Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory
Part 5 Non-reversible arithmetic and limits

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Abstract

Investigate and define non-reversible arithmetic in $*G$ and the real numbers. That approximation of an argument of magnitude, is arithmetic. For non-reversible multiplication we define a logarithmic magnitude relation $\succ\!\succ$. Apply the much-greater-than relation $\succ$ in the evaluation of limits. Consider L'Hopital's rule with infinitesimals and infinities, and in a comparison $f\ (z)\ g$ form.

1 Introduction

Two parallel lines may meet at infinity, or they may always be apart. Infinity is non-unique. We believe different number systems can co-exist by the non-unique nature of infinity.

We focus on the infinite case where the largest magnitude dominates, $a+b=a$ where $b \neq 0$.

For example, arbitrarily truncating a Taylor series. $f(x+h)|_{h=0} = f(x)+hf'(x)+\frac{h^2}{2}f^{(2)}(x)+ \ldots|_{h=0} = f(x)+hf'(x)|_{h=0}$.

That an infinite sum of the discarded lower order terms have no effect on the outcome is explained by sum convergence theory. However, if such a sum was for example bounded above by an infinitesimal, then the realization of the sum (a transfer) to real numbers would discard the infinitesimal terms, $\Phi \mapsto 0$.

> DU BOIS-REYMOND in his journal articles on the infinitary calculus is not much interested in a theory of sets as such, or even explicitly in sets on lines. He is interested in the nature of a linear continuum, but chiefly because he wants to consider a more general "continuum" of functions. He especially concerns himself with limit processes as they occur in a linear continuum, since he wants to consider limit processes among functions. [4, p.110]

In our previous papers we have identified the investigation of functions as working in a higher dimensional number system than $\mathbb{R}$, and fitted $*G$ to du Bois-Reymond's work, as a representation of the continuum that he was investigating.

Since we believe $*G$ is a field, multiplication and addition by non-zero elements are reversible prior to realization. For example truncation $*G \mapsto *G$. After this process, information is lost and in general you cannot go back.

What we demonstrate is that arithmetic 'is' non-reversible; and that this is a major aspect of analysis. Ultimately we see this as a way for working with transference, with reasoning by magnitude.

For example, we prove L'Hopital's rule Proposition 4.2 not with an equality but with an argument of magnitude. This, after reading the original translation of the rule is closer to the discovery than the formation with the mean value theorem. And this is a problem, that mathematics is being used not in an intuitive way, but as a means of proof. Newton did not publish the Principia with his calculus, but the traditional geometric arguments which are "extremely" difficult to use. Analysis is not complete without a more detailed investigation into arguments of magnitude.

2 Non-reversible arithmetic

The following is a complicated argument that associates finite mathematics with reversible processes, and infinitary calculus with non-reversible processes. This results in non-uniqueness in the additive operator, the consequences of which are profound.

To start, consider addition and multiplication. Given $x + 2$, then $(x + 2) - 2$ gives back x, the -2 undoing the $+2$ operation. Similarly for multiplication, if we start at 5, $5 \times 2 = 10$. Reversing, $10/2 = 5$. Similarly with powers, where logarithms and powers are each others inverse.

However, if the operator is not reversible you cannot undo the operator previously applied; it is as if the operation has disappeared. $n^2 + \ln n = n^2|_{n=\infty}$ leaves no evidence of adding $\ln n$, an infinity. We need a number system with infinities for this to occur. The much-greater-than relation with realization explains this.

Theorem 2.1. *$a, b \in *G$; $(a, b \neq 0)$ Assume transfer between Φ and 0.*

$$\textit{If } a + b = a \Leftrightarrow a \succ b.$$

Proof. $\delta \in \Phi$;

If $a + b = a$, $\frac{a}{a} + \frac{b}{a} = \frac{a}{a}$, $\frac{b}{a} = 0$, $0 \mapsto \Phi$, $\frac{b}{a} = \delta$, $a \succ b$.

If $a \succ b$, $\frac{b}{a} = \delta$. Consider $a + b = y$, $\frac{a}{a} + \frac{b}{a} = \frac{y}{a}$, $1 + \delta = \frac{y}{a}$, $\delta \mapsto 0$, $1 = \frac{y}{a}$, $a = y$, $a + b = a$ □

Corollary 2.1. *$a, b \in *G$; $b \neq 0$*

If $a+b = a$ then after the addition and realization in general, it is impossible to determine b.

Proof. By Definition 2.2. □

This is a fundamental statement about our numbers. We use Theorem 2.1 whenever we approximate. We approximate whenever we use infinitesimal or infinite numbers. By considering an alternative definition to the much-greater-than relation as an infinitesimal ratio ($a \succ b$ then $\frac{b}{a} \in \Phi$) is explained.

If $a \succ b$ is alternatively defined[2, p.19]: if $a - b = \infty$ then $a \succ b$. We cannot apply the transfer $\Phi \to 0$ then $a + b = a$ is not true in general. Consider $a = e^{x+1}$, $b = -e^x$, $e^{x+1} - e^x = (e-1)e^x|_{x=\infty} = \infty$ but $e^{x+1} \neq (e-1)e^x|_{x=\infty}$. This lacks the '$a+b=a$' property given in Theorem 2.1. Since this paper is concerned about non-reversibility we do not want to lose this property.

Proposition 2.1. *If $a_n \sim b_n|_{n=\infty}$ then there exists c_n: $a_n + c_n = b_n|_{n=\infty}$ where $c_n \prec a_n$.*

Proof. $a_n + c_n \; z \; b_n$, $a_n \; z \; b_n$ as $a_n + c_n = a_n$, $z = \sim$ □

Definition 2.1. *An Archimedean number system has no infinitesimals or infinities. A non-Archimedean number system is not an Archimedean number system.*

That is, a non-Archimedean number system has non-reversible arithmetic, see Definition 2.2. A consequence of the existence of ratios being an infinitesimal or an infinity.

A consequence of the Archimedean property are unique inverses, both additive and multiplicative, which allow unique solutions to equations involving addition and multiplication.

With the infinireals, as long as we do not approximate via the "realization" operation, and assuming non-zero numbers, we have unique inverses when adding and subtracting and unique inverses when multiplying or dividing.

However the non-Archimedean property is necessary for Theorem 2.1. We need an infinity of possibilities with addition before we can say an operation is non-reversible. However Archimedean systems can have non-reversible arithmetic.

Theorem 2.2. *Multiplication by 0 in $\mathbb{R}$ number systems is non-reversible.*

Proof. Let $x \in \mathbb{R}$. Consider the equation $x \times 0 = 0$. If you then ask what was the value of x, there is an infinity of solutions in $\mathbb{R}$. By Definition 2.2 the multiplication is non-reversible. □

The number zero collapses other numbers through multiplication. What was the original number being collapsed? For this reason 0 is considered separately from other numbers. By defining $0^0 = 0$ we can extend the number system further, see Example 2.5.

The reals, with which we are so familiar, form an Archimedean number system which excludes infinity.

This exclusion of infinity in many respects is illusionary, as an arbitrary large number acts as an infinity. This is exploited in proofs by retaining the property of being a finite number. For the statement "$\forall n > n_0$ the following is true ..." implicitly defines infinity.

Excluding that infinity implicitly exists even when ignored, the Archimedean property essentially means that there are no infinitesimals or infinities as elements.

There is fascination that both the finite perspective and infinite perspective can co-exist. Both can describe, even when they are contradictory viewpoints, that is, they are completely different views of the same event. 1. $a + b$ for finite numbers excluding zero always changes the sum. 2. $a + b = a$

If we add a cent to a million dollars we have a million dollars plus a cent. Alternatively, if we treat a million dollars as the infinity, then adding a cent does 'nothing' to the sum, and the sum remains unchanged, as its magnitude is not changed. Hence, the accountant and the businessman may have different views on the same transaction.

Looking at infinitary calculus with infinity as a point, consider $\ln n + n^2 = n^2|_{n=\infty}$. The term $\ln n|_{n=\infty} = \infty$ is infinite, but acts like a zero when added to a much larger infinity, a consequence of $\ln n \prec n^2|_{n=\infty}$. Replacing $\ln n$ with n gives a similar result where $n|_{n=\infty}$ acts as a zero element, when $n + n^2 = n^2|_{n=\infty}$. The additive identity is not unique. It should be apparent that there is an infinity of additive elements.

The same can also be true with regard to multiplication, where a multiplicative identity is not unique. That is, 1 is not the only element which, when multiplied, does not change the number.

Example 2.1. *The following demonstrates non-unique multiplicative identities. Let* $f = \infty, g = \infty, h = \infty$. *Consider* $f \cdot g \quad z \quad h$, $\ln(f \cdot g) \quad (\ln z) \quad \ln h$, $\ln f + \ln g \quad (\ln z) \quad \ln h$. *Let* $\ln f \succ \ln g$, *then* $\ln f + \ln g = \ln f$ *and* $\ln f \quad (\ln z) \quad \ln h$. *When reversing the process is possible,* $f \quad z \quad h$ *and* g *is a multiplicative identity.*

Definition 2.2. *We say that an arithmetic is non-reversible if there is an infinite number of additive identities or there is an infinite number of multiplicative identities.*

Let $g(n)$ *be one of the infinitely many additive identities. Let* $h(n)$ *be one of the infinitely many multiplicative identities.*

If one of the following is true, then we say that the arithmetic is non-reversible arithmetic.

$$f(n) + g(n) = f(n) \text{ or } f(n)h(n) = f(n)$$

Consider when lower order terms realized?

Theorem 2.1 proves non-reversible addition. From this, Theorem 3.1 proves non-reversible multiplication. These operations, particularly addition have wide applicability. However, we stress that this is only one possibility at infinity when via algebra and a transfer, $\Phi \mapsto 0$. That at infinity we can have non-uniqueness is further exploration of what constitutes the continuum.

While Theorem 2.2 multiplication by 0 in $\mathbb{R}$ is non-reversible, it is isolated. By excluding the element 0 in $\mathbb{R}$, multiplication and division are reversible. $\infty \times c = \infty$ when $c \neq 0$ has a similar property, but in $\mathbb{R}$, ∞ is excluded. $0 \times \infty$ is better considered in $*G$ before the infinitesimals or infinities are realized. The order in which numbers are realized matters.

Generally addition is easier for reasoning with non-reversible arithmetic than multiplication; probably a consequence of multiplication being defined as repeated addition, a more complex operation. Hence, for example, when applicable, $\prod_{k=1}^{n} a_k = e^{\sum_{k=1}^{n} \ln a_k}$ transforms a product to a power with a sum for reasoning.

Calculus has been extended in many ways to use infinity in calculations and theory because infinity is so useful. Calculus often states the numbers in $\mathbb{R}$ but then reasons in $*G$. Limits are a typical example. We generally agree with this as the utility of calculation is paramount.

However, just as we discuss the atom in teaching physics, because it describes fundamental properties, a teaching of the infinitesimal is warranted. Not the exclusion of its existence, which is fair to say is the current practice.

Infinitary calculus facilitates arguments with magnitude, and has the potential to reduce the use of inequalities in analysis, beyond what Non-Standard-Analysis or present analysis does. This could make problem-solving more accessible by reducing the technical difficulties associated with the use of inequalities.

For example, one may experience the problem of not knowing or not using the right inequality, and become stuck; without specific knowledge, no progress is likely to be made. This applies to specialized domains where networking may be required. If, however, the problem could be done without "networking", time would be saved.

As well as providing alternative arguments to problems, infinitary calculus can be combined with standard mathematical arguments and inequalities, leading to them being used in new ways.

Well how can this be achieved?

Simply put, by employing non-reversible arithmetic; that is, in number systems with non-Archimedean properties, non-reversible arguments can be made. This is indeed possible with the much-less-than ($\prec$) and much-greater-than ($\succ$) relations defined by du Bois-Reymond.

We have an equation that we would like to solve hence the need for a new number system.

Considering the scales of infinity, there does not exist $c \in \mathbb{R}$ with such a property to move between the infinities because c is finite.

Example 2.2. *Let $c \in \mathbb{R}$ then $c \prec \infty$, $\nexists c : cn^2 = n^3|_{n=\infty}$. Solving in $*G$, $cn^2 = n^3|_{n=\infty}$, $c = n|_{n=\infty}$, $c \in \Phi^{-1}$. This implies $c \notin \mathbb{R}$. If we restricted c to $\mathbb{R}$ then the equation would not have been solved, as c was infinite.*

An Archimedean number system cannot solve this. Just as we needed i to solve $x^2 = -1$, we need a non-Archimedean number system to solve the equation with infinities.

The definition of $\prec$ and little-o distinguish between infinities. $a \prec b$ if $\frac{a}{b} \in \Phi$. If two functions differ infinitely through division, then they must also differ infinitely through addition to a greater degree (See Theorem 2.1).

Depending on the context, lower-order-magnitude terms may be disregarded. $f(x) + g(x) = g(x)|_{x=\infty}$. $f(x) = \infty$, $g(x) = \infty$. Here $f(x)|_{x=\infty}$ acts as a zero identity element, even though $f(x)$ is not zero. However $f(x)$ has its magnitude dwarfed by the much larger $g(x)$, so $f(x)$ is negligible.

To avoid summing infinite collections of terms, the general restriction when applying the simplification $a_n + b_n = a_n|_{n=\infty}$ for infinitary or infinitesimal a_n and b_n is that the rule is only good for a finite sum of infinitesimals or infinities.

A sum of infinities or infinitesimals to infinity, can itself step up in orders. In other words, generally apply the simplification to a finite number of times. If you were to sum infinities or infinitesimals to infinity, you would need to integrate instead. Or apply truncation when sum convergence is known.

This is discussed in detail [9]. The assumption of independence of sums may by invalid at infinity [11]. Briefly, how we view finite mathematics may be completely different to how we view mathematics "at infinity" because it is a much larger space.

The advantages of such simplification can allow classes of functions to be reduced.

Example 2.3. $\{\frac{1}{x^2+\pi}, \frac{1}{x^2-3x}, \ldots\}|_{x=\infty}$ *simplify to considering* $\frac{1}{x^2}|_{x=\infty}$.

Applications include taking the limit which applies in summing infinitesimals to zero, hence truncating a series. The arguments can apply to diverging sums as well.

Use of extended calculus in $*G$ as a heuristic, for developing algorithms.

Example 2.4. *Developing an algorithm for approximating* $\sqrt{2}$. $x, x_n, \delta_n \in *G$; $\delta \in \Phi$.

$$\begin{aligned}
(x+\delta)^2 &= 2 \\
x^2 + 2x\delta + \delta^2 &= 2|_{\delta=0} \\
x^2 + 2x\delta &= 2|_{\delta=0} && (2x\delta \succ \delta^2|_{\delta=0}) \\
x_n^2 + 2x_n\delta_n &= 2|_{\delta=0} && \text{(Developing an iterative scheme.)} \\
\delta_n &= \frac{1}{x_n} - \frac{x_n}{2} && \text{(Solving for } \delta_n\text{)} \\
x_{n+1} &= x_n + \delta_n && \text{(Progressive sequence of } x_n\text{)}
\end{aligned}$$

If $\delta_n \to 0$ *then at infinity* δ_n *becomes an infinitesimal(see [10, Example 2.12], that is* $\delta_n|_{n=\infty} \in \Phi$, *then* $(x+\delta)^2 \simeq 2$ *is solved, and* $x_{n+1} \simeq x_n + \delta_n$ *has* x_n *converging. Starting the approximation with* $x_0 = 1.5$, x_5 *is correct to* 47 *places (for a numerical calculation with Maxima see [7]), where the algorithm was transferred from* $*G$ *to* $\mathbb{R}$. $(*G, \delta_n, x_n) \mapsto (\mathbb{R}, \delta_n, x_n)$, *an infinitesimal was promoted to a real number.*

Turning towards the number system, what is common is the operation of numbers at zero or infinity. That is where the numbers display non-Archimedean properties.

In fact, all function evaluation is at zero or infinity. Simply shift the origin. Zero and infinity form a number system, $\mathbb{R}_\infty$. The cardinality of $\mathbb{R}_\infty$ is infinitely larger than the cardinality of $\mathbb{R}$. Then the gossamer number system $*G$ is much larger than the real number system(the reals which are embedded within it).

In this paper, addition simplification is applied to solving relations by converting a series of relations to a sum, where lower order terms are discarded and the relations solved.

Definition 2.3. *Using the Iverson bracket notation (see [12, p.24])*

$$[f \ z \ g] = \begin{cases} 1 & \text{when the relation } f \ z \ g \text{ is true,} \\ 0 & \text{when then relation } f \ z \ g \text{ is false.} \end{cases}$$

In a more radical approach to demonstrate addition as a basis for building relations, we can define $0^0 = 0$ as another extension to the real number system, which then allows the building of the comparison greater than function (see Example 2.5).

Canceling the 0's is not allowed, as by definition this is now a non-reversible process, as we view either multiplication by 0 or multiplication by $\frac{0}{0}$ as collapsing the number to 0. We also get to test if a number is zero or not.

Example 2.5. *Non reversible mathematics to build the relations. See Definition 2.3*

$$[x > 0] = \frac{x + |x|}{2x}$$

When $x=0$, $\frac{0+|0|}{2\times 0} = \frac{0}{0} = 0$. *When* $x>0$, $\frac{x+|x|}{2x} = \frac{2x}{2x} = 1$. *When* $x<0$, $\frac{x+|x|}{2x} = \frac{0}{2x} = 0$.

$$[x \neq 0] = x^0$$

When $x=0$, $0^0 = 0$. *When* $x \neq 0$, $x^0 = 1$.

3 Logarithmic change

We introduce a relation, for better explaining magnitudes, and their comparison. While $f \succ g$ describes an infinity in the ratio of f and g, there could be a much larger change in the functions themselves.

If we consider the operations of addition, multiplication as repeated addition, a power as repeated multiplication, all these operations accelerate change. Conversely, subtraction, division, and logarithms of positive numbers to greater degrees decelerate change.

Consider the log function as undoing change, then applying to both sides of a relation and comparing, we can determine a much-greater than relationship, and 'if' one exists, we infer an infinity between the functions.

For example, in solving $f \; z \; g$ if we find a $\ln f \succ \ln g$ relationship. The undoing log operation revealed a much-greater-than relationship.

Definition 3.1. *We describe a logarithmic magnitude.*

$$\text{We say } f \succ\!\succ g \text{ when } \ln f \succ \ln g$$

$$\text{We say } f \prec\!\prec g \text{ when } \ln f \prec \ln g$$

$$\text{We say } f \text{ 'log dominates' } g \text{ when } f \succ\!\succ g.$$

$$\text{We say } f \text{ is 'log dominated' by } g \text{ when } f \prec\!\prec g.$$

The much-greater-than relation $f \succ g$ may have a log dominating relation $f \succ\!\succ g$ or be log dominated by g: $f \prec\!\prec g$ or no such relationship. That is the relations between the magnitude and the logarithmic magnitude are not necessarily in the same direction. An exception is when both positively diverge, see Proposition 3.1.

A logarithmic magnitude is like a derivative. A derivative's sign is not necessarily the same as the function's sign. The much greater than relation is independent in direction to the log dominating relation.

Logarithmic magnitude can describe non-reversible product arithmetic (Definition 2.2). $0 \cdot \infty$ indeterminate case arises in the calculation of the limit. We prove the non-reversible product as a consequence of non-reversible addition (see Theorem 3.1).

Example 3.1. *Consider* $x^n \; z \; n|_{n=\infty}$ *where* $|x| < 1$.

$$
\begin{aligned}
x^n \; z \; n|_{n=\infty} \\
\ln(x^n) \; (\ln z) \; \ln n|_{n=\infty} \\
n \ln x \; (\ln z) \; \ln n|_{n=\infty} \\
n \ln x \succ \ln n|_{n=\infty} & \qquad \text{(By Definition 3.1)} \\
x^n \succ\!\succ n|_{n=\infty}
\end{aligned}
$$

While $x^n \prec n|_{n=\infty}$ *as* $0 \prec \infty$, *the logarithmic magnitude of* x^n *is much greater than* n *with* $x^n \succ\!\succ n|_{n=\infty}$.

When simplifying products by non-reversible arithmetic, for example in the calculation of limits, rather than solve with products, reason by exponential and logarithmic functions which are each other inverses, converting the problem of multiplication to one with addition. $f \cdot g = e^{\ln(f \cdot g)} = e^{\ln f + \ln g}$. If possible, apply non-reversible arithmetic: $\ln f + \ln g = \ln f$ or $\ln f + \ln g = \ln g$.

Example 3.2. *When* $|x| < 1$, *evaluate* $x^n \cdot n|_{n=\infty}$.

This is an indeterminate form $0 \cdot \infty$. $x^n \cdot n = x^n|_{n=\infty}$ *is harder to understand than when the problem is reformed and when simplifying, non-reversible arithmetic applied on a sum and not a product.*

$$
\begin{aligned}
& x^n \cdot n|_{n=\infty} \\
&= e^{\ln(x^n \cdot n)}|_{n=\infty} \\
&= e^{n \ln x + \ln n}|_{n=\infty} & \qquad (n \succ \ln n \text{ then apply non-reversible arithmetic}) \\
&= e^{n \ln x}|_{n=\infty} & \qquad (n \ln x + \ln n = n \ln x|_{n=\infty}) \\
&= x^n|_{n=\infty}
\end{aligned}
$$

Theorem 3.1. *Non-reversibility in a product. Let* a *and* b *be positive.*

$$\text{If } a \succ\!\succ b \text{ then } a \cdot b = a$$

Proof. $ab = e^{\ln(ab)} = e^{\ln a + \ln b} = e^{\ln a} = a$, as $a \succ\!\succ b$ then $\ln a \succ \ln b$. □

Example 3.3. *If we know the log magnitude relationship, we may directly calculate.* $x^n \cdot n|_{n=\infty} = x^n|_{n=\infty}$ *as* $x^n \succ\!\succ n|_{n=\infty}$

Proposition 3.1. *Let* $f = \infty$, $g = \infty$. *If* $f \succ\!\succ g$ *then* $f \succ g$.

Proof. $f \succ\!\succ g$ then $\ln f \succ \ln g$. Since there is no smallest infinity, $f_2 = \ln f = \infty$, $g_2 = \ln g = \infty$. $f_2 \succ g_2$. By the following theorem: $a = \infty$, $b = \infty$, if $a \succ b$ then $e^a \succ e^b$ (see [8, Table 2]) then $f_2 \succ g_2$, $e^{f_2} \succ e^{g_2}$, $f \succ g$. □

4 Limits at infinity

Definition 4.1. *In context, we say* $f(x)|_{x=\infty}$ *then* $\sup \lim_{x\to\infty} f(x)$*, similarly* $\inf \lim_{x\to\infty} f(x)$

When the definition is put into a context such as a relation, since the condition is assumed to be true for all n at infinity (else the condition is false and a contradictory statement), the exact lower and upper bound language can optionally be excluded.

Example 4.1. *Condition* $\inf \lim_{n\to\infty} \rho_n > 1$ *becomes* $\rho_n|_{n=\infty} > 1$.

Similarly condition $\sup \lim_{n\to\infty} \rho_n < 1$ *becomes* $\rho_n|_{n=\infty} < 1$.

The following demonstrates the application of the notation and ideas discussed in this paper about limits and the more general at-a-point evaluation.

In computation of limits, infinity can be as useful in simplifying expressions as infinitesimals. So rather than dividing and forming the infinitesimals, instead apply arguments of magnitude. Let the user choose. The non-reversible arithmetic works either way.

Example 4.2. *A simple example will show this.* $\frac{3n+5}{5n}|_{n=\infty} = \frac{3n}{5n}|_{n=\infty} = \frac{3}{5}$ *The justification being* $3n + 5 = 3n|_{n=\infty}$

Example 4.3. *Apostol [1, 3.6.7],* $\lim_{x\to 0} \frac{x^2-a^2}{x^2+2ax+a^2}$*,* $a \neq 0$*,* $\frac{x^2-a^2}{x^2+2ax+a^2}|_{x=0}$*,* $= \frac{-a^2}{a^2} = -1$ *as* $-a^2 \succ x^2|_{x=0}$ *and similarly* $a^2 \succ 2ax \succ x^2|_{x=0}$

Example 4.4. *Apostol [1, 7.17.28],* $\lim_{x\to\infty} \{(x^5 + 7x^4 + 2)^c - x\}$*, Find c for non-zero limit (* $c = 0$ *may collapse to* $1 - x|_{x=\infty}$*). Using* $x^5 + 7x^4 + 2 = x^5|_{x=\infty}$ *as* $x^5 \succ x^4 \succ x^0|_{x=\infty}$*,* $(x^5 + 7x^4 + 2)^c - x|_{x=\infty} = x^{5c} - x|_{x=\infty} = b$ *then* $c = \frac{1}{5}$ *as the difference reduces the power by one to a finite value. I.e. a limit.*

Example 4.5. $(x^5(1 + \frac{7}{x} + \frac{2}{x^5}))^{\frac{1}{5}} - x|_{x=\infty} = (x^5(1 + \frac{7}{x}))^{\frac{1}{5}} - x|_{x=\infty}$
expand with the binomial theorem. $(1+x)^w = 1 + wx + w(w-1)\frac{x^2}{2!} + \ldots$*,* $(1+\frac{7}{x})^{\frac{1}{5}} = 1 + \frac{1}{5}\frac{7}{x} + \frac{1}{5}\frac{-4}{5}\frac{7^2}{x^2}\frac{1}{2} + \ldots$ *then* $x(1+\frac{7}{x})^{\frac{1}{5}} - x|_{x=\infty} = x + \frac{7}{5} + \frac{1}{5}\frac{-4}{5}\frac{7^2}{x}\frac{1}{2} + \ldots - x|_{x=\infty} = \frac{7}{5} + \frac{1}{5}\frac{-4}{5}\frac{7^2}{x}\frac{1}{2} + \ldots|_{x=\infty}$ $= \frac{7}{5}$

Arguments of magnitude are commonly used in calculations. Apostol [1, pp 289–290] discusses polynomial approximations used in the calculation of limits, where the relation is within the little-o variable.

Computing the terms separately lead to the indeterminate form 0/0; by computing the numerator and denominator as a coupled problem, leading magnitude terms may be subtracted (for example through factorization).

When we remove little-o the calculation is not cluttered. If you need to be exact include it, but if not then it may as well be omitted.

Using the identity $\frac{1}{1-x} = 1 + x + x^2 + \ldots$, $\frac{1}{1-(\frac{x^2}{2}-o(x^3))} = 1 + \frac{1}{2}x^2 - o(x^3)$ as $x \to 0$ becomes $\frac{1}{1-\frac{x^2}{2}} = 1 + \frac{x^2}{2}|_{x=0}$ Truncation is part of calculations context and assumed to be the case.

Applying the at-a-point notation to some limits. Since the series expressions have terms forming a scale of infinities, often only a fixed number of terms with the expansions need be used. Taylor series, the binomial expansion, trigonometric series and others can be viewed as not unique since they have an infinity of terms.

Example 4.6. $\frac{a^x-b^x}{x}|_{x=0} = (e^{x\ln a} - e^{x\ln b})\frac{1}{x}|_{x=0}$, *expanding the exponential series for the first three terms,* $\frac{a^x-b^x}{x}|_{x=0} = (1 + x\ln a + (x\ln a)^2\frac{1}{2} - (1 + x\ln b + (x\ln b)^2\frac{1}{2})\frac{1}{x}|_{x=0} = (x\ln a - x\ln b)\frac{1}{x}|_{x=0} = \ln a - \ln b = \ln\frac{a}{b}$

With known algebraic identities, such $n^{\frac{1}{n}}|_{n=\infty} = 1$ or $e = \frac{n}{(n!)^{\frac{1}{n}}}|_{n=\infty}$ can easily be used to solve limits.

Example 4.7. *[3, p.39, 2.3.23.b]* , $(\frac{(n!)^3}{n^{3n}e^{-n}})^{\frac{1}{n}}|_{n=\infty} = e(\frac{(n!)^3}{n^{3n}})^{\frac{1}{n}}|_{n=\infty} = e((\frac{n!}{n^n})^{\frac{1}{n}})^3|_{n=\infty} = e(\frac{(n!)^{\frac{1}{n}}}{n})^3|_{n=\infty} = e(e^{-1})^3 = e^{-2}$

The typical interchange between zero and infinity is useful.

$$x|_{x=0^+} = \frac{1}{n}|_{n=\infty} \text{ then } f(x)|_{x=0^+} = f(\frac{1}{n})|_{n=\infty}$$

Example 4.8. *[1, 7.17.18]* $\lim_{x\to 0^-}(1-2^x)^{\sin x} = \lim_{x\to 0^-}(1-2^x)^x$ *as* $\sin(x) = x$ *for small* x*. Show* $y = 1$*. Let* $y \in *G : y = (1-2^x)^x|_{x=0^-}$, $\ln y = x\ln(1-2^x)|_{x=0^-}$, $0\cdot\infty$ *form.*

With a log expansion, the problem can can be solved. $\ln(1-2^x)|_{x=0^-} = 2^x + \frac{(2^x)^2}{2} + \frac{(2^x)^3}{3} + \ldots|_{x=0^-} = 2^x|_{x=0^-}$. $\ln y = x\ln(1-2^x)|_{x=0^-} = x2^x|_{x=0^-}$, $y = (e^{2^x})^x|_{x=0^-} = (e^1)^x|_{x=0^-} = 1$

Example 4.9. *Solve* $\delta^\delta = y$, $\delta \in \Phi$. $\ln(\delta^\delta) = \ln y$, $\delta\ln\delta = \ln y$*. Noticing* $\delta\cdot\ln\delta = 0\cdot -\infty = \frac{1}{\infty}\cdot -\infty$*, which can be expressed as* $-\infty/\infty$ *and differentiated using L'Hopital's rule,* $\delta\ln\delta = \frac{\ln\delta}{1/\delta} = \frac{1/\delta}{-1/\delta^2} = -\delta = 0$ *by* $(*G,\Phi) \mapsto (\mathbb{R}, 0)$, $0 = \ln y$ *then* $y = 1$*.*

Since the magnitude $\{\prec, \preceq\}$ and other relations are defined in terms of ratios, when comparing two functions in a "multiplicative sense", we convert between the fraction and the comparison.

Proposition 4.1. *When* $\frac{f}{g} \in *G$ *and* z *is defined in a "multiplicative sense",* $g \neq 0$

$$\frac{f}{g}\ z\ 1 \Leftrightarrow f\ z\ g$$

Proof. Since no information is lost, and the operation is reversible, $\frac{f}{g} \cdot g \; z \; 1 \cdot g$, $f \; z \; g$. □

Consider the problem process $\frac{f}{g} \Rightarrow f \; z \; g$. $\frac{f}{g} = 1$, $(*G = 1) \mapsto (*G = *G)$. Though non-uniqueness also has advantages. $(*G \; z \; 1) \mapsto (*G \; z \; *G)$.

It is common for a problem to be phrased as if in $\mathbb{R}$ but in actuality is in $*G$. Then after algebraically manipulating in $*G$, the information has to transfer back to $\mathbb{R}$, or be phrased as such.

In fractional form $\frac{f}{g}$ then becomes particularly convenient to apply what we know about fractions to the comparison. For example we may transform the comparison to a point where L'Hopital's rule can be applied. With the extended number system $*G$, the indeterminate forms $0/0$ and ∞/∞ are expressed as Φ/Φ and Φ^{-1}/Φ^{-1} respectively.

We proceed with another proof of L'Hopital's rule where we use infinitary calculus theory and work in $*G$. L'Hopital's argument is interpreted in $*G$ (see Proposition 4.2) and the algebra is explained with non-reversible arithmetic, directly calculating the ratio (see [6]).

Lemma 4.1. *A ratio of infinitesimals is equivalent to a ratio of infinities.* $\frac{\Phi}{\Phi} \equiv \frac{\Phi^{-1}}{\Phi^{-1}}$

Proof. $f, g \in \Phi^{-1}$; $\frac{f}{g} = \frac{\frac{1}{g}}{\frac{1}{f}}$, but ; $\frac{1}{g}, \frac{1}{f} \in \Phi$; then $\frac{f}{g}$ of the form $\frac{\Phi}{\Phi}$. The implication in the other direction has a similar argument. $a, b \in \Phi$; $\frac{a}{b} = \frac{\frac{1}{b}}{\frac{1}{a}}$, but ; $\frac{1}{b}, \frac{1}{a} \in \Phi^{-1}$; then $\frac{a}{b}$ of the form $\frac{\Phi^{-1}}{\Phi^{-1}}$. □

Proposition 4.2. $f, g \in \Phi$; *If* $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ *exists then* $\frac{f(a)}{g(a)} = \frac{f'(a)}{g'(a)}$

Proof. Since $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, we can vary about $x = a$ in f and g. $\frac{f(a)}{g(a)} = \frac{f(a+h)}{g(a+h)}|_{h=0} = \frac{f(a)+f'(a)h}{g(a)+g'(a)h}|_{h=0}$. Choose $h \in \Phi : f'(a)h \succ f(a)$ and $g'(a)h \succ g(a)$. Then $f(a) + f'(a)h = f'(a)h|_{h=0}$, $g(a) + g'(a)h = g'(a)h|_{h=0}$. $\frac{f(a)+f'(a)h}{g(a)+g'(a)h}|_{h=0} = \frac{f'(a)h}{g'(a)h}|_{h=0} = \frac{f'(a)}{g'(a)}$ □

Theorem 4.1. *L'Hopital's rule (weak) in $*G$.*

Proof. The indeterminate form $0/0$ is represented by Φ/Φ in $*G$. A transfer $\Phi \mapsto 0$ confirms this. The indeterminate form ∞/∞ is represented as Φ^{-1}/Φ^{-1}. Similarly a transfer $\Phi^{-1} \mapsto \infty$ confirms this.

The indeterminate form Φ^{-1}/Φ^{-1} by Lemma 4.1 can be transformed to the indeterminate form Φ/Φ.

Apply Proposition 4.2 to the indeterminate form Φ/Φ. □

Theorem 4.2. *Comparison form of L'Hopital's rule. If f/g is in indeterminate form $\frac{\Phi}{\Phi}$ or $\frac{\Phi^{-1}}{\Phi^{-1}}$, when f'/g' exists then $f \; z \; g \Rightarrow f' \; z \; g'$ where $z \in \{\prec, \propto, \succ\}$.*

Proof. Equivalent to L'Hopital's rule. See Theorem 4.1. □

Example 4.10. $f' \succ g' \Rightarrow f \succ g$. $1 \succ \frac{1}{n}|_{n=\infty}$, $n \succ \ln n|_{n=\infty}$.

Example 4.11. *Applying L'Hopital's rule reaches a $0 \;\; z \;\; \infty$ form. Hence a much greater than or much less than relationship. Solve $\ln x \;\; z \;\; x^2|_{x=\infty}$. This is in indeterminate form $\infty \;\; z \;\; \infty$, differentiate. $\frac{1}{x} \;\; (Dz) \;\; 2x|_{x=\infty}$, $\frac{1}{x} \prec 2x|_{x=\infty}$, $Dz = \prec$, $z = \int \prec = \prec$ then $\ln x \prec x^2|_{x=\infty}$*

If the limit exists then the comparison and limit are solved for $f(x) \propto g(x)|_{x=a}$ as a standard application of L'Hopitals rule with the comparison notation.

Example 4.12. *Computing an indeterminate form 0/0. $\lim\limits_{x\to 2} \frac{3x^2+2x-16}{x^2-x-2}$, $3x^2+2x-16 \;\; z \;\; x^2-x-2|_{x=2}$, $0 \;\; z \;\; 0$ form then differentiate. $6x+2 \;\; (Dz) \;\; 2x-1|_{x=2}$, $14 \;\; (Dz) \;\; 3$, $\frac{3x^2+2x-16}{x^2-x-2}|_{x=2} = \frac{14}{3}$*

Example 4.13. *Indeterminate form $\infty/-\infty$. $\frac{v}{\ln v}|_{v=0} = \frac{1}{1/v}|_{v=0} = v|_{v=0} = 0$ then $v \prec \ln v|_{v=0}$. Since the relation could occur with the relation notation, $v \; z \; \ln v|_{v=0}$, $1 \; Dz \; \frac{1}{v}|_{v=0}$, $1 \; Dz \; \infty$, $Dz = \prec$, $z = \prec$.*

Example 4.14. *[2, p.8] . Show $P_m \succ Q_n$ when $m > n$, given $P_m(x) = \sum_{k=0}^{m} p_k x^k$ and $Q_n(x) = \sum_{k=0}^{n} q_k x^k$ for positive coefficients. Let $m = n+a$, $a > 0$. $P_m \;\; z \;\; Q_n|_{x=\infty}$, $\infty \;\; z \;\; \infty$, $D^n P_m \;\; (D^n z) \;\; D^n Q_n|_{x=\infty}$, $D^n P_m \;\; (D^n z) \;\; \alpha|_{x=\infty}$, $\beta x^a \;\; (D^n z) \;\; \alpha|_{x=\infty}$, $\beta x^a \succ \alpha|_{x=\infty}$, integrating n times preserves this relation and solves for z.*

Comparison can be in a "multiplicative sense", or an "additive sense". In the additive sense, we treat the expression more as a relation with addition and we may add and subtract, but drawing conclusions with much-larger-than relations may be problematic. $2x \; z \; 3x$, $0 \; z \; x$, $0 \prec \infty$, may then mistakingly draw the conclusion $2x \prec 3x|_{x=\infty}$. In the multiplicative sense, divide by x, $2x \prec 3x|_{x=\infty}$, $2 \prec 3$ is false. Both comparisons are beneficial.

Example 4.15. *Consider $x^2 \succ x|_{x=\infty}$. By L'Hopital, $x^2 \; z \; x$, $2x \; z \; 1$, $z = \succ$.*
By multiplicative z, $x^2 \; z \; x$, $\frac{x^2}{x} \; z \; 1$, $x \; z \; 1$, $z = \succ$.
However a variation, divide by x, $x \; z \; 1$, $1 \; z \; \frac{1}{x}$, realize the infinitesimal, $1 \; z \; 0$, $z = >$ does not solve for $\succ$. In 'realizing' the infinitesimal information is lost as in $\mathbb{R}$.

Example 4.16. *[5, WolframMathworld] An occasional example where L'Hopital's rule fails. Applying the rule swaps the arguments to opposite sides. Since the relation is equality, this is indeed true. $\frac{u}{(u^2+1)^{\frac{1}{2}}}|_{u=\infty}$, $u \;\; z \;\; (u^2+1)^{\frac{1}{2}}|_{u=\infty}$, $\frac{d}{du}u \;\; (Dz) \;\; \frac{d}{du}(u^2+1)^{\frac{1}{2}}|_{u=\infty}$, $1 \;\; z \;\; \frac{1}{2}(u^2+1)^{-\frac{1}{2}}2u|_{u=\infty}$, $(u^2+1)^{\frac{1}{2}} \;\; z \;\; u|_{u=\infty}$. Applying arguments of magnitude, $\frac{u}{(u^2+1)^{\frac{1}{2}}}|_{u=\infty} = \frac{u}{(u^2)^{\frac{1}{2}}}|_{u=\infty}$ $= \frac{u}{u}|_{u=\infty} = 1$*

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