A Nopreprint on the Pragmatic Logic of Fractions

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Abstract

A survey of issues concerning the pragmatic logic of fractions is presented, including a seemingly paradoxical calculation. The presence of nested ambiguity in the language of fractions is documented. Careful design of fraction related datatypes and of logics appropriate for such datatypes is proposed as a path towards novel resolution of these complications. The abstract datatype of splitting fractions is informally described. A rationale of its design is provided. A multi-threaded research plan on fractions is outlined.

Contents

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1 Introduction

Fractions constitute a difficult topic by all means. For instance it is not clear to which extent the concept of a fraction is a mathematical concept. Can it be true that fractions reside in the world of educational mathematics only, and that fractions don't exist in real mathematics, whatever that may be.

1.1 The intrinsic complexity of the concept of fraction

The concept of a fraction defeats definition in an ordinary sense. To appreciate this assessment of the state of affairs one may notice the following listing of viewpoints some of which are mutually inconsistent:

1. It is quite common to say that fractions are numbers.

2. If fractions are numbers then fractions are rational numbers. Indeed it is common to state that rational numbers are fractions and that fractions are rational numbers.

3. It is common to say that $\frac{1}{2}$ and 50% are the same. However, some would say that $\frac{1}{2}$ is 50% of 1 rather than merely 50%.

4. It is common to say that fractions represent rational numbers, so for instance the fractions $\frac{1}{2}$ and $\frac{2}{4}$ may be considered different representations of the same rational number.

5. Rational numbers are sets of pairs of integers with some (author dependent) constraints on the second integer. The pairs that occur in such sets are fractions. This is the mathematical dogma, so it seems, but the unavoidable conclusion that a fraction cannot be a rational number because

sets are well-founded in ZF set theory (and Aczel's non-well-founded set theory is unknown) is somehow missed.

6. The fractions $\frac{1}{2}$ and $\frac{2}{4}$ are different.

7. The fractions $\frac{1}{2}$, $1\colon 2$, 0.5, and $1/2$ are the same by all means.

8. The fractions $\frac{1}{2}$, 0.5, and $1/2$ are the same but $1\colon 2$ is a ratio which is not a fraction.

9. Decimal fractions are exact and therefore $1 + 1.199 + 1.301 = 1.500$.

10. Decimal fractions are approximate and therefore $1 + 1.199 + 1.301 = 1.5$.

11. Fractions may be considered expressions for rational numbers.

12. Fractions may simply be viewed as pairs of integers, or as pairs of integers with the second integer positive, or as the union of integers and the pairs of integers with the second integer positive and above 1.

13. There are many different (conceivable) datatypes in which fractions are contained, but mathematics pays no attention to datatypes. However, viewed from the perspective of datatype theory, fractions don't exist outside datatypes that contain them.

14. Although fractions are supposed to have a numerator and a denominator, (two natural attributes of a fraction so to say), for some reason schoolbooks never introduce functions `num(-)` (for the numerator) and `denom(-)` (for the denominator) which produce these attributes.

15. In the presence of functions `num(-)` and `denom(-)` the following (fallacious but seemingly paradoxical) chain of equations arises: $1 = \mathtt{num}(\frac{1}{2}) = \mathtt{denom}(\frac{2}{4}) = 2$. Pointing out what is wrong with this of $1 = 2$ is not entirely straightforward.

Padberg [20] exposes the viewpoint that fraction is a complex concept which allows for the coexistence of many different views on the subject, each of which a competent student should learn to master sooner or later. Some views are harder to master than others. Rollnik [22] suggests that a choice must be made and he chooses for Viewpoint 2 in the above listing, thereby dismissing most of the other viewpoints. In [15] one finds the view that fraction has 5 different interpretations (part-whole, ratio, operator, quotient, and measure (a collection of 5 interpretations attributed to Carolyn Kieran) and that for each interpretation there are several representations (that is representational formats, of which five are proposed: symbolic, area/region, number line, sets of objects, liquid measures), thus leading to a matrix of 25 ($= 5 \cdot 5$) entries. In this work different interpretations are supported by the same symbolic notation.[1]

[1]It is pointed out in the iTalk2learn project documentation that the so-called symbolic representation comprises a numerator and a denominator. Other representations have other

The spectrum of views on fractions differs from the spectrum of views on say, reals (working in ZF, ignoring intuitionistic and constructivist views) in that for reals one deals with visibly different mathematical constructions for structures that turn out to be isomorphic, which is considered a mathematical fact. In the case of fractions the different viewpoints primarily relate to different styles of mathematical presentation, rather than to different constructions in mathematics.

1.2 How important are fractions?

While the current importance of fractions can hardly be overstated, their future importance can hardly be properly assessed. Nowadays fractions feature in each syllabus on elementary mathematics and successful teaching about working with fractions aims at reaching a significant part of the world-wide population. This educational mega-task represents an ambition which is quite challenging and virtually unchallenged at the same time.

I see no intrinsic importance in fractions other than the fact that these appear in all teaching of elementary mathematics. I also think that mathematics teaching would be greatly simplified if one stopped talking about fractions. For instance if one could speak of rational numbers (or rationals for short) when meaning is of prior importance and of fracterms (see [10]) when expressions are meant, most of the conceptual difficulties would instantaneously disappear. Fracterms have a numerator and a denominator. Fracterms are subject to simplification, fracterms change when being simplified. 1/2 and 2/4 are different fracterms denoting the same rationals.

Now the paradoxical argument that $1 = \texttt{num}(\frac{1}{2}) = \texttt{denom}(\frac{2}{4}) = 2$ is resolved by pointing out that it features a serious typing error. If fracterms are explicity typed as such by writing say $(\frac{1}{2})_{ft}$, while the default typing of $\frac{1}{2}$ is as a rational, then $1 = \texttt{num}((\frac{1}{2})_{ft}) = \texttt{denom}((\frac{2}{4})_{ft}) = 2$ is clearly wrong because $(\frac{1}{2})_{ft} = (\frac{2}{4})_{ft}$ cannot be inferred from $\frac{1}{2} = \frac{2}{4}$.

But as things stand the issues concerning fractions cannot be solved that easily because of deeply rooted conventions in the use of the language of fractions. Whoever tries to simplify the subject by introducing either new terminology (such as rational versus fracterm), or by using existing jargon in a limited way only (fractional rational, versus fraction expression), or by choosing one of both defaults (fraction as rational versus fraction representation, or fraction as expression versus rational as its meaning), runs the risk of being ignored and even ridiculed by those who look at one's actions from a distance.

Mathematicians have the remarkable habit of classifying such worries always as a lack of understanding of their subject, while the opposite is true: the persistence of these complications reveals an ostentative lack of attention of the mathematical community for the interface of mathematics with language,

features which serve as attributes for all instances of that representation, for instance the whole-part representation uses a number of items. However, only the existence of both attributes of the symbolic notation is generally known. Further the description of 15 of the 20 non-symbolic representations makes use of symbolic representations.

notation, logic, and the conventions of usage. In my view mathematics has not in any way earned the right to escape from such questions. We need to ask what is information in times of information abundance, we need to ask what is democracy in times that elections abound, we need to ask questions about mathematics just as well.

Conceptual questions about fractions cannot be entirely separated from questions about the foundations of mathematics. But these subjects are not the same. These questions have not much common ground with Gödel's results, Hilbert's program, the consistency of ZFC, the need for constructive mathematics, the creative subject, or the status of nominalism in the appreciation of mathematical physics. For hard core foundations of mathematics a problem that can be solved by a mere renaming is not a significant problem. Indeed many aspects come into pay that have a minor role in the foundations of mathematics at best: the strength of conventions, issues of mass communication, various aspects of human factors, the psychological virtue of the presence of paradoxical reasoning patterns, the statistics of classroom success of various educational methods, the changing view on which part of mathematical awareness and practice will be automated and which part will be somehow consciously used by human agents.

1.3 Pragmatic logic of educational mathematics

It is not easy to find a name for a field of investigation that results when the mentioned worries concerning fractions are taken seriously as first class research questions (which I think they are). It is not a (real or potential) branch of the foundations of mathematics, it is not a (real or potential) branch of educational mathematics, it is not a branch of logic.

I will work under the heading of "pragmatic logic of educational mathematics". The phrase pragmatic logic is explained in detail in [18], and has been introduced by the Polish logician Kazimierz Adjukiewicz in [1] and earlier works. The phase pragmatic logic is criticized by Griffin in [14]. Griffin argues that practical logic would be have been more appropriate phrase in this case. To motivate my choice for pragmatic logic I may mention these arguments: (i) the methodology of teaching was considered a major area of application for pragmatic logic, (ii) pragmatic logic may but need not have an informal style whence the more common phrase informal logic is less appropriate for the topic at hand, and (iii) Pragmatic logic as investigated in the Lwow-Warzsawa School (often referred to as LWS) is liberal in its choice of techniques, (iv) this tradition brought forward one of the first paraconsistent logics as formulated by Jaskowski in 1949.[2]

[2]This matters because (in my view) paraconsistency paraconsistency provides an essential feature for pragmatic logics of fractions and other ingredients of elementary mathematics.

1.3.1 Pragmatic logic of fractions

Pragmatic logic of fractions constitutes a topic in the pragmatic logic of educational mathematics which I find particularly rewarding because of its links with as diverse aspects as: paraconsistent logic, 3 valued logics, short circuit logic, abstract datatypes, modularization, partial algebras, equational logic, term rewriting, term rewriting with conditions, term rewriting with priorities, regular rings, reduced rings, model theory of fields, and complexity of algorithms.

1.3.2 Pragmatic logic of bounded arithmetic

A second theme in the pragmatic logic of educational mathematics which I expect to be particularly rewarding emerges if one strives to replace infinite number systems by finite ones. Term rewriting and datatype theory are major tools for this theme, and principal question that emerges in different versions is this: how exactly is it the case that infinite systems are simpler than finite ones, cannot we redesign finitary arithmetics in such a way that advantages appear which have been lost, perhaps even without having been noticed, when moving to infinite systems.

1.3.3 Arithmetical datatypes

The investigation of datatypes for fractions belongs to the larger topic of arithmetical datatypes. In arithmetical datatypes on tries to design and specify old as well as novel mathematical structures with the techniques that were developed for datatype specification in informatics. This area leaves substantial room for future work.

2 Ambiguous fractions: a case of nested ambiguity

I will now focus on the contrast between two major options only, fractions as rationals and fractions as representations of rationals. Interestingly already this presentation of the contrast is asymmetric. Alternatively one may consider: fractions as expressions versus fractions as meanings of expressions. Now both presentations of the same contrast are quite similar but the point is that both provide opposite implicit suggestions regarding the default meaning or interpretation of the concept of fraction. The occurrence of ambiguity in the presentation of ambiguity about fractions is a phenomenon that I will refer to as nested ambiguity. The nested aspect of this particular ambiguity contributes to its stability and to the difficulty for solving it by means of the introduction of linguistic conventions or of novel terminology.

2.1 Fractions as numbers versus fractions as names of numbers

Fractions as numbers, and in particular as rational numbers is a conception of fractions which is advocated for instance by Rollnik in [22]. The idea is that fractions are a mere notational tool needed for working with rationals, and that many different representations exist, the variety of different fraction expressions constituting a single dimension of variation only. If one writes fraction expressions in various colors then red fractions may be confronted with blue fractions and so on. For Rollnik the existence of variation in naming is quite common and unrelated to numbers and for that reason it presents no incentive to introduce (and teach) a theory of names.

That the same fraction can be pointed at via different names is implicit in the notion of naming. Having a function `num(-)` that determines a numerator is not considered a necessity just as having a function that computes the color of the name of a fraction would be a redundant (and therefore confusing) feature. Having a clear perspective on the rational numbers as a mathematical construction is vital, however.

2.2 Fractions as expressions versus fractions as meanings of expressions

When viewing fractions as expressions contemplating the variety of their meanings becomes more important than focusing on a i single interpretation in isolation. The fraction $1/2$ may be viewed as a rational number, a real number, a complex number, an element of a Galois field and so on. In [10] an interpretation of fractions as expressions is provided under the name of fracpairs which is less abstract than the field of rationals.

Just as there is a significant variety in mathematical structures that allow the interoperation of fractions there is a significant variety in options for a syntax that allows the construction of expressions that are plausibly classified as fractions (now assuming that fractions are expressions by default). The existence of so-called term models implies that making a rigorous distinction between syntax and semantics may be almost irrelevant in some particular cases.

2.3 Fractions in a practical context: an issue that can be postponed

In educational mathematics one may hold the view that fractions and other mathematical types can be understood only by students who are supported in viewing and understanding how these objects play a role in meaningful operations and contexts. Preferably one wishes to illustrate how fractions may play a role in human decision taking[3] Now it seems to be undisputed that the rational

[3] I refer to [2] for my views on decision taking from which in principle a role for numbers and fractions in decision taking can be derived).

numbers have such a wide range of potential and actual applications that taking them for a topic that can be profitably studied in advance of any particular application is a reasonable strategy. But as an educational strategy it may fail with students who for some reason vastly prefer to abstract from the concrete to going this path in their mind in the opposite direction. Now I hold that reasoning about fractions is something that may lie in between of doing arithmetic and having it meaningfully applied.

For instance why not state that there is an ordering on fractions which yet has to be understood by a student. While it seems to be obvious that one may say to a person unable to drive a car that car driving is a competence that has yet to be acquired, it seems to be problematic to first list and discuss competences concerning numbers and fractions before actually acquiring these competences.

2.4 Fractions in a logical context: in need of clarification

The logical context of integers and fractions provides room for competences, in particular reasoning skills concerning numbers and fractions and operations on these, which can be acquired in advance of actual calculating skills. In consider the acquisition of such prerequisite competences to be more urgent than becoming fluent with contexts that require these skills. Why not explain a student the idea of addition, and subsequently explain in full detail how a selling point functionality works and thereafter discuss the question to what extent the ability to add two numbers is of any relevance for the sales process before the student is taught how to add in full and meticulous detail.

3 Towards a pragmatic logic of fractions

I wish to put forward the following list of working hypotheses:

1. The logic of common sense thinking about fractions is not known to the average mathematician (or teacher) with any accuracy. In my view that logic is not simply two valued classical predicate logic. It seems to involve at least the following ingredients:

    - 3 truth values (with as a third value $\perp$),
    - asymmetric (short circuit) logical connectives,
    - an error value (say a_V for additional value) for each sort V except the sort of Booleans,
    - two valued equational logic with and without error elements,
    - initial algebra specification for abstract datatypes,
    - complete term rewriting systems and canonical term algebras for the specification of concrete datatypes,

- the use of a_V to represent the result of functions which one might prefer to model intuitively as partial functions in the case that one insists to maintain a standard 2 valued equational logic instead of following the mentioned intuition of partiality,
- a chunck & permeate style backbone of paraconsistent logic,
- chunck & permeate style paraconsistent reasoning in order to link algebras with errors in a 2-valued logic setting with corresponding partial algebras in a setting of 3-valued logic,
- a non-strict conditional operator for each sort V.

2. Without having (that is creating in the mind of a student) a clear view on the logical basis of reasoning about numbers and fractions there is not much point in working towards (the awareness in the mind of a student of) an external (material or otherwise) context for working with numbers and fractions. In other words: the detailed analysis of logics of (working with and thinking about) fractions should be given precedence over the search of realistic contexts in which fractions are used (but not properly understood).

3. The logical context of numbers and fractions requires urgent attention in view of the many implicit occurrences of contexts that seem to allow that allow (but never in detail describe) reasoning patterns like $1 = \mathtt{num}(\frac{1}{2}) = \mathtt{denom}(\frac{2}{4}) = 2$.

4. Going for "real solutions" is crucial. Solutions to the challenge of invalidating the argument $1 = \mathtt{num}(\frac{1}{2}) = \mathtt{denom}(\frac{2}{4}) = 2$ require some kind of careful reasoning practice. But such a solutions is useless if its adoption in practice is very unlikely. That applies in my view to a proposal to equip fraction occurrences with additional typing information (allowing to distinguish a bias towards meaning from a bias towards expression).

5. The way forward involves the design of a family of datatypes each incorporating fractions in different ways. As examples I mention the following:

   (a) The meadow of rational numbers for which an algebraic specification has been given in [11] (with an improvement in [7]).[4]

   (b) The non-involutive meadows of [8].

   (c) The common meadow of rationals as defined in [9], for which an independent construction is given in [10].

   (d) The wheels as introduced in [23] and investigated in detail in [12, 13]. Wheels are a formalisation of Riemann's sphere of complex numbers.

[4]Meadow is merely a new label that was introduced in [11] for structures that were proposed and analysed in depth already in [17, 19]. Recently I the label meadow is used more generally for field-like structures that are equipped with either a unary function serving as a multiplicative inverse for most arguments, or a two place division operator.

(e) The datatype of fractions as it occurs in [6]. That datatype is quite different from the other datatypes in that it allows the addition of fractions with equal denominators only. Highlighting this construction is supported by having a better name for it at hand, and I will use the phrase "splitting fractions" for that datatype in recognition of the fact that it precisely captures the property of a fraction that it can be split in a numerator and a denominator. Below I will discuss the datatype of splitting fractions in more detail as an example.

(f) The transrational numbers (that is the rational part of the transreal numbers) as proposed by Anderson and discussed in [21].

6. Each datatype that has been designed in the context of educational mathematics will give rise to its own question about what is its most appropriate logic. The work of Shapiro [24] indicates that having a logic design tailor made for the particularities of a specific structure may be considered modern rather than being considered disappointingly unsystematic.

3.1 The datatype of splitting fractions

A splitting fraction is a fraction conceived in such a way that it can be split in a numerator and a denominator thereby producing a pair from which it can also be reconstructed. In [6][5] a datatype of fractions is designed in which sums of fracterms (fraction expressions) with different denominators lead to an error value. Addition of fracterms with equal denominator is possible in the usual manner.

I will call this datatype the splitting fraction datatype. Functions nominator and denominator are represented by way of functions $\texttt{num}(-)$ and $\texttt{denum}(-)$. In the presence of these functions the datatype for splitting fractions is fully abstract.

3.1.1 On adding apples and pears

It is conventional wisdom to assert that apples and pears cannot be added. But if one decomposes both objects in an array of numbers of molecules then componentwise addition becomes an option. Addition merely requires a theory of decomposition, a theory of addition at the decomposed level, and a theory of object reconstruction from decomposed status. It is the lack of a method for reconstructing an apple or pear from its spectrum of molecules that constitutes the fundamental obstacle to their addition.

Now splitting fractions focuses on a level of abstraction where the decomposition of say $1/3$ into two instances of $1/6$ is not supported, and neither is the reconstruction of $1/3$ from those fragments by means of addition, while the less conventional decomposition into 1 and 3 (in different respective roles) is provided for. That decomposition, however, gives little information on how to add

[5] I assume that readers are willing to consult [6] for relevant details which for that reason won't be repeated in the current text.

say 1/3 and 1/2. (Of course addition is usefully supported by the decomposing of 1/3 in two 1/6's and 1/2 in three 1/6's.)

3.1.2 The role of an error element

A design choice (I refer to [5] for more information concerning that phrase) that went into the description of the datatype of (splitting) fractions in [6] is that sorts are equipped with an error element (called a for additional element, thus following the convention of [9]). The reason for having an error element is that it helps to have all operations (and in particular division) total, which in turn allows to work with a classical two valued logic on top of the most conventional equational logic. More importantly, however, an error element is needed (unless partiality is preferred) to represent the result of the addition of fractions with unequal denominators.

The idea of splitting fractions is just as plausible if division by zero is considered undefined, and if accordingly the result of adding fractions with different denominators is considered undefined, merely the corresponding logic gets more involved (3-valued instead of 2-valued).

3.1.3 The role of an algebraic specification

It is unproblematic to provide algebraic specifications for the splitting rationals datatype. Obtaining good term rewriting properties is doable too, even much more easily than for any version of the rational numbers. These designs still have to be worked out in detail, however.

More importantly, however, just as the structure of rational numbers exceeds in importance each of its particular specifications, the structure of splitting fractions exceeds in terms of importance each of its specifications. In other words, the datatype of splitting fractions is not an ad hoc construction serving as the model of some specification that happens to have been designed for some accidental reason. On the contrary, as its role in understanding the logic of fractions is predictably very important, the datatype of splitting fractions and its family of specifications is designed in stages, with [6] featuring an initial stage of evolution of such specifications. In the current paper I merely put forward some comments about that particular datatype which might motivate subsequent efforts at giving more elegant or more efficient specifications.

3.2 Design alternatives

The design of a datatype of splitting fractions is not without alternatives. Here I will discuss one option for an alternative design in some detail.

3.2.1 Addition as usual

Splitting fractions allow component-wise multiplication, and semi-component-wise addition (assuming that arguments have equal denominators). Now one may have an alternative definition of addition:

$$y \neq v \to \frac{x}{y} + \frac{u}{v} = \frac{(x \cdot v) + (u \cdot y)}{y \cdot v}.$$

$$\frac{x}{y} + \frac{u}{y} = \frac{x + u}{y}.$$

Using this definition we find that $\texttt{denum}(1/2+1/3) = \texttt{denum}(2/6) = 6$ rather than a as in the datatype proposed in [6].[6] In this alternative setting one finds non-trivial equations (that is, different from what is valid in [6]) such as:

$$\frac{1}{2} + \frac{0}{1} = \frac{1}{2}.$$

Here the only deviation from rational numbers is that simplification of fractions is not allowed. Now it is not essential to use an error value, though after moving to a homomorphic image that equates fractions with their simplified versions an error element results as the value of $1/0$.

3.2.2 How to compare the alternatives

The alternative mentioned above is reasonable, though its advantage over the design of [6] is not entirely obvious. A disadvantage is that one needs to understand "full" fraction addition at the earliest stage. An advantage is that `num(-)` and `denum(-)` coexist with a homomorphic preimage of the common meadow of rationals.

Both designs feature in different paradigms: progressive introduction of features ([6]) and progressive use of abstractions (e.g. as in [10]).

Making the step to the rational numbers from this design of splitting fractions involves paraconsistent reasoning (following [6]) if one insists that both the splitting fractions and the rationals are simply called fractions. Alternatively, one may consider the clash between fraction simplification and the presence of `num(-)` and `denum(-)` as an instance of feature interaction (see [16]), which leads to yet another perspective on methods for solving such internal contradictions.

The research program just outlined asks for the systematic design of relevant datatypes in connection with rational numbers. Thus it is conceivable that more alternatives need to be investigated before a convincing design is found.

The very existence of these two design alternatives does not imply that either one is better. On the contrary the quality of either datatype of splitting fractions must be assessed in view of its application in a communicative context: how will it support the design of suitable logics, and how will the combination of datatype and logic support its application in a meaningful context, and finally how can all of this be exploited for educational purposes.

[6] Strictly speaking one may claim that the alternative is so different from the original datatype that making reference to [6] is pointless and that it is wrong to assert that both are formalisations of the same idea. Indeed one may deny that viewing both as design alternatives standing on an equal footing is justified. The terminology and methodology of datatypes will not be very informative for resolving such issues.

3.2.3 Yet another alternative

In fact many alternatives for the splitting fraction datatype of [6] can be found. For instance with $\texttt{gcd}(n,m)$ denoting the greatest common divisor of n and m and $\texttt{scm}(n,m)$ denoting the smallest common multiple of n and m, if both n and m are nonzero and non-error and error otherwise, and with / standing for integer division one may use the following defining equation for addition:

$$y \neq 0 \wedge y \neq a \wedge v \neq 0 \wedge v \neq a \to \frac{x}{y} + \frac{u}{v} = \frac{((x \cdot v) + (u \cdot y))/\texttt{gcd}(y,v)}{\texttt{scm}(y,v)}.$$

This alternative does away with the rather unnatural case distinction on denominator equality, but it does so at cost of an additional import and use of non-trivial integer arithmetic. At the time of writing I see no virtue in this alternative, but in a systematic approach to the design of datatypes for splitting fractions it clearly requires some attention.

3.3 Multi-threading and multi-tasking

In terms of the design of a research project on arithmetical datatypes with ambitions towards the development of educational applications there is no substitute for multi-threading (see [3] for my use of that term). Different threads can be devoted to different objectives, each equipped with one or more subthreads. A first multi-thread description for this purpose may look as follows:

Arithmetical datatypes. Development of a family of relevant datatypes and of their mathematical descriptions (algebraic specification) and analysis including the design of relevant logics for each individual datatype. This work must be carried out in a very classical style and must on the long run lead to scholarly published results. Subthreads include:

1. Classification of options for relevant classes of total algebras (involutive meadows, non-involutive meadows, common meadows, wheels, transrationals, and other yet unknown options).
2. Model theory, complexity of proof systems, logical complexity theory for specifications.
3. Term rewriting theory.
4. Development of novel datatypes (e.g. splitting fractions, or fracpairs).

Realistic context development. Development of a match between educational contexts and various datatypes indicating how in principle a particular datatype (or group of datatypes) might support some kind of educational activity. Subthreads include:

1. Development of examples involving money (notations for monies of account).

2. Examples connected with dimensions, notations for measurement, geometry, natural sciences, and engineering.
3. Examples in the context of economy, sales, and production.

Experimental educational practice. On the long run obtaining validated results is impossible without performing experimental work in an educational setting. This work requires close scrutiny in advance as it will involve human subjects who may not themselves be able to understand all consequences of the trials that they participate in. Currently no such work is planned.

Connecting with the philosophy of education. Development of a rationale for a pragmatic logic of arithmetical datatypes aiming at educational applications. Subthreads include:

1. The development and comparative analysis of competence levels in relation to relevant datatypes and in relation to the existing literature on reference levels for arithmetical competences.
2. Testing theory and practice for a variety of relevant competence levels.
3. Looking for integration with the philosophy of numbers that has informed thought on educational mathematics.

This work requires a systematic comparison with existing philosophies of education relevant for mathematics.

By working simultaneously in each of these threads the probability of mutual feedback is increased and every now and then the price needs to be paid that one of the threads made less progress than could have been made had not time and effort been spent in one or more of the other threads.

4 Concluding remarks

I have outlined a working program for the development of pragmatic logics for arithmetical datatypes with particular emphasis on datatypes relevant for the understanding of fractions. This document may serve as a rationale for a forthcoming multi-threaded project planning that combines conventional research methods aiming at scholarly publication with other activities that may lead to nopreprints and to other ways of expressing results.

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A Formalities and policy statements concerning this document

For information about Minstroom Research BV (hereafter MRbv) as well as for a rationale of the use of that affiliation I refer to the Appendices of [4].

This paper has MRbv document classification category B. The meaning of that labeling is explained in Appendx A of [4]. The classification is based on the following considerations.

1. The paper contains a subjective statement about plans and objectives which express my views to the extent that these are conclusive concerning how this subject wil be dealt with in MRbv (if at all).

2. The paper is meant to clarify the structuring and planning of work consisting of several threads at least one of which (mathematical and logical investigation of arithmetical datatypes) is carried out entirely outside MRbv, and at least one of which (experimental didactical work) will only be performed inside MRbv (if at all).

3. One of the threads mentioned in the multi-threaded architecture for work on the development of pragmatic logic for educational mathematics along the lines of the development of tailor made arithmetical datatypes consists of the collection and analysis of related literature in the area of mathematics education. That work can be done within the setting of MRbv and in the context sketched in this paper as long as its ambitions are limited to the development of a sound basis and context for the objective of reaching a productive match between technical work on arithmetical datatypes and potential educational practice.

This paper is a nopreprint in the sense of Appendix A of [4]. Nopreprint status of this work requires a defensive justification as indicated in [4]. My current justification in this particular case reads as follows:

1. There is no mathematical or logical fact mentioned or suggested that requires a proof or a validation. Most content is a matter of opinion only.

2. My use of the phrase "pragmatic logic" may be criticized. I suggest that readers will not use this phrase for the same or a similar meaning unless they have checked to their satisfaction that the use of this phrase is justified (in the case at hand). I have included a reference ([14]) to a critic of the use of the phrase.

3. My use of the phrase "splitting fraction" may be criticized for instance because other uses of that phrase can be found. But I have not noticed a standard meaning of the phrase "splitting fraction" that stands in the way of my use in the current paper. Whoever intends to use this phrase for the same objective and in a comparable setting must take full responsibility for that design choice, and by consequence is in no need to refer to my use of the phrase in this nopreprint.

   This is an essential feature of the idea of a nopreprint: that a reader who makes use of a result of the paper (such as a particular design choice) that is remotely in risk of being rejected in a peer reviewed publication process on grounds of scholarly validity should feel no need or even incentive to make reference to a document (i.e. this nopreprint) which proposes that very design choice but intentionally escapes from the peer review system (in advance of publication) for whatever reason that may or may not be known to or intelligible for the mentioned reader.

4. No prediction is made on the value of the results of this works for the objectives of educational mathematics. I have merely stated that my expectation that these results will be valuable serves as a rationale for my own efforts in this direction. Technical mathematical and logical work on arithmetical datatypes has been carried out before, and probably will be carried out in the future, without making reference to such particular expectations or rationales.