Hardy-Littlewood Conjecture and Exceptional real Zero

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Abstract. In this paper, we assume that Hardy-Littlewood Conjecture, we got a better upper bound of the exceptional real zero for a class of module.

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In this paper, we reveal the relationship between Hardy-Littlewood Conjecture and the exceptional real zero. Assume that Hardy-Littlewood Conjecture, we obtained a good results of the exceptional real zero.

In this paper, I generalize the results of my paper "An Application of Hardy-Littlewood Conjecture". q is the prime number and $q \equiv 3\,(mod\,4)$ is improved to q is odd square-free and $q \equiv 3\,(mod\,4)$. we know that the module q of the exceptional primitive real character are square-free when q is odd positive integer.

First, we give Hardy-Littlewood Conjecture.

Hardy-Littlewood Conjecture. When N is even integer and $N \geq 6$, we have

$$\sum_{\substack{3\leq p_1, p_2\leq N\\ p_1+p_2=N}} 1 \approx \frac{N}{\varphi(N)} \prod_{p\,\dagger N} \left(1 - \frac{1}{(p-1)^2}\right) \frac{N}{\log^2 N}$$

where p_1, p_2, p are the prime numbers, $\varphi(n)$ is Euler function.

Under the above conjecture, we have the following theorem

Theorem. Let q is odd square-free and $q \equiv 3\,(mod\,4)$, it has exceptional real character χ, and its Dirichlet $L(s,\chi)$ function has an exceptional real zero β. If Hardy-Littlewood Conjecture is correct, then there is a positive constant c , we have

$$\beta \leq 1 - \frac{c}{\log^2 q}$$

Now, we do some preparation work.

Lemma 1 . Assuming that the Hardy-Littlewood Conjecture. Let n is any positive integer, q is odd integer and large sufficiently, then

$$\sum_{\substack{3 \leq p_1, p_2 \leq 2nq \\ p_1 + p_2 = 2nq}} 1 \geq \frac{q}{\varphi(q)} \frac{2\,n\,q\,d}{(\log 2nq)^2}$$

where

$$d = \prod_{3 \leq p} \left(1 - \frac{1}{(p-1)^2}\right)$$

and p_1, p_2, p are the prime numbers.

Proof . By Hardy-Littlewood Conjecture, when N is large sufficiently, we have

$$\sum_{\substack{3 \leq p_1, p_2 \leq N \\ p_1 + p_2 = N}} 1 \geq \frac{N}{2\varphi(N)} \prod_{p \dagger N} \left(1 - \frac{1}{(p-1)^2}\right) \frac{N}{\log^2 N}$$

because

$$\prod_{p \nmid N}\left(1-\frac{1}{(p-1)^2}\right) \geq \prod_{3 \leq p}\left(1-\frac{1}{(p-1)^2}\right)$$

and

$$\frac{2nq}{\varphi(2nq)} = \prod_{p|2nq}\frac{p}{p-1} \geq \prod_{p|2q}\frac{p}{p-1} = \frac{2q}{\varphi(q)}$$

We choose $N = 2nq$, This completes the proof of Lemma 1 .

Lemma 2. Let m is positive integer and n is integer, then

$$\sum_{k=1}^{m} e\left(\frac{kn}{m}\right) = \begin{cases} m & if\ n \equiv 0\ (mod\ m) \\ 0 & otherwise \end{cases}$$

where $e(x) = e^{2\pi ix}$. The lemma 2 is obvious

Lemma 3. Let c_1 be the positive constant. if $(a, q) = 1$, then

$$\pi(x; q, a) = \frac{Lix}{\varphi(q)} - \frac{\chi(a)}{\varphi(q)}\int_2^x \frac{u^{\beta-1}}{\log u}du + O\left(x\exp(-c_1\sqrt{\log x})\right)$$

when there is an exceptional character χ modulo q and β is the concomitant zero. Where $Lix = \int_2^x \frac{du}{\log u}$ and $\exp(x) = e^x$

The lemma 3 follows from the References [2], Corollary 11.20 of the page 381

It is easy to see that

$$Lix = \int_2^x \frac{du}{\log u} = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

and

$$\int_2^x \frac{u^{\beta-1}}{\log u} du = \frac{x^\beta}{\beta \log x} + O\left(\frac{x^\beta}{\log^2 x}\right)$$

Lemma 4. if χ is a primitive character modulo m, then

$$\sum_{k=1}^{m} \chi(k) e\left(\frac{nk}{m}\right) = \overline{\chi}(n)\tau(\chi)$$

where $\tau(\chi) = \sum_{k=1}^{m} \chi(k) e(\frac{k}{m})$.

The lemma 4 follows from the References [1], the page 47.

Lemma 5. if m is odd square-free and χ is a primitive real character modulo m, then

$$\tau(\chi) = \begin{cases} \sqrt{m} & if\ m \equiv 1\ (mod\ 4) \\ i\sqrt{m} & if\ m \equiv 3\ (mod\ 4) \end{cases}$$

The lemma 5 follows from the References [1], the theorem 3.3 of the page 49.

Lemma 6. We give the value of two sums, they are used in the proof of the Theorem.

(1)

$$\sum_{k=1}^{q} \left(\sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} e\left(\frac{ak}{q}\right) \right)^2 = \sum_{k=1}^{q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} e\left(\frac{ak}{q}\right) \sum_{\substack{b=1 \\ (b,q)=1}}^{q-1} e\left(\frac{bk}{q}\right)$$

$$= \sum_{k=1}^{q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \sum_{\substack{b=1 \\ (b,q)=1}}^{q-1} e\left(\frac{(a+b)k}{q}\right) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \sum_{\substack{b=1 \\ (b,q)=1}}^{q-1} \sum_{k=1}^{q} e\left(\frac{(a+b)k}{q}\right)$$

$$= q \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \sum_{\substack{b=1 \\ (b,q)=1,\, a+b=q}}^{q-1} 1 = q\,\varphi(q)$$

(2)

$$\sum_{k=1}^{q} \chi(k) \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} e\left(\frac{ak}{q}\right) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \sum_{k=1}^{q} \chi(k) e\left(\frac{ak}{q}\right) = \tau(\chi) \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \overline{\chi}(a) = 0$$

where χ is the primitive character modulo q,

This completes the proof of Lemma 6 .

PROOF OF THEOREM.

The first part.

By Lemma 2, when $x \geq q^4$, we have

$$\sum_{k=1}^{q} \left(\sum_{3 \leq p \leq x} e\left(\frac{kp}{q}\right)\right)^2 = \sum_{k=1}^{q} \sum_{3 \leq p_1 \leq x} \sum_{3 \leq p_2 \leq x} e\left(\frac{k(p_1+p_2)}{q}\right)$$

$$= \sum_{3\le p_1\le x}\sum_{3\le p_2\le x}\sum_{k=1}^{q} e\left(\frac{k(p_1+p_2)}{q}\right) = q \sum_{\substack{3\le p_1,p_2\le x\\ p_1+p_2\equiv 0\,(q)}} 1 \ge q \sum_{n=1}^{[\frac{x}{2q}]} \sum_{\substack{3\le p_1,p_2\le x\\ p_1+p_2=2nq}} 1$$

by Lemma 1 , the above formula

$$\ge \frac{q^2}{\varphi(q)} \sum_{n=1}^{[\frac{x}{2q}]} \frac{2nqd}{\log^2 2nq} \ge \frac{q^2}{\varphi(q)} \sum_{n=1}^{[\frac{x}{2q}]} \frac{2nqd}{\log^2 x} \ge \frac{2dq^3}{\varphi(q)\log^2 x} \sum_{n=1}^{[\frac{x}{2q}]} n$$

$$= \frac{2dq^3}{\varphi(q)\log^2 x} \cdot \frac{[\frac{x}{2q}]([\frac{x}{2q}]+1)}{2} \ge \frac{q\,d\,x^2}{4\varphi(q)\log^2 x} + O\left(\frac{xq^2}{\varphi(q)\log^2 x}\right)$$

The second part.

When $1 \le k \le q$, we have

$$\sum_{3\le p\le x} e\left(\frac{pk}{q}\right) = \sum_{\substack{3\le p\le x\\ (p,q)=1}} e\left(\frac{pk}{q}\right) + O(\log q) = \sum_{\substack{a=1\\ (a,q)=1}}^{q} e\left(\frac{ak}{q}\right) \sum_{\substack{3\le p\le x\\ p\equiv a(q)}} 1 + O(\log q)$$

by Lemma 3 and Lemma 4, the above formula

$$= \sum_{\substack{a=1\\ (a,q)=1}}^{q} e\left(\frac{ak}{q}\right)\left(\frac{Lix}{\varphi(q)} - \frac{\chi(a)}{\varphi(q)}\int_2^x \frac{u^{\beta-1}}{\log u}du + O\left(x\exp(-c_1\sqrt{\log x})\right)\right) + O(\log q)$$

$$= \frac{Lix}{\varphi(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{ak}{q}\right) - \frac{\tau(\chi)\chi(k)}{\varphi(q)} \int_2^x \frac{u^{\beta-1}}{\log u} du + O\left(qx \exp(-c_1\sqrt{\log x})\right)$$

where χ is the exceptional primitive real character modulo q.

therefore

$$\left(\sum_{3\leq p\leq x} e\left(\frac{pk}{q}\right)\right)^2 = \left(\frac{Lix}{\varphi(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{ak}{q}\right)\right)^2$$

$$-\frac{2\tau(\chi)\chi(k)Lix}{\varphi^2(q)} \left(\sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{ak}{q}\right)\right) \int_2^x \frac{u^{\beta-1}}{\log u} du + \left(\frac{\chi(k)\tau(\chi)}{\varphi(q)} \int_2^x \frac{u^{\beta-1}}{\log u} du\right)^2$$

$$+O\left(q^2x^2 \exp(-c_2\sqrt{\log x}\right)$$

By Lemma 5 and Lemma 6, we have

$$\sum_{k=1}^{q} \left(\sum_{3\leq p\leq x} e\left(\frac{pk}{q}\right)\right)^2 = \frac{q}{\varphi(q)} (Lix)^2 - \frac{q}{\varphi(q)} \left(\int_2^x \frac{u^{\beta-1}}{\log u} du\right)^2$$

$$+O\left(q^3x^2\exp(-c_2\sqrt{\log x}\right)$$

$$=\frac{q\,x^2}{\varphi(q)\log^2 x}-\frac{q\,x^{2\beta}}{\varphi(q)\beta^2\log^2 x}+O\left(\frac{q\,x^2}{\varphi(q)\log^3 x}+q^3x^2\exp(-c_2\sqrt{\log x}\right)$$

We synthesize the first part and second part, we have

$$\frac{q\,d\,x^2}{4\varphi(q)\log^2 x}\le\frac{q\,x^2}{\varphi(q)\log^2 x}-\frac{q\,x^{2\beta}}{\varphi(q)\beta^2\log^2 x}$$

$$+O\left(\frac{x^2}{\log^3 x}+\frac{xq^2}{\varphi(q)\log^2 x}+q^3x^2\exp(-c_2\sqrt{\log x})\right)$$

$$\frac{d\,x^2}{4\log^2 x}\le\frac{x^2}{\log^2 x}-\frac{x^{2\beta}}{\beta^2\log^2 x}+O\left(\frac{x^2}{\log^3 x}+\frac{x\,q}{\log^2 x}+q^3x^2\exp(-c_2\sqrt{\log x})\right)$$

$$\frac{d}{4}\le 1-\frac{x^{2\beta-2}}{\beta^2}+O\left(\frac{1}{\log x}+\frac{q}{x}+q^3\exp(-c_3\sqrt{\log x})\right)$$

we take $\log x=(\frac{4}{c_3}\log q)^2$, then

$$x^{2\beta-2}\le 1-\frac{d}{4}+\frac{c_4}{\log^2 q}$$

we take $\log q\ge\sqrt{\frac{8c_4}{d}}$, then

$$x^{2\beta-2} \leq 1 - \frac{d}{8}$$

$$\beta - 1 \leq \frac{\log(1-\frac{d}{8})}{2\log x} = -\frac{\log(\frac{8}{8-d})}{2\log x}$$

therefore

$$\beta \leq 1 - \frac{c}{\log^2 q}$$

This completes the proof of Theorem.

Because $\tau(\chi_4) = 2i$ and $\tau(\chi_8) = 2\sqrt{2}$, when q is odd square-free, by same method, for module $4q$,$q \equiv 1(mod4)$ and module $8q$,$q \equiv 3(mod4)$, we have the same conclusion.

REFERENCES

[1] Henryk Iwaniec, Emmanuel Kowalski, *Analytic Number Theory,* American mathematical Society, 2004.

[2] Hugh L. Montgomery, Robert C. Vaughan, *Multiplicative Number Theory I. Classical Theory,* Cambridge University Press, 2006.