n}\int \frac{n-2}{n}\,dn = \frac{2m'}{n}[n-2\ln n+c]$$

. Taking n=3 we get:

$c = 3 - 2\ln 3$.

The chromatic-complete ratio of the original lollipop graph is $f(n) = \frac{1}{n}$

So that $g(n) = 1 - 2f(n)$.

5. Conclusion

In this paper we used the idea of the difference between the chromatic numbers of two graphs on the same number of vertices, and the significance of the complete graph, to formulate the chromatic-complete difference ratio which allowed for the investigation of the domination effect that the chromatic number of a graph G with respect to the chromatic number complete graph when a large number of vertices are involved-known as the chromatic-complete domination effect. We found that this domination effect took on a value on the interval [0,1] for all classes of graphs, with graphs with a large complete subgraph taking on the left hand end-point, while the dumbbell graph, consisting two copies of $K_{\frac{n}{2}}$ joined by and edge, takes on the average domination value of $\frac{1}{2}$.

We attached the average degree to the Riemann integral of this chromatic-complete difference ratio to determine chromatic-complete difference areas associated with classes of graphs, and applied the above ideas to the complement of classes of graphs and found that the relationship between the chromatic-complete difference ratio g(x), of the complete-split bipartite graph, and the chromatic-complete difference ratio f(x), of it complement, resulted in the differential equation:

$g'(n)+g(n)=1-[f(n)n]^{-1}+\frac{n^{-2}}{f(n)}$ with general solution:

$g(n)=1-[f(n)n]^{-1}+ce^{-n}$

Converging to the chromatic-complete difference asymptote of the complete-split bipartite graph.

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