Ratio test and a generalization with convergence sums

Chelton D. Evans and William K. Pattinson

Abstract

For positive series convergence sums we generalise the ratio test in $*G$ the gossamer numbers. Via a transfer principle, within the tests we construct variations. However, most significantly we connect and show the generalization to be equivalent to the boundary test. Hence, the boundary test includes the generalized tests: the ratio test, Raabe's test, Bertrand's test and others.

1 Introduction

Of the convergence sums [3] $\sum a_n|_{n=\infty}$, there exists two generalizations of convergence and divergence tests involving the ratio of successive terms. Both are equivalent, and different expansions of $\frac{a_n}{a_{n+1}}$ and $\frac{a_{n+1}}{a_n}$.

For example consider Raabe's test(Theorem 2.4) for convergence. If $n(\frac{a_n}{a_{n+1}} - 1)|_{n=\infty} > 1$ is associated with the $\frac{a_n}{a_{n+1}}$ generalization, and $n(\frac{a_{n+1}}{a_n} - 1)|_{n=\infty} < -1$ is associated with the $\frac{a_{n+1}}{a_n}$ generalization. Both tests can be rearranged to show the other.

We find by considering the sum in the more detailed number system with infinitesimals and infinities in $*G$ [1], that we can multiply and divide terms in the ratio and generalized tests, thus modifying the tests. For example, express the ratio test not as a ratio, but as a comparison without fractions, with no denominators.

In the final comparison of the test, the numbers are projected to the extended real numbers, $*G \mapsto \overline{\mathbb{R}}$. The extension is used to include cases, where for example in the ratio test, the ratio is 0 where the denominator is an infinity. Multiplying the denominator out results in a comparison of two numbers which differ by an infinity, hence cannot be compared in the reals.

While the ratio test is phrased as being in $\mathbb{R}$, in actuality the ratio test is a ratio between infinitesimals and infinities, none of which exist in $\mathbb{R}$ and the ratio is projected back to $\mathbb{R}$.

The standard ratio test does this via the limit, where the infinitesimals and infinities are realised in the ratio via a transfer [2]. For example, $\frac{1}{n+1}$ and $\frac{1}{n}$ are infinitesimals, their limit is 1 and corresponds to the indeterminate case for the ratio test. Their ratio, $\frac{n}{n+1} = \frac{n+1}{n+1} - \frac{1}{n+1}$ $= 1 - \frac{1}{n+1}_{n=\infty} = 1$. The infinitesimal is realized in this process.

Working in $*G$ gives the flexibility to consider the ratio test as not set in stone, but where terms can be multiplied and divided. This is something which the real number system alone, is not suited to, because it does not have infinitesimals or a transfer principle. Though, as we have seen, the transfer principle is applied via limits.

Finally we make a direct connection with this generalized ratio test and the boundary test [4], and show their equivalence.

The boundary test can be proved from the generalized ratio test, or vice versa.

2 The ratio test and variations

We now consider the tests from the number system's perspective. Consider the ratio test and generalizations in $*G$. The test can be algebraically reformed, and a transfer principle used to apply back to $\mathbb{R}$.

A sum $\sum a_n|_{n=\infty}$, by having a negative gradient $\frac{da_n}{dn} < 0$ and being positive, does not have to converge. However, the ratio test is in part a gradient test; we can transform the test to the continuous variable as a first derivative test. The ratio test is, therefore both a gradient and a magnitude test, the gradient being a necessary but not a sufficient condition.

We say $(\overline{\mathbb{R}}, <)$ to mean $(*G, <) \mapsto (\overline{\mathbb{R}}, <)$, as the algebra is in $*G$ and transferred to $\overline{\mathbb{R}}$, with $*G \mapsto \overline{\mathbb{R}}$ as the last step. By symmetry of the relation $<$, $(\overline{\mathbb{R}}, <)$ implies $(\overline{\mathbb{R}}, >)$.

These examples demonstrate the equivalence of the ratio test and modified ratio test.

Example 2.1. *Consider* $\sum \frac{1}{n}$ *by the ratio test, Theorem 2.1. Let* $a_n = \frac{1}{n}$, $\frac{a_{n+1}}{a_n}|_{n=\infty} = \frac{n}{n+1}|_{n=\infty} = 1$ *the indeterminate case.*

Using the modified ratio test, Theorem 2.2, a_{n+1} z a_n, $\frac{1}{n+1} < \frac{1}{n}$, *however, transferring this to* $\mathbb{R}$, $0 < 0$ *is a contradiction, hence this is also an indeterminate case.* $(*G, <) \not\mapsto (\mathbb{R}, <)$.

Example 2.2. *By the ratio test,* $a_n = \frac{1}{e^n}$, $\frac{a_{n+1}}{a_n}|_{n=\infty} = \frac{e^n}{e^{n+1}}|_{n=\infty} = \frac{1}{e} < 1$ *converges.*

By the modified ratio test, a_{n+1} z a_n, $\frac{1}{e^{n+1}} < \frac{1}{e^n}|_{n=\infty}$, $\frac{1}{e} < 1$, *converges.* $(*G, <) \mapsto (\mathbb{R}, <)$.

By considering the ratio test in a higher dimension, $*G$ with infinitesimals and infinities, the test does not have to be as a ratio. We can multiply and divide the terms, then by a transfer principle realize the test in $\overline{\mathbb{R}}$. Since the variations may be used as convergence tests, all have been stated as theorems.

Theorem 2.1. *Let* $a_n \in *G$, $(\overline{\mathbb{R}}, <)$.

$$\text{If } \frac{a_{n+1}}{a_n} < 1 \text{ then } \sum a_n|_{n=\infty} = 0 \text{ converges.}$$

$$\text{If } \frac{a_{n+1}}{a_n} > 1 \text{ then } \sum a_n|_{n=\infty} = \infty \text{ diverges.}$$

Proof. Given $z \in \{<, >\}$, in $*G$, $\frac{a_{n+1}}{a_n}|_{n=\infty} \; z \; 1$, $a_{n+1} \; z \; a_n|_{n=\infty}$, apply Theorem 2.2. □

Theorem 2.2. *$a_n \in *G$; $(\overline{\mathbb{R}}, <)$.*

$$\text{If } a_{n+1} < a_n \text{ then } \sum a_n|_{n=\infty} = 0 \text{ converges.}$$

$$\text{If } a_{n+1} > a_n \text{ then } \sum a_n|_{n=\infty} = \infty \text{ diverges.}$$

Proof. Given $z \in \{<, >\}$, in $*G$, $a_{n+1} \; z \; a_n|_{n=\infty}$, $a_{n+1} - a_n \; z \; 0|_{n=\infty}$, $\frac{da_n}{dn}|_{n=\infty} \; z \; 0$, convert to the continuous domain, $\frac{da(n)}{dn}|_{n=\infty} \; z \; 0$, apply Theorem 2.3. □

Theorem 2.3. *$a(n) \in *G$; $(\overline{\mathbb{R}}, <)$.*

$$\text{If } \frac{da(n)}{dn}|_{n=\infty} < 0 \text{ then } \int a(n)\, dn|_{n=\infty} = 0 \text{ converges.}$$

$$\text{If } \frac{da(n)}{dn}|_{n=\infty} > 0 \text{ then } \int a(n)\, dn|_{n=\infty} = \infty \text{ diverges.}$$

Proof. Substitute $m = 1$ into Theorem 3.1 which is equivalent to Theorem 3.2 with the inequalities inverted. The equality case is discarded. □

Proof. Although more complex, we find another proof combining integrating over relations and the transfer condition.

Consider a particle undergoing constant deceleration, where the particle cannot move backwards it will stop (In the infinitesimal domain, the particle can still be moving). The area swept by the particle has similarly stopped.

Expressing the conditions. Let $s(n) = \int a(n)\, dn$. Deceleration in $\mathbb{R}$: $\frac{d^2 s(n)}{dn^2} < 0$. The particle can only move forward. In $*G$: $\frac{ds(n)}{dn} \geq 0$.

$$\begin{aligned}
\frac{d^2 s(n)}{dn^2}|_{n=\infty} &< 0 && \text{(Integrating)} \\
0 \leq \frac{ds(n)}{dn}|_{n=\infty} &< c && \text{(c is positive, integrating)} \\
0 \leq s(n) &< cn + c_2|_{n=\infty} && (cn|_{n=\infty} \succ c_2) \\
0 \leq s(n) &< cn|_{n=\infty} && \text{(transfer preserving inequality as } (\overline{\mathbb{R}}, <)) \\
0 \leq s(n)|_{n=\infty} &< \infty && \text{($s(n)$ converges)}
\end{aligned}$$

Consider when the particle is under constant acceleration.

$$
\begin{aligned}
\frac{d^2s(n)}{dn^2}|_{n=\infty} &> 0 && \text{(Integrating)}\\
\frac{ds(n)}{dn}|_{n=\infty} &\geq c && \text{(c is positive, integrate)}\\
s(n) &\geq cn + c_2|_{n=\infty} && (cn + c_2 = cn|_{n=\infty})\\
s(n) &\geq \infty && \text{($s(n)$ diverges)}
\end{aligned}
$$

□

By threading a continuous function through the monotonic sequence a_n we can show the above to be the ratio test. The gradient of $a(n)$ is the curvature of $s(n)$.

Consider a sum of positive terms. Then the sum, by always having terms added, is increasing. Threading a continuous function through the series, the function $a(n)$ is always positive. $s(n) = \int a(n)\, dn$, $\frac{ds(n)}{dn} = \frac{d}{dn} \int a(n)\, dn = a(n) > 0$ is true in $*G$.

This is the continuous version of a sum with a negative sequence derivative, $\frac{a_{n+1}}{a_n} < 0$, $a_{n+1} < a_n$, $a_{n+1} - a_n < 0$, $\frac{da_n}{dn} < 0$, $\frac{da(n)}{dn} < 0$. The area or distance traveled by the particle is finite, and in the same way the sum is finite and converges.

Example 2.3. $\sum \frac{1}{n}|_{n=\infty} = \infty$ *is known to diverge. The ratio test fails to determine convergence. Let* $a_n = \frac{1}{n}$, $\frac{a_{n+1}}{a_n}|_{n=\infty} = \frac{n}{n+1}|_{n=\infty} = 1$ *is indeterminate.*

In working with the higher dimension $*G$ which includes the infinireals, when we realize and apply the tests, a less than relationship with infinitesimals is not a less than relationship in $\mathbb{R}$.

$$
\begin{aligned}
a_{n+1} \;\; z \;\; a_n|_{n=\infty} && \text{(Theorem 2.2)}\\
\frac{1}{n+1} \;\; z \;\; \frac{1}{n}|_{n=\infty} && \\
\frac{1}{n+1} < \frac{1}{n}|_{n=\infty} && \text{(Realizing the infinitesimals)}\\
0 < 0 \;\; \textit{contradicts} && \text{(Indeterminate result)}\\
&& \text{(Alternatively multiply the denominators out.)}\\
n < n+1|_{n=\infty} && \text{(Realizing the infinities)}\\
\infty \not< \infty && \text{(Indeterminate result)}
\end{aligned}
$$

The tests are the same and in their variation almost trivially similar to the classic ratio test. However it is nice to do things in different ways.

The limit ratio test, in its application can be varied as a ratio expression, multiplying and dividing the numerator and denominator. Rather than seeing the test set in stone, you can manipulate it. At times this is trivial, in other instances this becomes a way to transform tests.

Example 2.4. *Determine the convergence or divergence of* $\sum \frac{1\cdot 3\cdot \ldots (2n-1)}{3\cdot 6\cdot \ldots (3n)}|_{n=\infty}$.

Let $a_n = \frac{1\cdot 3\cdot \ldots (2n-1)}{3\cdot 6\cdot \ldots (3n)}$, $a_{n+1}\ z\ a_n|_{n=\infty}$, $\frac{1\cdot 3\cdot 5\cdot \ldots (2(n+1)-1)}{3\cdot 6\cdot \ldots (3(n+1))}\ z\ \frac{1\cdot 3\cdot \ldots (2n-1)}{3\cdot 6\cdot \ldots (3n)}|_{n=\infty}$, $\frac{2n+1}{3n+3}\ z\ 1|_{n=\infty}$, $2n+1\ z\ 3n+3|_{n=\infty}$ $1 < n+3|_{n=\infty}$ *and by Theorem 2.2 the series converges.*

When $\frac{a_{n+1}}{a_n}|_{n=\infty} = 1$, expressed as $a_{n+1} \simeq a_n$, use Raabe's test.

Theorem 2.4. *Raabe's test 1.*

$$n(\frac{a_n}{a_{n+1}} - 1)|_{n=\infty} = \begin{cases} > 1 & \text{then } \sum a_n|_{n=\infty} = 0 \text{ is convergent,} \\ < 1 & \text{then } \sum a_n|_{n=\infty} = \infty \text{ is divergent.} \end{cases}$$

Proof. Rearrange expression to one line. Let $z \in \{<, >\}$. $n(\frac{a_n}{a_{n+1}} - 1)|_{n=\infty}\ z\ 1$, $na_n - na_{n+1}\ z\ a_{n+1}$, $na_n - (n+1)a_{n+1}\ z\ 0$, prove by Theorem 2.6. □

Theorem 2.5. *Raabe's test 2.*

$$n(\frac{a_{n+1}}{a_n} - 1)|_{n=\infty} = \begin{cases} < -1 & \text{then } \sum a_n|_{n=\infty} = 0 \text{ is convergent,} \\ > -1 & \text{then } \sum a_n|_{n=\infty} = \infty \text{ is divergent.} \end{cases}$$

Proof. Rearrange expression to one line. Let $z \in \{<, >\}$. $n(\frac{a_{n+1}}{a_n} - 1)\ z\ -1$, $n(a_{n+1} - a_n)\ z\ -a_n$, $na_{n+1} - (n-1)a_n\ z\ 0$, relable index a_{n+1} to a_n, $na_n - (n-1)a_{n-1}\ z\ 0$, $(n+1)a_{n+1} - na_n\ z\ 0$, $na_n - (n+1)a_{n+1}\ (-z)\ 0$, and prove by Theorem 2.6. □

Theorem 2.6. *Raabe's test 3. In* $*G$ *and* $(\overline{\mathbb{R}}, <)$.

$$na_n - (n+1)a_{n+1}|_{n=\infty} = \begin{cases} > 0 & \text{then } \sum a_n|_{n=\infty} = 0 \text{ is convergent,} \\ < 0 & \text{then } \sum a_n|_{n=\infty} = \infty \text{ is divergent.} \end{cases}$$

If $na_n - (n+1)a_{n+1} > 0|_{n=\infty}$ *then* $\sum a_n|_{n=\infty} = 0$ *is convergent. If* $na_n - (n+1)a_{n+1} < 0|_{n=\infty}$ *then* $\sum a_n|_{n=\infty} = \infty$ *is divergent.*

Proof. $m = 0$ in Theorem 3.2, $\frac{a_n}{a_{n+1}}\ z\ 1 + \frac{1}{n}$, $\frac{na_n}{a_{n+1}}\ z\ n+1$, $na_n - (n+1)a_{n+1}\ z\ 0$. Case $z = >$ converges. $z = <$ diverges. □

Theorem 2.7. *See [8, 3.2.16], reformed with at-a-point notation.*

$$n \ln \frac{a_n}{a_{n+1}}|_{n=\infty} = \begin{cases} > 1 & \text{then } \sum a_n|_{n=\infty} = 0 \text{ is convergent,} \\ < 1 & \text{then } \sum a_n|_{n=\infty} = \infty \text{ is divergent.} \end{cases}$$

Proof. Rearrange into Raabe's theorem. Let $z \in \{<, >\}$.

$n \ln \frac{a_n}{a_{n+1}} \; z \; 1|_{n=\infty}$, $\ln \frac{a_n}{a_{n+1}} \; z \; \frac{1}{n}|_{n=\infty}$, $\frac{a_n}{a_{n+1}} > e^{\frac{1}{n}}|_{n=\infty}$, $a_n \; z \; a_{n+1} e^{\frac{1}{n}}|_{n=\infty}$. Substitute $e = (\frac{n+1}{n})^n|_{n=\infty}$ into the inequality, $a_n \; z \; a_{n+1}((\frac{n+1}{n})^n)^{\frac{1}{n}}|_{n=\infty}$, $a_n \; z \; a_{n+1}\frac{n+1}{n}|_{n=\infty}$, $na_n - (n+1)a_{n+1} \; z \; 0|_{n=\infty}$. This is Raabe's test, Theorem 2.6. □

3 A Generalized test

The ratio test can be generalized to produce other tests with the sum of the boundary functions. Each test involves higher order terms.

In the preceding discussion we proved Raabe's test(Theorem 2.6) by transforming the theorem into the ratio test.

Knopp [7, p.129] referred to a generalization of the ratio test Theorem 3.1, saying "only the test for $k = 0$ and at most $k = 1$ have any practical importance." Presumably this is because the ratio and Raabe tests are most often used.

Theorem 3.1. *[7, p.129] with m terms.*

$$[\frac{a_{n+1}}{a_n} - 1 + (\frac{1}{n} + \frac{1}{n \ln n} + \ldots + \frac{1}{n \ln n \ldots \ln_m n})] n \cdot \ln n \ldots \ln_m n = \begin{cases} < 0 & \text{then } \sum a_v \text{ is convergent,} \\ \geq 0 & \text{then } \sum a_v \text{ is divergent.} \end{cases}$$

Proof. By Proposition 3.1, rearrange to Theorem 3.2 which is subsequently proved. □

Constructing a ratio of $\frac{a_n}{a_{n+1}}$ instead of $\frac{a_{n+1}}{a_n}$ leads to a different, but equivalent formation, Theorem 3.2. See Proposition 3.1.

Theorem 3.2.

$$\frac{a_n}{a_{n+1}} - (1 + \frac{1}{n} + \frac{1}{n \ln n} + \ldots + \frac{1}{n \ln n \ldots \ln_m n})|_{n=\infty} = \begin{cases} > 0 & \text{then } \sum a_n|_{n=\infty} = 0 \text{ converges,} \\ \leq 0 & \text{then } \sum a_n|_{n=\infty} = \infty \text{ diverges.} \end{cases}$$

Definition 3.1. *An undefined sum has a value of* 0. *E.g.* $\sum_{k=2}^{1} x = 0$

Then when $m = -1$, $\sum_{k=0}^{m} \frac{1}{\ln_k} = 0$ Restating Theorem 3.2 with sum notation, we can define the sum to produce the ratio and higher order tests.

$$\frac{a_n}{a_{n+1}} - (1 + \sum_{k=0}^{m} \frac{1}{\ln_k})|_{n=\infty} = \begin{cases} > 0 & \text{then } \sum a_n|_{n=\infty} = 0 \text{ converges,} \\ \leq 0 & \text{then } \sum a_n|_{n=\infty} = \infty \text{ diverges.} \end{cases}$$

Successive values of m from -1 produce the tests. For example, with $m = -1$ and the remove of equality for the divergence case, gives the ratio test.

m	Comparison of terms	Test
-1	$\frac{a_n}{a_{n+1}} \, z \, 1$	Ratio test
0	$\frac{a_n}{a_{n+1}} \, z \, 1 + \frac{1}{n}$	Raabe's test
1	$\frac{a_n}{a_{n+1}} \, z \, 1 + \frac{1}{n} + \frac{\rho_n}{n \ln n}$	Bertrand's test [6]

Table 1: Tests

The table entry for Bertrand's test excluded the p-series as this is another test. $\rho > 1$ and $\rho < 1$ for the largest values of the sums become $1 + \frac{1}{n} + \frac{1}{n \ln n} > 1$ and $1 + \frac{1}{n} + \frac{1}{n \ln n} < 1$ respectively. These are the only cases that need to be considered, as ρ is just a real number. The assumption being $\rho \prec n \ln n|_{n=\infty}$, hence it could be factored to a real number greater than 1.

The generalized ratio test is proved by transforming the test to the boundary test, which we assume is true. By doing this, the boundary test is shown to be very general, and useful in proving other tests.

Proof. **Theorem 3.2** Assume the boundary test is true. Using algebra we transform the generalized ratio test into the boundary test.

$$
\begin{aligned}
\frac{a_n}{a_{n+1}} \; z \; 1 + \sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k}|_{n=\infty} && \text{(Generalized ratio)} \\
a_n \; z \; a_{n+1}(1 + \sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k}|_{n=\infty}) && \\
a_n - a_{n+1} \; z \; a_{n+1}(\sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k}|_{n=\infty}) && \text{(Interpet the difference as a derivative [5])} \\
-\frac{da_{n+1}}{dn} \; z \; a_{n+1}(\sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k}|_{n=\infty}) && \\
-\frac{da}{dn} \; z \; a(\sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k}|_{n=\infty}) && \text{(Convert to the continuous domain)} \\
-\int \frac{1}{a} da \; z \; \int \sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k} dn|_{n=\infty} && \text{(Separation of variables integral)} \\
-\ln a \; z \; \sum_{i=0}^{m} \int \frac{1}{\prod_{k=0}^{i} \ln_k} dn|_{n=\infty} && \\
-\ln a \; z \; \sum_{i=0}^{m} \ln_{i+1}|_{n=\infty} && \\
\ln a \; (-z) \; -\ln(\prod_{i=0}^{m} \ln_i)|_{n=\infty} && \text{(Raising to a base of } e \text{ does not change the relation)} \\
a \; (-z) \; \frac{1}{\prod_{i=0}^{m} \ln_i}|_{n=\infty} && \text{(The boundary test [4])} \\
a_n \; (-z) \; \frac{1}{\prod_{i=0}^{m} \ln_i}|_{n=\infty} && \text{(Convert to a series)}
\end{aligned}
$$

The $-z$ is correct, as the generalized ratio test defined z in the opposite direction. □

Corollary 3.1. *The boundary test and the generalized ratio test are equivalent.*

Proof. Since the algebra transformation from the ratio test to the boundary test is reversible, by starting from the boundary test and, in reverse order to the previous proof of Theorem 3.2, proceed to the generalized ratio test, hence both tests are equivalent. □

Proposition 3.1. *Theorem 3.1 and Theorem 3.2 are equivalent.*

Proof.

$$(\frac{a_{n+1}}{a_n} - 1 + \frac{1}{n} + \frac{1}{n \ln n} + \ldots + \frac{1}{n \ln n \ldots \ln_m n})n \ln n \ldots \ln_m n \; z \; 0$$

$$(\frac{a_{n+1}}{a_n} - 1)\prod_{j=0}^{m} \ln_j + (\ln_1 \cdot \ln_2 \ldots \ln_m + \ln_2 \cdot \ln_3 \cdot \ldots \cdot \ln_m + \ldots + 1) \; z \; 0$$

$$\frac{a_{n+1} - a_n}{a_n} + \sum_{j=0}^{m} \frac{1}{\prod_{k=0}^{j} \ln_k} \; z \; 0$$

$$\sum_{j=0}^{m} \frac{1}{\prod_{k=0}^{j} \ln_k} \; z \; -\frac{1}{a_n}\frac{da_n}{dn}$$

$$-\frac{da}{dn} \; z \; a(\sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k}|_{n=\infty})$$

(reversing to form the other ratio test)

$$-\frac{da_{n+1}}{dn} \; z \; a_{n+1}(\sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k}|_{n=\infty})$$

$$a_n - a_{n+1} \; z \; a_{n+1}(\sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k}|_{n=\infty})$$

$$a_n \; z \; a_{n+1}(1 + \sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k}|_{n=\infty})$$

$$\frac{a_n}{a_{n+1}} \; z \; 1 + \sum_{i=0}^{m} \frac{1}{\prod_{k=0}^{i} \ln_k}|_{n=\infty}$$

Reversing the above implies the other test. Hence both tests are equivalent. □

References

[1] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 1 Gossamer numbers*

[2] C. D. Evans, W. K. Pattinson, *Extending du Bois-Reymond's Infinitesimal and Infinitary Calculus Theory Part 4 The transfer principle*

[3] C. D. Evans, W. K. Pattinson, *Convergence sums at infinity a new convergence criteria*

[4] C. D. Evans, W. K. Pattinson, *The Boundary test for positive series*

[5] C. D. Evans, W. K. Pattinson, *Convergence sums and the derivative of a sequence at infinity*

[6] From MathWorld–A Wolfram Web Resource, *Bertrand's Test*, available at `http://mathworld.wolfram.com/BertrandsTest.html`

[7] Konrad Knopp translated by F. Bagemihl, *Infinite sequences and series*, New York Dover Publications, 1956

[8] W.J. Kaczor, M. T. Nowak, *Problems in Mathematical Analysis I*, AMS, Providence , 2000.

RMIT University, GPO Box 2467V, Melbourne, Victoria 3001, Australia
chelton.evans@rmit.edu.au