Is axial anomaly really an anomaly?

Murod Abdukhakimov
murod.abdukhakimov@gmail.com

1 Introduction

The conventional description of the axial anomaly is as follows:

Step 1: A massless, non-interacting Dirac field satisfies the equation[1]:

$$\begin{aligned} \partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} &= 0 \\ \partial_{\mu\dot{\nu}}\xi^{\mu} &= 0 \end{aligned} \tag{1.1}$$

Due to symmetries of the field, both vector current P^μ and axial current P_A^μ obey continuity equations (i.e. both currents are conserved):

$$\begin{aligned} \partial_\mu P^\mu &= 0 \\ \partial_\mu {P_A}^\mu &= 0 \end{aligned} \tag{1.2}$$

Then a coupling to a vector gauge field is introduced, and both symmetries continue to hold.

Step 2: The second step is to apply quantization, necessarily followed by regularization and renormalization. That makes preservation of both symmetries impossible. It apperas that it is possible to preserve conservation of the vector current or axial current (or a linear combination of the two) but not both.

Since the preservation of both symmetries is impossible, *a choice is made* to presevre vector current *only*

$$\partial_\mu P^\mu = 0 \tag{1.3}$$

while the divergence of the axial current is required by quantum perturbation theory to be proportional to the well known electromagnetic field invariant:

$$\partial_\mu {P_A}^\mu \sim \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \sim \mathbf{E}\cdot\mathbf{B} \tag{1.4}$$

The results agree with experiment in a sense that only vector current is actually conserved in nature.

[1] We extensively use *spinorial form* of equations here, see **Annex I: Spinor calculus** for details.

The concept of axial anomaly is fully accepted by modern physics, but nearly all physicists find it weird.

Once we know from experiment that axial current is not conserved, why do we start with equation (1.1) that preserves axial symmetry? Do we intentionally "make even number of mistakes" to obtain the correct result?

Is there a field equation (other than (1.1)) allowing to directly derive the correct expression for the divergence of the axial current (1.4)? In this paper we will find such equation and describe some amazing consequences emerging from it.

2 Eliminating anomaly

Our strategy will be based on finding the new field equation that will directly lead us to (1.4).

Since (1.4) contains electromagnetic filed invariant, it is obvious that the field equation shall include electromagnetic fields. The simplest Lorentz invariant equation that meets this requirement is as follows[2]:

$$\begin{aligned} \partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} + \ i f^{\mu}_{\nu}\ \xi^{\nu} &= 0 \\ \partial_{\mu\dot{\nu}}\xi^{\mu} + \ i \dot{f}^{\dot{\mu}}_{\dot{\nu}}\ \eta_{\dot{\mu}} &= 0 \end{aligned} \tag{2.1}$$

In this equation the spinor and co-spinor fields are coupled *via* second rank electromagnetic field spinors (see Section 4.5). The conjugate equations can be written as follows:

$$\begin{aligned} \partial^{\mu\dot{\nu}}\eta_{\mu} - \ i \dot{f}^{\dot{\nu}}_{\dot{\mu}}\ \xi^{\dot{\mu}} &= 0 \\ \partial_{\mu\dot{\nu}}\xi^{\dot{\nu}} - \ i f^{\nu}_{\mu}\ \eta_{\nu} &= 0 \end{aligned} \tag{2.2}$$

2.1 Conservation of the vector current

The momentum density vector of the spinorial matter field can be defined as a sum of *left* and *right* spinorial (chiral) currents (see Section 4.4)

$$\{P_{\mu} = p_{\mu} + \hat{p}_{\mu}\} \rightarrow P^{\mu\dot{\nu}} = \ p^{\mu\dot{\nu}} + \hat{p}^{\mu\dot{\nu}} \tag{2.3}$$

where

$$\begin{aligned} p^{\mu\dot{\nu}} &= \ \xi^{\mu}\xi^{\dot{\nu}} \\ \hat{p}^{\mu\dot{\nu}} &= \ \eta^{\mu}\eta^{\dot{\nu}} \end{aligned} \tag{2.4}$$

[2]In literature it is known as *Pauli coupling*.

Accordingly the divergence of the momentum density vector P_μ

$$\partial_\mu P^\mu = \partial_{\mu\dot{\nu}} P^{\mu\dot{\nu}} = \partial_{\mu\dot{\nu}} p^{\mu\dot{\nu}} + \partial_{\mu\dot{\nu}} \hat{p}^{\mu\dot{\nu}} \tag{2.5}$$

Using field equations (2.1 - 2.2) we can find that

$$\begin{aligned} \partial_{\mu\dot{\nu}} p^{\mu\dot{\nu}} &= \ \partial_{\mu\dot{\nu}} \left[\xi^\mu \xi^{\dot{\nu}}\right] = \left(\left[\partial_{\mu\dot{\nu}} \xi^\mu\right] \xi^{\dot{\nu}} + \xi^\mu \left[\partial_{\mu\dot{\nu}} \xi^{\dot{\nu}}\right]\right) = -\ i \left(\dot{f}^{\dot{\mu}}_{\dot{\nu}} \eta_{\dot{\mu}} \xi^{\dot{\nu}} - \xi^\mu f^\nu_\mu \eta_\nu\right) \\ \partial_{\mu\dot{\nu}} \hat{p}^{\mu\dot{\nu}} &= \ \partial^{\mu\dot{\nu}} \left[\eta_\mu \eta_{\dot{\nu}}\right] = \left(\left[\partial^{\mu\dot{\nu}} \eta_\mu\right] \eta_{\dot{\nu}} + \eta_\mu \left[\partial^{\mu\dot{\nu}} \eta_{\dot{\nu}}\right]\right) = +\ i \left(\dot{f}^{\dot{\nu}}_{\dot{\mu}} \xi^{\dot{\mu}} \eta_{\dot{\nu}} - \eta_\mu f^\mu_\nu \xi^\nu\right) \end{aligned} \tag{2.6}$$

from what we conclude that momentum of the spinorial field is *conserved* due to field equations:

$$\partial_\mu P^\mu = 0 \tag{2.7}$$

and hence, (1.3) is satisfied.

2.2 Reducing spinor space

Axial vector current ${P_A}^\mu$ is defined, as usual, as a difference of spinorial (chiral) currents p^μ and $\hat{p}^\mu$

$${P_A}^\mu = p^\mu - \hat{p}^\mu \tag{2.8}$$

From (2.6) we can derive expression for divergence of the axial current:

$$\partial_\mu {P_A}^\mu = \partial_{\mu\dot{\nu}} {P_A}^{\mu\dot{\nu}} = \partial_{\mu\dot{\nu}} p^{\mu\dot{\nu}} - \partial_{\mu\dot{\nu}} \hat{p}^{\mu\dot{\nu}} = -\ 2i \left(\dot{f}^{\dot{\mu}}_{\dot{\nu}} \eta_{\dot{\mu}} \xi^{\dot{\nu}} - \xi^\mu f^\nu_\mu \eta_\nu\right) \tag{2.9}$$

Our next step would be aiming at making (2.9) consistent with (1.4).

This can be achieved by reducing the total space of (4-component) spinors to the subspace of spinors satisfying certain Lorentz invariant condition.

This can be done in the way similar to allocating the subspaces of Dirac and/or Majorana spinors.

For instance, the subspace of *Majorana* spinors is allocated by requiring that all spinors of this subspace are eigenvectors of the charge conjugation operator.

Dirac spinors are also defined as eigenvectors of 4-momentum operator in spinor space:

$$\gamma_\mu P^\mu \ u(p) = m \ u(p) \tag{2.10}$$

Similarly, we will demand that spinorial matter fields ξ and $\dot{\eta}$ are *eigenvectors* of the second rank electromagnetic field spinors f^{μ}_{ν} and $\dot{f}^{\dot{\mu}}_{\dot{\nu}}$ correspondingly:

$$
\begin{aligned}
f^{\mu}_{\nu}\ \xi^{\nu} &= \ \lambda\ \xi^{\mu} \\
\\
\dot{f}^{\dot{\mu}}_{\dot{\nu}}\ \eta_{\dot{\mu}} &= \bar{\lambda}\ \eta_{\dot{\nu}}
\end{aligned}
\tag{2.11}
$$

With the condition (2.11) our matter field equations (2.1) become very simple

$$
\begin{aligned}
\partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} +\ i\lambda\ \xi^{\mu} &= 0 \\
\partial_{\mu\dot{\nu}}\xi^{\mu}\ +\ i\bar{\lambda}\ \eta_{\dot{\nu}} &= 0
\end{aligned}
\tag{2.12}
$$

and replicate the structure of the *free* Dirac equation (see (6.2)) where *constant* mass term m is replaced by the *variable* "mass density" terms λ and $\bar{\lambda}$.

Taking account the explicit form of electromagnetic field spinors f^{μ}_{ν} and $\dot{f}^{\dot{\mu}}_{\dot{\nu}}$ (see (4.42) - (4.43)) one can see that eigenvalues λ and $\bar{\lambda}$ are well known electromagnetic field invariants:

$$
\begin{aligned}
\lambda_{\pm} &= \ \pm\sqrt{(F^1)^2 + (F^2)^2 + (F^3)^2} \qquad {\lambda_{\pm}}^2 = \ E^2 - B^2 - 2i\ \mathbf{E}\cdot\mathbf{B} \\
\overline{\lambda}_{\pm} &= \ \pm\sqrt{(\ \bar{F^1})^2 + (\ \bar{F^2})^2 + (\ \bar{F^3})^2} \qquad {\overline{\lambda}_{\pm}}^2 = \ E^2 - B^2 + 2i\ \mathbf{E}\cdot\mathbf{B}
\end{aligned}
\tag{2.13}
$$

Hence, from the analogy with free Dirac equation we can say that in our model electromagnetic field invariants play the roles of mass densities[3]. These mass densities are variable and complex valued, but we will show that it does not lead to any inconsistencies and/or non-physical solutions.

The Lorentz invariant condition (2.11) establish strong connection between electromagnetic and spinorial (matter) fields. One of the consequences is that the momentum 4-vector P_{μ} (i.e. vector current (2.3)) becomes an eigenvector of the stress-energy tensor of the electromagnetic field (5.10):

$$
\begin{aligned}
t^{\mu\dot{\rho}}_{\nu\dot{\sigma}}\ p^{\nu\dot{\sigma}} &= (\dot{f}^{\dot{\rho}}_{\dot{\sigma}}\ \xi^{\dot{\sigma}})\,(f^{\mu}_{\nu}\ \xi^{\nu}) = |\lambda|^{\,2}\ p^{\mu\dot{\rho}} \\
t^{\mu\dot{\rho}}_{\nu\dot{\sigma}}\ \hat{p}^{\nu\dot{\sigma}} &= (\dot{f}^{\dot{\rho}}_{\dot{\sigma}}\ \eta^{\dot{\sigma}})\,(f^{\mu}_{\nu}\ \eta^{\nu}) = |\lambda|^{\,2}\ \hat{p}^{\mu\dot{\rho}}
\end{aligned}
\tag{2.14}
$$

[3]We do not claim that we discovered the mass origin other than Higgs boson. But at least we've found the application of electromagnetic field invariants (2.13) in physics.

$$t^{\mu\dot{\rho}}_{\nu\dot{\sigma}}\ P^{\nu\dot{\sigma}} = |\lambda|^{\,2}\ P^{\mu\dot{\rho}} \tag{2.15}$$

The eigenvalue of the stress-energy tensor is *real valued* and, of course, expressed *via* electromagnetic field invariants (2.13).

2.3 Eigenvectors

Let us now derive the expressions for the eigenvectors of the electromagnetic field spinors f^{μ}_{ν} and $\dot{f}^{\dot{\mu}}_{\dot{\nu}}$.

Consider arbitrary point Q at the space-time. For the sake of convenience we can choose the reference frame (denoted as $M_{\perp}$) in such a way that the fields $\mathbf{E}$, $\mathbf{B}$ at the point Q will be orthogonal to the axis $\mathbf{e}_3$.

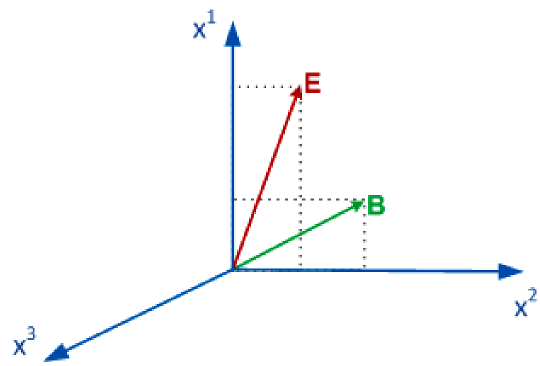

Figure 1: Spatial dimensions of the frame $M_{\perp}$

There is, of course, infinite number of such frames, but all the considerations presented in this section are valid for any of these frames.

In the reference frame $M_{\perp}$ the expression for spinor f^{μ}_{ν} at the point Q will be (see (4.42)):

$$f^{\mu}_{\nu} = \left[\begin{array}{cc} 0 & F^1 - iF^2 \\ F^1 + iF^2 & 0 \end{array} \right] \tag{2.16}$$

because $F^3 = 0$ at the point Q.

One can easily check now that two spinors ξ_+ and ξ_- defined as

$$\xi_\pm = \begin{bmatrix} \xi^1_\pm \\ \\ \xi^2_\pm \end{bmatrix} = \begin{bmatrix} \pm\sqrt{F^1 - iF^2} \\ \\ \sqrt{F^1 + iF^2} \end{bmatrix} \tag{2.17}$$

will be eigenvectors of the matrix (2.16) at the point Q:

$$f^\mu_\nu \; \xi^\nu_\pm = \; \lambda_\pm \; \xi^\mu_\pm \tag{2.18}$$

Hence, with the special choice of the reference frame we can write an explicit expression for the components of the spinorial field ξ satisfying condition (2.11). The expressions for the field components in all other frames can be obtained by the appropriate Lorentz transformations.

Similarly one can show that at the reference frame $M_\perp$ two co-spinors $\dot{\eta}_+$ and $\dot{\eta}_-$ defined as

$$\dot{\eta}_\pm = \begin{bmatrix} \eta_{\pm\dot{1}} \\ \\ \eta_{\pm\dot{2}} \end{bmatrix} = \begin{bmatrix} \pm\sqrt{\bar{F}^1 - i\bar{F}^2} \\ \\ \sqrt{\bar{F}^1 + i\bar{F}^2} \end{bmatrix} \tag{2.19}$$

will satisfy the condition

$$f^{\dot{\mu}}_{\dot{\nu}} \; \eta_{\pm\dot{\mu}} = \bar{\lambda}_\pm \; \eta_{\pm\dot{\nu}} \tag{2.20}$$

at the point Q.

Using (2.17) and (2.19) one can find that in the frame $M_\perp$

$$\begin{aligned} \xi^1_\pm \xi^{\dot{1}}_\pm &= \sqrt{(F^1 - iF^2)\,(\bar{F}^1 + i\bar{F}^2)} = \eta^1_\pm \eta^{\dot{1}}_\pm \\ \xi^2_\pm \xi^{\dot{2}}_\pm &= \sqrt{(F^1 + iF^2)\,(\bar{F}^1 - i\bar{F}^2)} = \eta^2_\pm \eta^{\dot{2}}_\pm \\ \xi^1_\pm \xi^{\dot{2}}_\pm &= \pm\sqrt{(F^1 - iF^2)\,(\bar{F}^1 - i\bar{F}^2)} = -\eta^1_\pm \eta^{\dot{2}}_\pm \\ \xi^2_\pm \xi^{\dot{1}}_\pm &= \pm\sqrt{(F^1 + iF^2)\,(\bar{F}^1 + i\bar{F}^2)} = -\eta^2_\pm \eta^{\dot{1}}_\pm \end{aligned} \tag{2.21}$$

at the point Q. From this we can see that in the frame $M_\perp$ the components of the spinorial (chiral) currents p_μ and $\hat{p}^\mu$ (see (2.4)) satisfy the relationships

$$\begin{aligned} p_{\pm 0} &= \hat{p}^0_\pm, \qquad & p_{\pm 1} &= \hat{p}^1_\pm \\ p_{\pm 2} &= \hat{p}^2_\pm, \qquad & p_{\pm 3} &= -\hat{p}^3_\pm \end{aligned} \tag{2.22}$$

and the total momentum density P_μ (vector current) has only two non-zero components in the frame $M_\perp$

$$\begin{aligned} P_{\pm 0} &= 2\ p_{\pm 0} = 2\ \hat{p}^0_\pm, \qquad & P_{\pm 1} &= 0 \\ P_{\pm 2} &= 0, \qquad & P_{\pm 3} &= 2\ p_{\pm 3} = -2\ \hat{p}^3_\pm \end{aligned} \tag{2.23}$$

From (2.21) we can derive the "mass square" of the momentum density 4-vector P_μ, which is *invariant* under Lorentz transformations and hence has the same value in all reference frames:

$$P^\mu P_\mu = 4|\lambda|^2 \tag{2.24}$$

It is worth noting that the momentum density vector P_μ is always *time-like*, and its time-like component P_0 is always positive, hence no solutions with negative energies are allowed.

From (2.21) we can find the values of the following Lorentz invariants:

$$\begin{aligned} \xi^\mu_\pm\ \eta_{\pm\mu} &= \pm 2\ \lambda_\pm \\ \xi^{\dot\mu}_\pm\ \eta_{\pm\dot\mu} &= \pm 2\ \bar\lambda_\pm \end{aligned} \tag{2.25}$$

In further sections we will also use the following identities:

$$\begin{aligned} \xi^\mu_-\ \eta_{+\mu} &= 0 \\ \xi^{\dot\mu}_-\ \eta_{+\dot\mu} &= 0 \end{aligned} \tag{2.26}$$

2.4 Divergence of axial current

Let us now assume that fields ξ and $\dot\eta$ are the eigenvectors of the electromagnetic field spinors f^μ_ν and $\dot f^{\dot\mu}_{\dot\nu}$, i.e. condition (2.11) is satisfied. Then using (2.11) and (2.25) the expressions (2.6) can be written as

$$\begin{aligned} \partial_\rho p^\rho_\pm &= -i\left(\lambda_\pm \xi^\mu_\pm \eta_{\pm\mu} - \bar\lambda_\pm \eta_{\dot\mu}\xi^{\dot\mu}\right) = \mp\ 2i\ ({\lambda_\pm}^2 - {\bar\lambda_\pm}^2) = \ \mp\ 8\ \mathbf{E}\cdot\mathbf{B} \\ \partial_\rho \hat{p}^\rho_\pm &= +i\left(\lambda_\pm \xi^\mu_\pm \eta_{\pm\mu} - \bar\lambda_\pm \eta_{\dot\mu}\xi^{\dot\mu}\right) = \pm\ 2i\ ({\lambda_\pm}^2 - {\bar\lambda_\pm}^2) = \ \pm\ 8\ \mathbf{E}\cdot\mathbf{B} \end{aligned} \tag{2.27}$$

From (2.27) we conclude that spinorial (chiral) currents p^μ and $\hat{p}^\mu$ are *conserved independently* when $\mathbf{E}\cdot\mathbf{B} = 0$, i.e. when imaginary parts of the squared electromagnetic field invariants λ^2 and $\bar\lambda^2$ are zero. Particularly, this is the case of *orthogonal* ($\boldsymbol{E} \perp \boldsymbol{B}$) electric and magnetic fields (see **Annex IV** for details).

From (2.27) one can see that the 4-divergence of the axial vector current equals to

$$\partial_\rho P_{A\pm}{}^\rho = \partial_\rho p^\rho_\pm - \partial_\rho \hat{p}^\rho_\pm = \mp\, 4i\, \left(\lambda_\pm{}^2 - \overline{\lambda}_\pm{}^2\right) = \mp\, 16\;\; \mathbf{E}\cdot\mathbf{B} \tag{2.28}$$

This makes our model fully consistent with (1.4).

3 Physical applications

Let's have another look at our new equation

$$\begin{aligned} \partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} &= \; - \, i f^\mu_\nu \; \xi^\nu \\ \partial_{\mu\dot{\nu}}\xi^\mu &= \; - \, i f^{\dot{\mu}}_{\dot{\nu}} \; \eta_{\dot{\mu}} \end{aligned} \tag{3.1}$$

and eigenvector condition

$$\begin{aligned} f^\mu_\nu \; \xi^\nu &= \; \lambda \; \xi^\mu \\ f^{\dot{\mu}}_{\dot{\nu}} \; \eta_{\dot{\mu}} &= \bar{\lambda} \; \eta_{\dot{\nu}} \end{aligned} \tag{3.2}$$

Particle physicists *hate* equations of this sort because they don't know how to quantize them. Or maybe some physicists know, but anyway it requires tremendous effort.

Why quantizing everything? Let us first see what we can do with the equation (3.1) as it is.

3.1 Approach to solving self-action problem

In this paper we will consider the well known particle's self-action problem in electrodynamics. The electromagnetic field produced by the charged particle is acting on particle itself. This action on the source impacts the electromagnetic field, and so on, and so on.

The major difficulty is in finding the *balance* between the two equations responsible for evolutions of the electromagnetic field and it's source: the *Maxwell equations* and the (spinorial) *matter field equations* correspondingly.

This balance can be achieved due to strong connection between spinorial matter fields and electromagnetic field (3.2). Using the approach developed by Belinfante and Ohanian [7] we will demonstrate that with (3.2) the matter field equation (3.1) can be *reduced to* Maxwell equations, so that evolutions of the electromagnetic field and it's source will be *synchronized*.

We will consider three field configurations with different symmetries:

- *Transverse plane waves* corresponding to plane electromagnetic waves in vacuum (*photons*),
- *Fields with axial symmetry* that can be associated with stable charged fermions (such as *electron*)
- *Longitudinal plane waves.* Such fields can be associated with *neutrino* since they satisfy Majorana condition.

In all cases we will use the same matter field equation (3.1), and will make no assumptions other than symmetry properties listed above.

Due to chosen symmetries of the field configurations, at every point of the space-time the electric and magnetic field vectors **E**, **B** are orthogonal to one of the basis vectors.

This enables us to use in our calculations the explicit expressions for spinorial filed components (2.17) and (2.19) derived in Section 2.3.

We will use them to write down the explicit expressions for the matter field equation (3.1) and Maxwell equations for each symmetry type, and then we will require the *equivalence* of Maxwell and matter field equations.

This would enable us to express *charge densities* of the spinorial matter fields *via* electromagnetic fields, similarly to expressions of their *mass densities* derived in previous sections (see e.g. (2.24)).

3.2 Transverse plane waves

Consider transverse plane waves propagating in the direction of the axis $\mathbf{e}_3$. In each point the electric and magnetic field vectors **E**, **B** are orthogonal to the axis $\mathbf{e}_3$:

$$\begin{gathered} \mathbf{E},\mathbf{B} \perp \ \mathbf{e}_3 \\ F^3 = \bar{F}^3 \equiv 0 \end{gathered} \tag{3.3}$$

As shown in the **Annex V**, the matter wave equations (3.1) will have the following form:

$$\begin{gathered} (\partial_1 - i\partial_2)\,(F^1 + iF^2) = +\ 2\ i\ \bar{\lambda}\ \sqrt{F^1 + iF^2}\ \sqrt{\bar{F}^1 - i\bar{F}^2} \\ (\partial_0 - \partial_3)\,(F^1 - iF^2) = 0 \\ (\partial_0 + \partial_3)\,(F^1 + iF^2) = 0 \\ (\partial_1 + i\partial_2)\,(F^1 - iF^2) = +\ 2\ i\ \bar{\lambda}\ \sqrt{F^1 - iF^2}\ \sqrt{\bar{F}^1 + i\bar{F}^2} \end{gathered} \tag{3.4}$$

With expressions (2.21 - 2.23) for the momentum density of the matter field P_μ in the frame $M_\perp$ we can rewrite (3.4) in the following form:

$$
\begin{aligned}
(\partial_1 - i\partial_2)\,(F^1 + iF^2) &= i\;\bar{\lambda}\;\;(P^0 + P^3) \\
(\partial_0 - \partial_3)\,(F^1 - iF^2) &= 0 \\
(\partial_0 + \partial_3)\,(F^1 + iF^2) &= 0 \\
(\partial_1 + i\partial_2)\,(F^1 - iF^2) &= i\;\bar{\lambda}\;\;(P^0 - P^3)
\end{aligned}
\tag{3.5}
$$

At the same time, the Maxwell equations for transverse plane waves are as follows (see Section 5.2.2):

$$
\begin{aligned}
(\partial_1 - i\partial_2)\,(F^1 + iF^2) &= J^0 + J^3 \\
(\partial_0 - \partial_3)\,(F^1 - iF^2) &= 0 \\
(\partial_0 + \partial_3)\,(F^1 + iF^2) &= 0 \\
(\partial_1 + i\partial_2)\,(F^1 - iF^2) &= J^0 - J^3
\end{aligned}
\tag{3.6}
$$

From comparison of (3.5) and (3.6) we conclude that in the case of the transverse plane waves there can be established the following relationship between the charge density current J^μ and the momentum density P^μ:

$$
J^\mu = i\;\bar{\lambda}\;P^\mu \tag{3.7}
$$

In this expression the electromagnetic field invariant $(i\bar{\lambda})$ plays the role of the electromagnetic *charge density scalar*[4].

Generally $\bar{\lambda}$ is complex valued, hence allowing for *both* non-zero electric and magnetic charge densities.

The expression for the Lorentz force density (5.12) acting on matter fields is also derived in the **Annex V**:

$$
\Lambda_{\mu\dot{\nu}} = -i\left(\lambda^2 - \bar{\lambda}^2\right) P_{A\mu\dot{\nu}} \tag{3.8}
$$

It is interesting that the Lorentz force $\mathcal{F}^\mu$ is proportional to the axial vector current ${P_A}^\mu$.

[4] It is clear that $(-i\lambda)$ plays the same role for anti-particles. Again, we do not claim that we've discovered the origin of electric charge, but at least we've found another application of electromagnetic field invariants (2.13) in physics.

From (3.8) we can see that Lorentz force vanishes when $\mathbf{E} \cdot \mathbf{B} = 0$, i.e. when imaginary parts of the squared electromagnetic field invariants λ^2 and $\bar{\lambda}^2$ are zero.

Particularly, this is the case of electromagnetic waves "in vacuum" (*photons*, see below). When Lorentz force is zero, the momentum density of the matter field remains constant in the course of particle's motion, hence allowing for *uniform* motion of the particle.

In the case of plane electromagnetic waves "in vacuum" ($\boldsymbol{E} \perp \boldsymbol{B}$, $E = B$) we have $\lambda = \bar{\lambda} = 0$, and matter field equations (3.5) coincide with the "source-free" Maxwell equations (see 5.16). In this case the momentum density P^μ of the matter field is non-zero, while the charge density current J^μ is zero.

In this sense the electromagnetic wave in vacuum is not actually "source-free", i.e. despite of zero charge density there exist a spinorial matter field which is the source of electromagnetic field.

3.3 Axial symmetry

Unlike free Dirac equation, our equation (3.1) allows for configuration with axial symmetry.

Indeed, as shown by Erwin Schrödinger in 1930, the free Dirac equation (see (6.2)) only allows for trembling motion, also known as *Zitterbewegung*. The difference between axial symmetry and Zitterbewegung is illustrated on Figure 2.

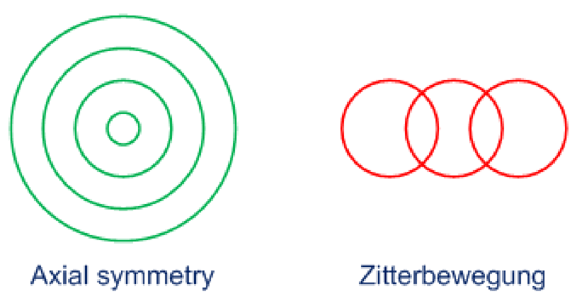

Figure 2: Axial symmetry vs Zitterbewegung

Due to constant mass terms in free Dirac equation, the field oscillates with the same amplitude at every point.

In our model the "mass terms" are variable, hence allowing for axial symmetry where the oscillation amplitude depends on a distance to the symmetry axis.

As shown in the Annex VI, by introducing polar cylindrical coordinates

$$\begin{aligned} x^2 &= \rho \cos\varphi \\ x^3 &= \rho \sin\varphi \\ x^1 &= x^1 \end{aligned} \tag{3.9}$$

our axially symmetric matter field equation (3.1) can be written in the following form:

$$\begin{gathered} \tfrac{1}{\rho}F^\rho + (\partial_1 - i\partial_\rho)(F^1 + iF^\rho) = +\ 2\ i\ \bar{\lambda}\ \sqrt{\bar{F}^1 - i\bar{F}^\rho}\ \sqrt{F^1 + iF^\rho} + \tfrac{i}{\rho}F^1 \\ \left(\partial_0 - \tfrac{1}{\rho}\partial_\varphi\right)(F^1 - iF^\rho) = 0 \\ \left(\partial_0 + \tfrac{1}{\rho}\partial_\varphi\right)(F^1 + iF^\rho) = 0 \\ \tfrac{1}{\rho}F^\rho + (\partial_1 + i\partial_\rho)(F^1 - iF^\rho) = +\ 2\ i\ \bar{\lambda}\ \sqrt{\bar{F}^1 + i\bar{F}^\rho}\ \sqrt{F^1 - iF^\rho} - \tfrac{i}{\rho}F^1 \end{gathered} \tag{3.10}$$

The Maxwell equations in cylindrical coordinates can be written as follows (see Section 5.2.2):

$$\begin{gathered} \tfrac{1}{\rho}F^\rho + (\partial_1 - i\partial_\rho)(F^1 + iF^\rho) = J^0 + J^\varphi \\ \left(\partial_0 - \tfrac{1}{\rho}\partial_\varphi\right)(F^1 - iF^\rho) = 0 \\ \left(\partial_0 + \tfrac{1}{\rho}\partial_\varphi\right)(F^1 + iF^\rho) = 0 \\ \tfrac{1}{\rho}F^\rho + (\partial_1 + i\partial_\rho)(F^1 - iF^\rho) = J^0 - J^\varphi \end{gathered} \tag{3.11}$$

From comparison with (3.11) we conclude that (3.10) will coincide with Maxwell equations if we define:

$$\begin{gathered} J^0 - J^\varphi = i\ \bar{\lambda}\,(P^0 - P^\varphi) - i\tfrac{1}{\rho}F^1 \\ J^0 + J^\varphi = i\ \bar{\lambda}\,(P^0 + P^\varphi) + i\tfrac{1}{\rho}F^1 \end{gathered} \tag{3.12}$$

where J^0, J^φ and P^0, P^φ are non-zero components of charge density and momentum density correspondingly.

It is interesting that in axially symmetric case

$$\begin{gathered} J^0 = i\ \bar{\lambda}\ P^0 \\ J^\varphi = i\ \bar{\lambda}\ P^\varphi + 2i\ \tfrac{1}{\rho}F^1 \end{gathered} \tag{3.13}$$

and hence the ratio $\frac{J^\varphi}{J^0}$ is not equal to the ratio $\frac{P^\varphi}{P^0}$. Consequently, the "velocity of charge" is not the same as "velocity of mass" anymore[5].

3.4 Neutrino model

3.4.1 General considerations

So far we have always been considering the matter field equations in the following forms:

$$\begin{aligned}\partial^{\mu\dot{\nu}}\eta_{\pm\dot{\nu}} &= -\, i\ \lambda_\pm\ \xi_\pm{}^\mu \\ \partial_{\mu\dot{\nu}}\xi_\pm{}^\mu &= \ -\, i\ \bar{\lambda}_\pm\ \eta_{\pm\dot{\nu}}\end{aligned} \tag{3.14}$$

In (3.14) spinor ξ_+ is coupled with co-spinor $\dot{\eta}_+$, and spinor ξ_- is coupled with co-spinor $\dot{\eta}_-$.

Let us now consider a field configuration where spinor ξ_- is coupled with co-spinor $\dot{\eta}_+$. The matter field equations will be written as

$$\begin{aligned}\partial^{\mu\dot{\nu}}\eta_{+\dot{\nu}} &= -\, i\ \lambda_-\ \xi_-{}^\mu \\ \partial_{\mu\dot{\nu}}\xi_-{}^\mu &= \ -\, i\ \bar{\lambda}_+\ \eta_{+\dot{\nu}}\end{aligned} \tag{3.15}$$

or, taking account that $\lambda_\pm = -\lambda_\mp$ (see (2.13))

$$\begin{aligned}\partial^{\mu\dot{\nu}}\eta_{+\dot{\nu}} &= -\, i\ \lambda_-\ \xi_-{}^\mu \\ \partial_{\mu\dot{\nu}}\xi_-{}^\mu &= \ +\, i\ \bar{\lambda}_-\ \eta_{+\dot{\nu}}\end{aligned} \tag{3.16}$$

According to (2.17) and (2.19) the components of the spinor and co-spinor fields will be

$$\begin{aligned}\xi_-^1 &= -\sqrt{F^1 - iF^2} \quad \eta_{+\dot{1}} = +\sqrt{\overline{F^1} - i\overline{F^2}} \\ \xi_-^2 &= +\sqrt{F^1 + iF^2} \quad \eta_{+\dot{2}} = +\sqrt{\overline{F^1} + i\overline{F^2}}\end{aligned} \tag{3.17}$$

and hence the Lorentz invariant *Majorana condition* will be satisfied:

$$\begin{aligned}\xi^1 &= -\ \eta_2 \\ \xi^2 &= +\ \eta_1\end{aligned} \tag{3.18}$$

[5]In our model the charge is not *nailed* to mass.

We put (3.17) in the matter field equations (3.16)

$$
\begin{aligned}
&\begin{bmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{bmatrix} \begin{bmatrix} \bar{\xi}^2 \\ -\bar{\xi}^1 \end{bmatrix} = - \, i \; \lambda_- \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} \\
&\begin{bmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{bmatrix} \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} = + \, i \; \bar{\lambda}_- \begin{bmatrix} \bar{\xi}^2 \\ -\bar{\xi}^1 \end{bmatrix}
\end{aligned} \tag{3.19}
$$

and after expansion of the formulas and complex conjugation of the first pair of equations we obtain

$$
\begin{aligned}
&\left.\begin{array}{r} (\partial_0 + \partial_3)\, \xi^2 - (\partial_1 + i\partial_2)\, \xi^1 = + \, i \; \bar{\lambda}_- \; \bar{\xi}^1 \\ (\partial_1 - i\partial_2)\, \xi^2 - (\partial_0 - \partial_3)\, \xi^1 = + \, i \; \bar{\lambda}_- \; \bar{\xi}^2 \end{array}\right\} \\
&\left.\begin{array}{r} (\partial_0 - \partial_3)\, \xi^1 - (\partial_1 - i\partial_2)\, \xi^2 = + \, i \; \bar{\lambda}_- \; \bar{\xi}^2 \\ -\,(\partial_1 + i\partial_2)\, \xi^1 + (\partial_0 + \partial_3)\, \xi^2 = + \, i \; \bar{\lambda}_- \; \bar{\xi}^1 \end{array}\right\}
\end{aligned} \tag{3.20}
$$

From (3.20) it is clear that, due to Majorana condition, the two matter field equations (3.15) become equivalent to each other, hence only one of these equations is *independent*.

Let us now find the expressions for divergences of spinorial currents and momentum density.

With matter field equations (3.16) one can easily find that

$$
\begin{aligned}
\partial_{\mu\dot{\nu}} p^{\mu\dot{\nu}} &= \partial_{\mu\dot{\nu}} \left[\xi^\mu \xi^{\dot{\nu}}\right] = \left(\left[\partial_{\mu\dot{\nu}} \xi^\mu\right] \xi^{\dot{\nu}} + \xi^\mu \left[\partial_{\mu\dot{\nu}} \xi^{\dot{\nu}}\right]\right) = - \, i \; \left(\bar{\lambda}_+ \; \eta_{+\dot{\nu}} \xi_-{}^{\dot{\nu}} - \lambda_+ \eta_{+\mu} \xi_-{}^{\mu}\right) = 0 \\
\partial_{\mu\dot{\nu}} \hat{p}^{\mu\dot{\nu}} &= \partial^{\mu\dot{\nu}} \left[\eta_\mu \eta_{\dot{\nu}}\right] = \left(\left[\partial^{\mu\dot{\nu}} \eta_\mu\right] \eta_{\dot{\nu}} + \eta_\mu \left[\partial^{\mu\dot{\nu}} \eta_{\dot{\nu}}\right]\right) = - \, i \; \left(\bar{\lambda}_- \; \eta_{+\dot{\nu}} \xi_-{}^{\dot{\nu}} - \; \lambda_- \; \xi_-{}^{\mu} \eta_{+\mu}\right) = 0
\end{aligned} \tag{3.21}
$$

In (3.21) we used the invariant properties (2.26).

Consequently we conclude that both spinorial currents p_μ and $\hat{p}_\mu$, as well as momentum density current P_μ are conserved.

3.4.2 Longitudinal plane waves

It is known (see [6]) that if $\mathbf{E} \cdot \mathbf{B} \neq 0$, i.e. electric and magnetic fields vectors $\mathbf{E}$, $\mathbf{B}$ are *not* orthogonal to each other, there exist a reference frame where these vectors are parallel to each other: $\mathbf{E} \parallel \mathbf{B}$.

Let's denote this frame as $M_\parallel$ and assume for simplicity that both $\mathbf{E}$ and $\mathbf{B}$ are directed along the axis $\mathbf{e}_1$:

$$F^1 \neq 0 \quad F^2 = F^3 \ = 0 \tag{3.22}$$

In the frame $M_\parallel$ the components of the spinors (3.17) will have the form:

$$\begin{aligned} \xi^1_- &= -\sqrt{F^1} \quad \eta_{+\dot{1}} = +\sqrt{\bar{F}^1} \\ \xi^2_- &= +\sqrt{F^1} \quad \eta_{+\dot{2}} = +\sqrt{\bar{F}^1} \end{aligned} \tag{3.23}$$

If we denote

$$\zeta = \sqrt{F^1} \tag{3.24}$$

then spinorial currents (2.4) can be written as

$$\begin{aligned} p_0 = +\, \zeta\bar{\zeta} \quad p_1 = -\, \zeta\bar{\zeta} \\ p_2 = 0 \qquad\quad p_3 = 0 \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \hat{p}^0 = +\, \zeta\bar{\zeta} \quad \hat{p}^1 = +\, \zeta\bar{\zeta} \\ \hat{p}^2 = 0 \qquad\quad \hat{p}^3 = 0 \end{aligned} \tag{3.26}$$

Consequently, spatial parts of both spinorial currents p_μ and $\hat{p}_\mu$, as well as momentum density vector P_μ, are *opposite in direction to the axis* $\mathbf{e}_1$, while the momentum density 4-vector is isotropic: $P^\mu P_\mu = 0$. This is the first indication that, *in spite of* non-zero "mass term" λ in the matter field equations, the neutrino field is "moving" at the speed of light.

Let us now rewrite the matter field equations (3.20) in the frame $M_\parallel$.

$$\begin{aligned} (\partial_0 + \partial_3)\, \zeta + (\partial_1 + i\partial_2)\, \zeta &= -\ i\ \bar{\lambda}_-\ \bar{\zeta} \\ (\partial_1 - i\partial_2)\, \zeta + (\partial_0 - \partial_3)\, \zeta &= +\ i\ \bar{\lambda}_-\ \bar{\zeta} \end{aligned} \tag{3.27}$$

By adding and subtracting these equations we obtain:

$$\begin{aligned} (\partial_0 + \partial_1)\, \zeta &= 0 \\ (\partial_3 + i\partial_2)\, \zeta &= -2\ i\ \bar{\lambda}_-\ \bar{\zeta} \end{aligned} \tag{3.28}$$

The first equationin (3.28) means that the field ζ is "moving" at the speed of light in the direction opposite to the axis $\mathbf{e}_1$. That means that in the frame $M_\parallel$ the neutrino field is a *longitudinal* wave, i.e. the wave propagating parallel to the direction of the fields **E**, **B**.

The second equation in (3.28) can be further expressed in terms of the field ζ taking account that $\bar{\lambda}_{-} = -\left(\bar{\zeta}\right)^2$:

$$(\partial_3 + i\partial_2)\,\zeta = +\,2\ i\ \left(\bar{\zeta}\right)^3 \tag{3.29}$$

The Maxwell equations in the chosen frame $M_{\|}$ will have the form

$$\begin{aligned}
(\partial_1 + i\partial_2)\,F^1 &= \ J^0 - J^3 && (i)\\
(\partial_0 - \partial_3)\,F^1 &= \ -J^1 + iJ^2 && (ii)\\
(\partial_0 + \partial_3)\,F^1 &= \ -J^1 - iJ^2 && (ii)\\
(\partial_1 - i\partial_2)\,F^1 &= \ J^0 + J^3 && (iv)
\end{aligned} \tag{3.30}$$

By adding all equations we will obtain:

$$(\partial_0 + \partial_1)\,F^1 = \ J^0 - J^1 \tag{3.31}$$

By adding and subtracting equations $[(i) - (ii) + (iii) - (iv)]$ we obtain

$$(\partial_3 + i\partial_2)\,F^1 = -J^3\ - iJ^2 \tag{3.32}$$

Hence the Maxwell equations will be consistent with matter field equations if the following relationships are satisfied:

$$\begin{aligned}
J^0 - J^1 &= 0\\
-J^3\ - iJ^2 &= 4\ i\ \zeta\ \left(\bar{\zeta}\right)^3
\end{aligned} \tag{3.33}$$

In Section 3.2 we have demonstrated that in the case of the "source-free" transverse plane electromagnetic waves the charge density was zero while the momentum density of the matter field was non-zero.

From (3.33) we conclude that in our model of neutrino the components of the charge density J^2 and J^3 might be non-zero while the components of the momentum density P^2 and P^3 are both zero.

4 Annex I: Spinor calculus

4.1 Basic notations

In this paper we use the *spinor calculus* developed by B. van der Waerden, G.E. Uhlenbeck and O. Laporte. This is because many spinorial equations are much simpler than the corresponding tensorial equations. This applies equally to Maxwell and Dirac equations.

Below we describe the basic notations used in this paper.

The metric of the Minkowski space-time is defined as following:

$$g_{\alpha\beta} = \; g^{\alpha\beta} = diag \;\; (+,-,-,-)\,, \quad \alpha,\beta = 0,1,2,3 \tag{4.1}$$

We use the following representations for Pauli matrices

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.2}$$

$$\acute{\sigma}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; \acute{\sigma}_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \; \acute{\sigma}_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \; \acute{\sigma}_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.3}$$

and Dirac's gamma matrices:

$$\begin{gathered}
\gamma_0 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \quad \gamma_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \\
\gamma_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \gamma_3 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}
\end{gathered} \tag{4.4}$$

4.2 Spinors and co-spinors

In spinor notation the (four-component) wave function of the fermion field ψ is considered as a formal sum of first rank spinor and first rank co-spinor fields:

$$\psi(x) = \{\xi,\ \dot{\eta}\} = \ \{\xi,\ 0\} + \{0,\ \dot{\eta}\} = \begin{pmatrix} \xi^1(x) \\ \xi^2(x) \\ \eta_{\dot{1}}(x) \\ \eta_{\dot{2}}(x) \end{pmatrix} \tag{4.5}$$

where

$$\psi \in \ C^2 \oplus \ \dot{C}^2; \quad \xi \in \ C^2; \quad \dot{\eta} \ \in \ \dot{C}^2 \tag{4.6}$$

Under Lorentz transformation first rank spinors and co-spinors are transformed as follows:

$$\xi'^{\mu} = \; u^{\mu}_{\nu} \; \xi^{\nu}$$
$$\eta'_{\dot\mu} = \; (\; \overline{u}^{-1})^{\dot\nu}_{\dot\mu} \; \eta_{\dot\nu} \tag{4.7}$$

where matrix $u \in SL\ (2, C)$ can be presented in the following form:

$$u = \left[\begin{array}{cc} u^1_1 & u^1_2 \\ u^2_1 & u^2_2 \end{array} \right] = \exp\left[\; (-b_k + \; i\ r_k)\, \sigma_k\right], \quad r_k, b_k \in R, \quad k = 1, 2, 3 \tag{4.8}$$

Any quantities transforming like the products $\xi^{\mu}\xi^{\nu}$, $\eta_{\dot\mu}\eta_{\dot\nu}$, $\xi^{\mu}\eta_{\dot\nu}$ are called second rank spinors and denoted by $a^{\mu\nu}$, $b_{\dot\mu\dot\nu}$, $c^{\mu}_{\dot\nu}$ correspondingly. Analogously one can define the spinors of higher ranks.

Transition from subscript to superscript spinor indices is established by means of Lorentz-invariant spinors $\epsilon_{\mu\nu}$ and $\epsilon^{\dot\mu\dot\nu}$:

$$\epsilon_{\mu\nu} = \left[\begin{array}{cc} 0 & +1 \\ -1 & 0 \end{array} \right] \qquad \epsilon^{\dot\mu\dot\nu} = \left[\begin{array}{cc} 0 & -1 \\ +1 & 0 \end{array} \right] \tag{4.9}$$

$$\xi_{\mu} = \; \epsilon_{\mu\nu}\xi^{\nu}, \qquad \eta^{\dot\mu} = \; \epsilon^{\dot\mu\dot\nu}\eta_{\dot\nu} \tag{4.10}$$

$$\xi_1 = \; \xi^2, \qquad \xi_2 = \; - \; \xi^1$$
$$\eta^{\dot 1} = \; - \; \eta_{\dot 2}, \qquad \eta^{\dot 2} = \; \eta_{\dot 1} \tag{4.11}$$

One can show easily that complex conjugates of spinors transform as co-spinors, and *vice versa*, so that we can denote

$$\xi_{\dot\mu} = \bar\xi_{\mu}$$
$$\eta_{\nu} = \bar\eta_{\dot\nu} \tag{4.12}$$

As in the usual tensor algebra, the only covariant operations are *multiplication* and *contraction*. For instance, from the spinors $a^{\rho}_{\dot\mu\dot\nu}$ and $b^{\dot\sigma}_{\alpha\beta}$ we can form the spinor of the 6^{th} rank

$$c^{\rho\dot\sigma}_{\dot\mu\dot\nu\alpha\beta} = a^{\rho}_{\dot\mu\dot\nu}\; b^{\dot\sigma}_{\alpha\beta} \tag{4.13}$$

or the spinor of the 4^{th} rank

$$c^{\rho}_{\dot\mu\alpha\beta} = a^{\rho}_{\dot\mu\dot\nu}\; b^{\dot\nu}_{\alpha\beta} \tag{4.14}$$

or the spinor of the 2^{nd} rank

$$c_{\dot{\mu}\beta} = a^{\alpha}_{\dot{\mu}\dot{\nu}}\ b^{\dot{\nu}}_{\alpha\beta} \tag{4.15}$$

The following two rules are essential for calculations:

$$a_\mu b^\mu = -a^\mu b_\mu \tag{4.16}$$

and

$$a^\mu b_\mu c_\nu + a_\mu b_\nu c^\mu + a_\nu b^\mu c_\mu = 0 \tag{4.17}$$

An immediate consequence of (4.16) is that any spinor of odd rank has absolute value zero:

$$a_\mu a^\mu = 0 \qquad a_{\lambda\mu\nu} a^{\lambda\mu\nu} = 0 \tag{4.18}$$

4.3 World vectors and tensors

Any vector and/or tensor of the Minkowski space-time can be expressed in a spinor form:

$$\{x^\alpha\} \rightarrow \{S_{\mu\dot{\nu}}\}:\ (S_{\mu\dot{\nu}}) =\ x^\alpha \sigma^T_\alpha =\ x_\alpha \sigma^{\alpha\ T} \tag{4.19}$$

$$\{x^\alpha\} \rightarrow \left\{S^{\mu\dot{\nu}}\right\}:\ \left(S^{\mu\dot{\nu}}\right) =\ x^\alpha \acute{\sigma}_\alpha =\ x_\alpha \acute{\sigma}^\alpha \tag{4.20}$$

or, equivalently

$$\left(S^{\mu\dot{\nu}}\right) = \begin{pmatrix} S^{1\dot{1}} & S^{1\dot{2}} \\ S^{2\dot{1}} & S^{2\dot{2}} \end{pmatrix} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \tag{4.21}$$

$$(S_{\mu\dot{\nu}}) = \begin{pmatrix} S_{1\dot{1}} & S_{1\dot{2}} \\ S_{2\dot{1}} & S_{22} \end{pmatrix} = \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} = \begin{pmatrix} S^{2\dot{2}} & -S^{2\dot{1}} \\ -S^{1\dot{2}} & S^{1\dot{1}} \end{pmatrix} \tag{4.22}$$

The determinants of the matrices $S_{\mu\dot{\nu}}$ and $S^{\mu\dot{\nu}}$ are equal to $x^\mu x_\mu$ and remain invariant under $SL(2,C)$ transformations. The following rule is also essential for calculations:

$$S_{\sigma\dot{\nu}} S^{\lambda\dot{\nu}} = \delta^\lambda_\sigma\,(x^\mu x_\mu) \qquad S_{\sigma\dot{\nu}} S^{\sigma\dot{\lambda}} = \delta^{\dot{\lambda}}_{\dot{\nu}}\,(x^\mu x_\mu) \tag{4.23}$$

In the spinor notation the gradient co-vector ($\partial_\mu = \frac{\partial}{\partial x^\mu}$) is transformed into the following matrices:

$$\left(\partial^{\mu\dot{\nu}}\right) = \begin{bmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{bmatrix} =\ \partial_0 + \partial_1\sigma_1 + \partial_2\sigma_2 + \partial_3\sigma_3 \tag{4.24}$$

$$(\partial_{\mu\dot{\nu}}) = \left[\begin{array}{cc} \partial_0 - \partial_3 & -\partial_1 - i\partial_2 \\ -\partial_1 + i\partial_2 & \partial_0 + \partial_3 \end{array} \right] = \partial_0 - \partial_1\sigma_1^T - \partial_2\sigma_2^T - \partial_3\sigma_3^T \tag{4.25}$$

From (4.23) we immediately conclude that

$$\partial_{\mu\dot{\nu}}\partial^{\lambda\dot{\nu}} = \delta_\mu^\lambda\,(\partial^\mu\partial_\mu) \qquad \partial_{\mu\dot{\nu}}\partial^{\mu\dot{\lambda}} = \delta_{\dot{\nu}}^{\dot{\lambda}}\,(\partial^\mu\partial_\mu) \tag{4.26}$$

and for any 4-vector V^μ represented (according to (4.21 - 4.22)) by second rank spinor $S^{\mu\dot{\nu}}$ the 4-divergence is written in the following form:

$$\partial_{\mu\dot{\nu}}S^{\mu\dot{\nu}} = \partial_\mu V^\mu \tag{4.27}$$

4.4 Spinorial currents

Alternatively, any spinor and co-spinor can be used to construct the world vector. We will call such vectors *spinorial currents*.

Consider arbitrary spinor ξ that can be expressed as a matrix with one column and two rows. We denote Hermitian conjugate matrix as ξ^+. Then we can construct the following world vector:

$$p_\mu = \frac{1}{2}\,\left(\xi^+\sigma_\mu\xi\right) \tag{4.28}$$

$$\begin{array}{ll} p_0 = \frac{1}{2}\left(\xi^+\xi\right) = \frac{1}{2}\left(\overline{\xi^1}\xi^1 + \overline{\xi^2}\xi^2\right) & p_1 = \frac{1}{2}\left(\xi^+\sigma_1\xi\right) = \frac{1}{2}\left(\overline{\xi^2}\xi^1 + \overline{\xi^1}\xi^2\right) \\ p_2 = \frac{1}{2}\left(\xi^+\sigma_2\xi\right) = \frac{i}{2}(\overline{\xi^2}\xi^1 - \overline{\xi^1}\xi^2) & p_3 = \frac{1}{2}\left(\xi^+\sigma_3\xi\right) = \frac{1}{2}\left(\overline{\xi^1}\xi^1 - \overline{\xi^2}\xi^2\right) \end{array} \tag{4.29}$$

One can easily check that $p_\mu p^\mu \equiv 0$, and using (4.7 - 4.8) we can see that vector $p_\mu = \frac{1}{2}\left(\xi^+\sigma_\mu\xi\right)$ transforms as *covariant* vector.

Following the general rule (4.21) the spinor current p_μ can be expressed as

$$p^{\mu\dot{\nu}} = \left[\begin{array}{cc} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{array} \right] = \left[\begin{array}{cc} \xi^1\overline{\xi^1} & \xi^1\overline{\xi^2} \\ \xi^2\overline{\xi^1} & \xi^2\overline{\xi^2} \end{array} \right] = \left[\begin{array}{cc} \xi^1\xi^{\dot{1}} & \xi^1\xi^{\dot{2}} \\ \xi^2\xi^{\dot{1}} & \xi^2\xi^{\dot{2}} \end{array} \right] \tag{4.30}$$

Similarly, we can construct *contravariant* vector from co-spinor $\dot{\eta}$:

$$\hat{p}^\mu = \frac{1}{2}\left(\dot{\eta}^+\acute{\sigma}^\mu\dot{\eta}\right) \tag{4.31}$$

$$\hat{p}^0 = \tfrac{1}{2}\left(\dot{\eta}^{+}\dot{\eta}\right) = \tfrac{1}{2}\left(\overline{\eta_{\dot{1}}}\eta_{\dot{1}} + \overline{\eta_{\dot{2}}}\eta_{\dot{2}}\right) \qquad \hat{p}^1 = \tfrac{1}{2}\left(\dot{\eta}^{+}\acute{\sigma}^1\dot{\eta}\right) = \tfrac{1}{2}\left(\overline{\eta_{\dot{2}}}\eta_{\dot{1}} + \overline{\eta_{\dot{1}}}\eta_{\dot{2}}\right)$$
$$\hat{p}^2 = \tfrac{1}{2}\left(\dot{\eta}^{+}\acute{\sigma}^2\dot{\eta}\right) = \tfrac{i}{2}\left(\overline{\eta_{\dot{2}}}\eta_{\dot{1}} - \overline{\eta_{\dot{1}}}\eta_{\dot{2}}\right) \qquad \hat{p}^3 = \tfrac{1}{2}\left(\dot{\eta}^{+}\acute{\sigma}^3\dot{\eta}\right) = \tfrac{1}{2}\left(\overline{\eta_{\dot{1}}}\eta_{\dot{1}} - \overline{\eta_{\dot{2}}}\eta_{\dot{2}}\right) \tag{4.32}$$

Vector $\hat{p}^\mu$ is also isotropic: $\hat{p}^\mu\hat{p}_\mu \equiv 0$. Using (4.21) it can be expressed in spinor form:

$$\hat{p}^{\mu\dot{\nu}} = \begin{bmatrix} \hat{p}^0 - \hat{p}^3 & -\hat{p}^1 + i\hat{p}^2 \\ -\hat{p}^1 - i\hat{p}^2 & \hat{p}^0 + \hat{p}^3 \end{bmatrix} = \begin{bmatrix} \overline{\eta_{\dot{2}}}\eta_{\dot{2}} & -\overline{\eta_{\dot{2}}}\eta_{\dot{1}} \\ -\overline{\eta_{\dot{1}}}\eta_{\dot{2}} & \overline{\eta_{\dot{1}}}\eta_{\dot{1}} \end{bmatrix} = \begin{bmatrix} \eta^1\eta^{\dot{1}} & \eta^1\eta^{\dot{2}} \\ \eta^2\eta^{\dot{1}} & \eta^2\eta^{\dot{2}} \end{bmatrix} \tag{4.33}$$

Using vectors constructed from spinor and co-spinor, one can form a new vector that will not be isotropic. Such vector is usually defined as a bilinear form of the four-component wave function of the fermion field ψ (*Dirac current*):

$$P_\mu = -\frac{1}{2}\left(\psi^{+}\gamma_0\gamma_\mu\psi\right) \tag{4.34}$$

where field ψ^+ is a Hermitian conjugate of ψ:

$$\psi = \begin{bmatrix} \xi^1 \\ \xi^2 \\ \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{bmatrix}, \quad \psi^{+} = \begin{bmatrix} \overline{\xi^1} & \overline{\xi^2} & \overline{\eta_{\dot{1}}} & \overline{\eta_{\dot{2}}} \end{bmatrix} \tag{4.35}$$

Using (4.34) and Dirac's gamma matrices (4.4) one can easily check that

$$\begin{aligned} P_0 &= \tfrac{1}{2}\left(\overline{\xi^1}\xi^1 + \overline{\xi^2}\xi^2\right) + \tfrac{1}{2}\left(\overline{\eta_{\dot{1}}}\eta_{\dot{1}} + \overline{\eta_{\dot{2}}}\eta_{\dot{2}}\right) = p_0 + \hat{p}^0 \\ P_1 &= \tfrac{1}{2}\left(\overline{\xi^2}\xi^1 + \overline{\xi^1}\xi^2\right) - \tfrac{1}{2}\left(\overline{\eta_{\dot{2}}}\eta_{\dot{1}} + \overline{\eta_{\dot{1}}}\eta_{\dot{2}}\right) = p_1 - \hat{p}^1 \\ P_2 &= \tfrac{i}{2}\left(\overline{\xi^2}\xi^1 - \overline{\xi^1}\xi^2\right) - \tfrac{i}{2}\left(\overline{\eta_{\dot{2}}}\eta_{\dot{1}} - \overline{\eta_{\dot{1}}}\eta_{\dot{2}}\right) = p_2 - \hat{p}^2 \\ P_3 &= \tfrac{1}{2}\left(\overline{\xi^1}\xi^1 - \overline{\xi^2}\xi^2\right) - \tfrac{1}{2}\left(\overline{\eta_{\dot{1}}}\eta_{\dot{1}} - \overline{\eta_{\dot{2}}}\eta_{\dot{2}}\right) = p_3 - \hat{p}^3 \end{aligned} \tag{4.36}$$

or

$$P_\mu = p_\mu + g_{\mu\nu}\hat{p}^\nu \tag{4.37}$$

According to (4.21), world vector P_μ can be expressed as second rank spinor $P^{\mu\dot{\nu}}$:

$$\{P_\mu\} \rightarrow P^{\mu\dot{\nu}} = p^{\mu\dot{\nu}} + \hat{p}^{\mu\dot{\nu}} \tag{4.38}$$

4.5 Electromagnetic fields

In tensor algebra electromagnetic field strengths are expressed in the form of *antisymmetric* second rank electromagnetic field tensor $F_{\mu\nu}$.

Similarly, in spinor calculus electromagnetic field strengths are expressed in the form of two complex conjugated *symmetric* second rank spinors $f_{\mu\nu}$ and $\dot{f}_{\dot{\mu}\dot{\nu}}$ that realize irreducible representation of the $SL(2, C)$ group (see [2, 3, 4]):

$$\left.\begin{aligned} f_{\mu\nu} &= f_{\nu\mu} \\ \dot{f}_{\dot{\mu}\dot{\nu}} &= \dot{f}_{\dot{\nu}\dot{\mu}} \end{aligned}\right\} \quad \textit{symmetry condition}$$

$$\dot{f}^{\dot{\mu}}_{\dot{\nu}} = \overline{f^{\mu}_{\nu}} \qquad \textit{neutrality condition} \tag{4.39}$$

In the expression above we used (4.10) to transform *symmetric* spinors $f_{\mu\nu}$ and $f_{\dot{\mu}\dot{\nu}}$ to *traceless* spinors f^{μ}_{ν} and $\dot{f}^{\dot{\mu}}_{\dot{\nu}}$:

$$f^{\mu}_{\nu} = \epsilon^{\mu\rho} f_{\rho\nu}, \qquad f^{\mu}_{\mu} = 0 \tag{4.40}$$

Due to symmetry of the spinors the field has only 3 complex components

$$f_{11}, \qquad f_{12} = f_{21}, \qquad f_{22} \tag{4.41}$$

This property enables us to introduce the structure of 3-dimentional complex space for electromagnetic field spinors

$$f^{\mu}_{\nu} = \begin{bmatrix} f^1_1 & f^1_2 \\ f^2_1 & f^2_2 \end{bmatrix} = \begin{bmatrix} F^3 & F^1 - iF^2 \\ F^1 + iF^2 & -F^3 \end{bmatrix} = F^k \sigma_k, \qquad k = 1, 2, 3 \tag{4.42}$$

where "coordinates" F^k can be decomposed into real and imaginary parts

$$\boldsymbol{F} = \boldsymbol{E} - i\boldsymbol{B} \tag{4.43}$$

From (4.42) one can see that matrices f^{μ}_{ν} belong to the Lie algebra of the group $SL(2, C)$.

5 Annex II: Maxwell equations in spinor form

Many spinorial equations are much simpler than the corresponding tensorial equations. This applies equally to Maxwell and Dirac equations, as we will demonstrate in the further sections.

Maxwell equations have the following spinor form:

$$\partial^{\nu\dot{\rho}} f^{\mu}_{\nu} = \ S^{\mu\dot{\rho}}, \qquad \partial^{\mu\dot{\rho}} \dot{f}^{\dot{\lambda}}_{\dot{\rho}} = \ \dot{S}^{\mu\dot{\lambda}} \tag{5.1}$$

Here we use two spinorial forms of the *electromagnetic current density*: $S^{\mu\dot{\rho}}$ and $\dot{S}^{\mu\dot{\lambda}}$

$$\begin{gathered} S_{\mu\dot{\nu}} = \begin{bmatrix} S_{1\dot{1}} & S_{1\dot{2}} \\ S_{2\dot{1}} & S_{2\dot{2}} \end{bmatrix} = \begin{bmatrix} J^0 + J^3 & J^1 + iJ^2 \\ J^1 - iJ^2 & J^0 - J^3 \end{bmatrix} \\ \dot{S}_{\mu\dot{\nu}} = \begin{bmatrix} \dot{S}_{1\dot{1}} & \dot{S}_{1\dot{2}} \\ \dot{S}_{2\dot{1}} & \dot{S}_{2\dot{2}} \end{bmatrix} = \begin{bmatrix} \dot{J}^0 + \dot{J}^3 & \dot{J}^1 + i\dot{J}^2 \\ \dot{J}^1 - i\dot{J}^2 & \dot{J}^0 - \dot{J}^3 \end{bmatrix} \end{gathered} \tag{5.2}$$

These two spinors are Hermitian conjugates of each other

$$\begin{bmatrix} \dot{S}_{1\dot{1}} & \dot{S}_{1\dot{2}} \\ \dot{S}_{2\dot{1}} & \dot{S}_{2\dot{2}} \end{bmatrix} = \begin{bmatrix} \bar{S}_{1\dot{1}} & \bar{S}_{2\dot{1}} \\ \bar{S}_{1\dot{2}} & \bar{S}_{2\dot{2}} \end{bmatrix} \tag{5.3}$$

Complex vectors J^k and $\dot{J}^k$ corresponding to spinors $S^{\mu\dot{\rho}}$ and $\dot{S}^{\mu\dot{\lambda}}$ are complex conjugated to each other and can be decomposed into *electric* and *magnetic* current densities

$$\begin{gathered} J^k = {J_e}^k - i{J_m}^k \\ \dot{J}^k = \dot{J}^k_e - i\dot{J}^k_m = \bar{J}^k \end{gathered} \qquad k = 0, 1, 2, 3 \tag{5.4}$$

To be convinced that any of the complex conjugated spinorial equations (5.1) explicitly correspond to Maxwell equations, we can rewrite, e.g. the first equation (using (4.10) and (4.16)) in the form

$$S_{\mu\dot{\nu}} = - \ \partial_{\rho\dot{\nu}} f^{\rho}_{\mu} \tag{5.5}$$

represent it in matrix form

$$\begin{bmatrix} \partial_{1\dot{1}} & \partial_{2\dot{1}} \\ \partial_{1\dot{2}} & \partial_{2\dot{2}} \end{bmatrix} \begin{bmatrix} f^1_1 & f^1_2 \\ f^2_1 & f^2_2 \end{bmatrix} = - \ \begin{bmatrix} S_{1\dot{1}} & S_{2\dot{1}} \\ S_{1\dot{2}} & S_{2\dot{2}} \end{bmatrix} \tag{5.6}$$

and then expand this expression using (4.25), (4.42), (4.43), (5.2) and (5.4):

$$\begin{aligned} &(\partial_0 - \partial_3)\left(E^3 - iB^3\right) + (-\partial_1 + i\partial_2)\left(E^1 - iB^1 + iE^2 + B^2\right) = -\left({J_e}^0 - i{J_m}^0 + {J_e}^3 - i{J_m}^3\right) \\ &(\partial_0 - \partial_3)\left(E^1 - iB^1 - iE^2 - B^2\right) - (-\partial_1 + i\partial_2)\left(E^3 - iB^3\right) = -\left({J_e}^1 - i{J_m}^1 - i{J_e}^2 - {J_m}^2\right) \\ &(-\partial_1 - i\partial_2)\left(E^3 - iB^3\right) + (\partial_0 + \partial_3)\left(E^1 - iB^1 + iE^2 + B^2\right) = -\left({J_e}^1 - i{J_m}^1 + i{J_e}^2 + {J_m}^2\right) \\ &(-\partial_1 - i\partial_2)\left(E^1 - iB^1 - iE^2 - B^2\right) - (\partial_0 + \partial_3)\left(E^3 - iB^3\right) = -\left({J_e}^0 - i{J_m}^0 - {J_e}^3 + i{J_m}^3\right) \end{aligned} \tag{5.7}$$

By separating real and imaginary parts of the equations (5.7), we obtain Maxwell equations in vector form

$$
\begin{aligned}
&div\ \boldsymbol{E} = {J_e}^0 \qquad curl\ \boldsymbol{B} - \dot{\boldsymbol{E}} = \boldsymbol{J}_e \\
&div\ \boldsymbol{B} = {J_m}^0 \quad -curl\ \boldsymbol{E} - \dot{\boldsymbol{B}} = \boldsymbol{J}_m
\end{aligned}
\tag{5.8}
$$

The conservation of charge is a consequence of the Maxwell equations. The continuity equation for the current density reads (see (4.27) and (4.26))

$$\partial_k\ J^k = \partial_{\mu\dot{\nu}} S^{\mu\dot{\nu}} = \partial_{\mu\dot{\nu}} \partial^{\lambda\dot{\nu}} f^{\mu}_{\lambda} = \delta^{\lambda}_{\mu} \left(\partial^{\rho}\partial_{\rho}\right) f^{\mu}_{\lambda} = \left(\partial^{\rho}\partial_{\rho}\right) f^{\mu}_{\mu} = 0 \tag{5.9}$$

since f^{μ}_{ν} is a traceless matrix: $f^{\mu}_{\mu} = 0$.

5.1 Lorentz force density

Now we can use Maxwell equations (5.1) to derive the expression for the Lorentz force spinor. We first introduce the *stress-energy density spinor* of the electromagnetic field

$$T^{\delta\dot{\rho}}_{\mu\dot{\nu}} = f^{\delta}_{\mu}\ \dot{f}^{\dot{\rho}}_{\dot{\nu}} \tag{5.10}$$

and then consider the expression

$$\partial_{\delta\dot{\rho}}\ T^{\delta\dot{\rho}}_{\mu\dot{\nu}} = \partial_{\delta\dot{\rho}} \left[f^{\delta}_{\mu}\ \dot{f}^{\dot{\rho}}_{\dot{\nu}} \right] = \dot{f}^{\dot{\rho}}_{\dot{\nu}} \left[\partial_{\delta\dot{\rho}}\ f^{\delta}_{\mu} \right] + f^{\delta}_{\mu} \left[\partial_{\delta\dot{\rho}}\ \dot{f}^{\dot{\rho}}_{\dot{\nu}} \right] = \Lambda_{\mu\dot{\nu}} \tag{5.11}$$

where the *force density* spinor

$$\Lambda_{\mu\dot{\nu}} = - \left[\dot{f}^{\dot{\rho}}_{\dot{\nu}}\ S_{\mu\dot{\rho}} + f^{\delta}_{\mu}\ \dot{S}_{\delta\dot{\nu}} \right] \tag{5.12}$$

Of course, the force density spinor $\Lambda_{\mu\dot{\nu}}$ corresponds to the Lorentz force density 4-vector $\mathcal{F}^{\mu}$:

$$\Lambda_{\mu\dot{\nu}} = \left[\begin{array}{cc} \Lambda_{1\dot{1}} & \Lambda_{1\dot{2}} \\ \Lambda_{2\dot{1}} & \Lambda_{2\dot{2}} \end{array} \right] = \left[\begin{array}{cc} \mathcal{F}^0 + \mathcal{F}^3 & \mathcal{F}^1 + i\mathcal{F}^2 \\ \mathcal{F}^1 - i\mathcal{F}^2 & \mathcal{F}^0 - \mathcal{F}^3 \end{array} \right] \tag{5.13}$$

5.2 Electromagnetic fields with special symmetries

In further sections we will need the expressions for Maxwell equations for the systems with special symmetries. Particularly we will need such expressions for:

- Transverse plane electromagnetic waves, and
- Stationary fields with axial symmetry

5.2.1 Transverse plane waves

By definition, in transverse plane waves the directions of vectors **E**, **B** are orthogonal to the direction of wave propagation. For simplicity we can choose axis $\mathbf{e}_3$ parallel to the direction of the wave propagation. In this case we will have:

$$\begin{aligned} F^3 &\equiv 0 \\ J^1 = J^2 &\equiv 0 \end{aligned} \tag{5.14}$$

at all times and all points in space.

The Maxwell equations will be reduced to the following expressions:

$$\begin{aligned} (\partial_1 - i\partial_2)\,(F^1 + iF^2) &= J^0 + J^3 \\ (\partial_0 - \partial_3)\,(F^1 - iF^2) &= 0 \\ (\partial_0 + \partial_3)\,(F^1 + iF^2) &= 0 \\ (\partial_1 + i\partial_2)\,(F^1 - iF^2) &= J^0 - J^3 \end{aligned} \tag{5.15}$$

In absence of charged currents the right hand sides of all the equations vanish, and we obtain

$$\begin{aligned} (\partial_1 - i\partial_2)\,(F^1 + iF^2) &= 0 \\ (\partial_0 - \partial_3)\,(F^1 - iF^2) &= 0 \\ (\partial_0 + \partial_3)\,(F^1 + iF^2) &= 0 \\ (\partial_1 + i\partial_2)\,(F^1 - iF^2) &= 0 \end{aligned} \tag{5.16}$$

5.2.2 Stationary field configurations with axial symmetry

Now we consider the stationary field configurations with axial symmetry. We introduce the polar cylindrical coordinate system $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\} \to \{\mathbf{e}_1,\ \mathbf{e}_\rho,\ \mathbf{e}_\varphi\ \}$ (see Figure 3):

$$
\begin{aligned}
x^2 &= \rho\cos\varphi \\
x^3 &= \rho\sin\varphi \\
x^1 &= x^1
\end{aligned}
$$

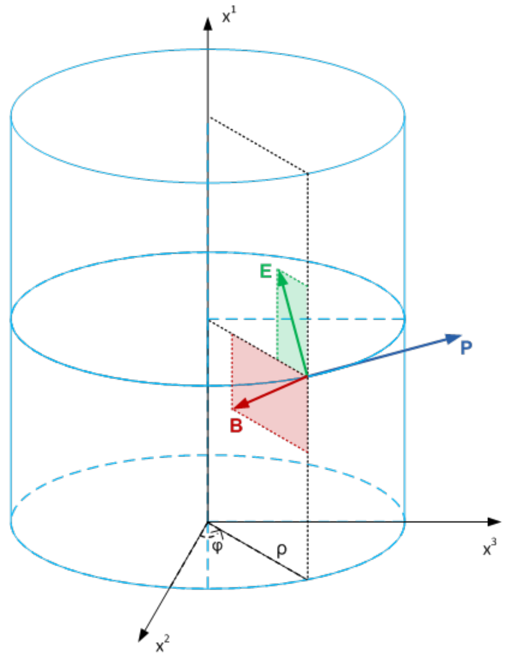

Figure 3. Axially symmetric field configuration

and require that

$$
\begin{aligned}
\partial_0 &\equiv 0 \quad \textit{stationarity} \\
\partial_\varphi &\equiv 0 \quad \textit{axial symmetry}
\end{aligned}
\tag{5.17}
$$

This ensures that field configuration is symmetric w.r.t. rotations around axis $\mathbf{e}_1$ and is not changing over time.

Due to axial symmetry, electric and magnetic fields $\mathbf{E}$, $\mathbf{B}$ in each point belong to the plane $\varphi = const$ (i.e. $F^\varphi \equiv 0$), and charge density current is parallel to direction of $\mathbf{e}_\varphi$ (i.e. $J^\rho = J^1 \equiv 0$).

Then Maxwell equations in cylindrical coordinates can be written as follows:

$$
\begin{gathered}
\tfrac{1}{\rho}F^\rho + (\partial_1 - i\partial_\rho)\,(F^1 + iF^\rho) = J^0 + J^\varphi \\
\left(\partial_0 - \tfrac{1}{\rho}\partial_\varphi\right)(F^1 - iF^\rho) = 0 \\
\left(\partial_0 + \tfrac{1}{\rho}\partial_\varphi\right)(F^1 + iF^\rho) = 0 \\
\tfrac{1}{\rho}F^\rho + (\partial_1 + i\partial_\rho)\,(F^1 - iF^\rho) = J^0 - J^\varphi
\end{gathered}
\tag{5.18}
$$

6 Annex III: Dirac equations in spinor form

The Dirac equations were first written in spinor form by G.E. Uhlenbeck and O. Laporte in 1931 [2]. According to (4.5) the four wave functions of Dirac correspond to spinor of the first rank ξ^{μ} and co-spinor of the first rank $\eta_{\dot{\nu}}$, and Dirac equations become

$$\begin{aligned} \left(\partial^{\mu\dot{\nu}} + ie\ \Phi^{\mu\dot{\nu}}\right)\ \eta_{\dot{\nu}}\ +\ im\ \xi^{\mu} =\ 0 \\ \left(\partial_{\mu\dot{\nu}} + ie\ \Phi_{\mu\dot{\nu}}\right) \xi^{\mu} +\ im\ \eta_{\dot{\nu}} = 0 \end{aligned} \qquad m \in R \tag{6.1}$$

where $\Phi^{\mu\dot{\nu}}$ is a spinor obtained from electromagnetic potential four-vector A_{μ} using general rule (4.21).

The free Dirac equation in spinorial form can be written as

$$\begin{aligned} \partial^{\mu\dot{\nu}}\eta_{\dot{\nu}}\ +\ im\ \xi^{\mu} =\ 0 \\ \partial_{\mu\dot{\nu}}\xi^{\mu} +\ im\ \eta_{\dot{\nu}} = 0 \end{aligned} \tag{6.2}$$

7 Annex IV: Transition to the rest frame

Let us briefly discuss the properties of the momentum densities P^{μ} in the special cases of *orthogonal* ($\boldsymbol{E} \perp \boldsymbol{B}$) electromagnetic fields. The case of *parallel* ($\boldsymbol{E} \parallel \boldsymbol{B}, \quad E = B$) electromagnetic fields is considered in connection with *neutrino* model in the Section 3.4.

Let us first consider the case $\boldsymbol{E} \perp \boldsymbol{B}$, $E > B$.

In this case the squares of invariants of the electromagnetic fields are positive real numbers:

$$\begin{aligned} {\lambda_{\pm}}^2 =\ E^2 - B^2 >\ 0 \\ {\overline{\lambda}_{\pm}}^2 =\ E^2 - B^2 > 0 \end{aligned} \tag{7.1}$$

It is easy to check that in the frame $M_{\perp}$ the non-zero components of the momentum density 4-vector P_{μ} will be

$$\begin{aligned} P_0 = +\ 4E \\ P_3 = -\ 4B \end{aligned} \tag{7.2}$$

if the pair of vectors $\{\mathbf{E}, \mathbf{B}\}$ has the same orientation as basis vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$, and

$$\begin{aligned} P_0 = +\ 4E \\ P_3 = +\ 4B \end{aligned} \tag{7.3}$$

for inverse orientation of the pair of vectors $\{\mathbf{E}, \mathbf{B}\}$.

It is well known that if $\boldsymbol{E} \perp \boldsymbol{B}$, $E > B$, there is a reference frame where magnetic field $\boldsymbol{B}$ vanish [6]. Let's denote this frame as M_E.

From (7.2 - 7.3) we can see that only time-like component P_0 of the momentum density has non-zero value in the frame M_E. In this sense frame M_E might be considered as a "*rest frame*" of the momentum density P_μ.

Similarly we can show that in the case of $\boldsymbol{E} \perp \boldsymbol{B}$, $E < B$ the squares of invariants of the electromagnetic fields are negative real numbers:

$$
\begin{aligned}
{\lambda_\pm}^2 &= \; E^2 - B^2 < \; 0 \\
{\overline{\lambda}_\pm}^2 &= \; E^2 - B^2 < 0
\end{aligned}
\tag{7.4}
$$

and in the frame $M_\perp$ the non-zero components of the momentum 4-vector P_μ will be

$$
\begin{aligned}
P_0 &= +\, 4B \\
P_3 &= -\, 4E
\end{aligned}
\tag{7.5}
$$

if the pair of vectors $\{\mathbf{E}, \mathbf{B}\}$ has the same orientation as basis vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$, and

$$
\begin{aligned}
P_0 &= +\, 4B \\
P_3 &= +\, 4E
\end{aligned}
\tag{7.6}
$$

for inverse orientation of the pair of vectors $\{\mathbf{E}, \mathbf{B}\}$.

Similarly, in the reference frame M_B where electric field $\boldsymbol{E}$ vanish, only time-like component P_0 of the momentum density has non-zero value.

In the case of $\boldsymbol{E} \perp \boldsymbol{B}$, $E = B$ we have $\lambda = \bar{\lambda} = 0$, and the momentum density is *isotropic* in all reference frames: $P^\mu P_\mu = 4|\lambda|^2 = 0$.

8 Annex V: Transverse plane waves

As mentioned above, we choose the coordinates system in such a way that in each point

$$
\begin{aligned}
&\mathbf{E}, \mathbf{B} \perp \; \mathbf{e}_3 \\
&F^3 = \bar{F}^3 \equiv 0
\end{aligned}
\tag{8.1}
$$

Let us now consider the matter wave equations (2.12) written in the form

$$\partial^{\mu\dot\nu}\eta_{\dot\nu} = -\ i\ \lambda\ \xi^{\mu}$$
$$\partial_{\mu\dot\nu}\xi^{\dot\nu} = +\ i\ \lambda\ \ \eta_{\mu} \tag{8.2}$$

We expand these equations using (4.24), (4.25), (2.17) and (2.19)

$$\begin{aligned}
&(\partial_0+\partial_3)\sqrt{\bar F^1 - i\bar F^2} + (\partial_1 - i\partial_2)\sqrt{\bar F^1 + i\bar F^2} = -\ i\ \lambda\ \sqrt{F^1 - iF^2} && (i)\\
&(\partial_1+i\partial_2)\sqrt{\bar F^1 - i\bar F^2} + (\partial_0 - \partial_3)\sqrt{\bar F^1 + i\bar F^2} = -\ i\ \lambda\ \sqrt{F^1 + iF^2} && (ii)\\
&(\partial_0-\partial_3)\sqrt{\bar F^1 + i\bar F^2} + (-\partial_1 - i\partial_2)\sqrt{\bar F^1 - i\bar F^2} = +\ i\ \lambda\ \sqrt{F^1 + iF^2} && (iii)\\
&(-\partial_1+i\partial_2)\sqrt{\bar F^1 + i\bar F^2} + (\partial_0 + \partial_3)\sqrt{\bar F^1 - i\bar F^2} = +\ i\ \lambda\ \sqrt{F^1 - iF^2} && (iv)
\end{aligned} \tag{8.3}$$

By adding (i) and (iv) we obtain

$$(\partial_0+\partial_3)\left(\bar F^1 - i\bar F^2\right) = 0 \tag{8.4}$$

Similarly, by adding (ii) and (iii) we obtain

$$(\partial_0-\partial_3)\left(\bar F^1 + i\bar F^2\right) = 0 \tag{8.5}$$

With the two remaining equations the whole system can be written as

$$\begin{aligned}
(\partial_1+i\partial_2)\left(\bar F^1 - i\bar F^2\right) &= -\ 2\ i\ \lambda\ \sqrt{F^1 + iF^2}\ \sqrt{\bar F^1 - i\bar F^2}\\
(\partial_0-\partial_3)\left(\bar F^1 + i\bar F^2\right) &= 0\\
(\partial_0+\partial_3)\left(\bar F^1 - i\bar F^2\right) &= 0\\
(\partial_1-i\partial_2)\left(\bar F^1 + i\bar F^2\right) &= -\ 2\ i\ \lambda\ \sqrt{F^1 - iF^2}\ \sqrt{\bar F^1 + i\bar F^2}
\end{aligned} \tag{8.6}$$

and complex conjugated equations will have the form

$$\begin{aligned}
(\partial_1-i\partial_2)\left(F^1 + iF^2\right) &= +\ 2\ i\ \bar\lambda\ \sqrt{F^1 + iF^2}\ \sqrt{\bar F^1 - i\bar F^2}\\
(\partial_0-\partial_3)\left(F^1 - iF^2\right) &= 0\\
(\partial_0+\partial_3)\left(F^1 + iF^2\right) &= 0\\
(\partial_1+i\partial_2)\left(F^1 - iF^2\right) &= +\ 2\ i\ \bar\lambda\ \sqrt{F^1 - iF^2}\ \sqrt{\bar F^1 + i\bar F^2}
\end{aligned} \tag{8.7}$$

With expressions (2.21 - 2.23) for the momentum density of the matter field P_μ in the frame $M_\perp$ we can rewrite (8.7) in the following form:

$$
\begin{aligned}
(\partial_1 - i\partial_2)\,(F^1 + iF^2) &= i\ \bar{\lambda}\ \ (P^0 + P^3) \\
(\partial_0 - \partial_3)\,(F^1 - iF^2) &= 0 \\
(\partial_0 + \partial_3)\,(F^1 + iF^2) &= 0 \\
(\partial_1 + i\partial_2)\,(F^1 - iF^2) &= i\ \bar{\lambda}\ \ (P^0 - P^3)
\end{aligned} \tag{8.8}
$$

From comparison of (8.7) and (5.15) we conclude that in the case of the transverse plane waves:

- Matter field equations are reduced to the Maxwell equations, and
- The charge density current J^μ is expressed via electromagnetic field invariant $\bar{\lambda}$ and momentum density P^μ in the following way:

$$
J^\mu = i\ \bar{\lambda}\ P^\mu \tag{8.9}
$$

Hence we conclude that, in the case of the transverse plane waves, electromagnetic field invariant $(i\bar{\lambda})$ plays the role of the electromagnetic *charge density* scalar (it is clear that $(-i\lambda)$ plays the same role for anti-particles). Generally $\bar{\lambda}$ is complex valued, hence allowing for *both* non-zero electric and magnetic charge densities.

Now we can find the expression for the Lorentz force density (5.12) acting on matter fields.

(8.9) can be written as

$$
\begin{aligned}
S_{\mu\dot{\nu}} &= i\bar{\lambda}\ P_{\mu\dot{\nu}} = i\bar{\lambda}\,(p_{\mu\dot{\nu}} + \hat{p}_{\mu\dot{\nu}}) = i\bar{\lambda}\left(\xi_\mu \xi_{\dot{\nu}} + \eta_\mu \eta_{\dot{\nu}}\right) \\
\dot{S}_{\mu\dot{\nu}} &= -i\lambda\ P_{\mu\dot{\nu}} = -i\lambda\,(p_{\mu\dot{\nu}} + \hat{p}_{\mu\dot{\nu}}) = -i\lambda\left(\xi_\mu \xi_{\dot{\nu}} + \eta_\mu \eta_{\dot{\nu}}\right)
\end{aligned} \tag{8.10}
$$

Using (2.11) we find that

$$
\begin{aligned}
f_\mu^\delta \dot{S}_{\delta\dot{\nu}} &= -i\lambda\ (f_\mu^\delta \xi_\delta \xi_{\dot{\nu}} + f_\mu^\delta \eta_\delta \eta_{\dot{\nu}}) = -i\lambda^2\left(-\xi_\mu \xi_{\dot{\nu}} + \eta_\mu \eta_{\dot{\nu}}\right) \\
\dot{f}_{\dot{\nu}}^{\dot{\rho}} S_{\mu\dot{\rho}} &= i\bar{\lambda}\ (\xi_\mu \dot{f}_{\dot{\nu}}^{\dot{\rho}} \xi_{\dot{\rho}} + \eta_\mu \dot{f}_{\dot{\nu}}^{\dot{\rho}} \eta_{\dot{\rho}}) = i\bar{\lambda}^2\left(-\xi_\mu \xi_{\dot{\nu}} + \eta_\mu \eta_{\dot{\nu}}\right)
\end{aligned} \tag{8.11}
$$

and the Lorentz force density becomes

$$
\Lambda_{\mu\dot{\nu}} = -\left[\dot{f}_{\dot{\nu}}^{\dot{\rho}} S_{\mu\dot{\rho}} + f_\mu^\delta \dot{S}_{\delta\dot{\nu}}\right] = -i\left(\lambda^2 - \bar{\lambda}^2\right)\left(\xi_\mu \xi_{\dot{\nu}} - \eta_\mu \eta_{\dot{\nu}}\right) = -i\left(\lambda^2 - \bar{\lambda}^2\right) P_{A\mu\dot{\nu}} \tag{8.12}
$$

It is interesting that the Lorentz force $\mathcal{F}^\mu$ is proportional to the axial vector current ${P_A}^\mu$ (see Section 5.5). From (8.12) we can see that Lorentz force vanishes when $\mathbf{E} \cdot \mathbf{B} = 0$, i.e. when imaginary parts of the squared electromagnetic field invariants λ^2 and $\bar{\lambda}^2$ are zero. Particularly, this is the case of electromagnetic waves "in vacuum" (i.e. *photons*, see below. When Lorentz force is zero, the momentum density of the matter field remains constant in the course of particle's motion, hence allowing for *uniform* motion of the particle.

In the case of plane electromagnetic waves "in vacuum" ($\boldsymbol{E} \perp \boldsymbol{B}$, $E = B$) we have $\lambda = \bar{\lambda} = 0$, and matter field equations (8.8) coincide with the "source-free" Maxwell equations (5.16). In this case the momentum density P^μ of the matter field is non-zero, while the charge density J^μ is zero. In this sense the electromagnetic wave is not actually "source-free".

9 Annex VI: Stationary field configurations with axial symmetry

In Cartesian basis matter field equations (2.1)

$$
\begin{aligned}
\partial^{\mu\dot{\nu}}\eta_{\dot{\nu}} &= -\, i\ \lambda\ \xi^\mu \\
\partial_{\mu\dot{\nu}}\xi^\mu &= -\, i\ \bar{\lambda}\ \eta_{\dot{\nu}}
\end{aligned}
\tag{9.1}
$$

can be written in the following matrix form

$$
\begin{aligned}
(\partial_0 + \partial_1\sigma_1 + \partial_2\sigma_2 + \partial_3\sigma_3)\,\dot{\eta} &= -\, i\ \lambda\ \xi \\
(\partial_0 - \partial_1\sigma_1 - \partial_2\sigma_2 - \partial_3\sigma_3)\,\xi &= -\, i\ \bar{\lambda}\ \dot{\eta}
\end{aligned}
\tag{9.2}
$$

where

$$
\xi = \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} \qquad \dot{\eta} = \begin{bmatrix} \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{bmatrix}
\tag{9.3}
$$

By introducing polar cylindrical coordinates (see Figure 3, Section 3.2.2)

$$
\begin{aligned}
x^2 &= \rho\cos\varphi \\
x^3 &= \rho\sin\varphi \\
x^1 &= x^1
\end{aligned}
\tag{9.4}
$$

we will have the following expressions for partial derivatives

$$
\begin{aligned}
\partial_0 &= \partial_0 \\
\partial_2 &= \cos\varphi\ \partial_\rho - \tfrac{\sin\varphi}{\rho}\ \partial_\varphi \\
\partial_3 &= \sin\varphi\ \partial_\rho + \tfrac{\cos\varphi}{\rho}\ \partial_\varphi \\
\partial_1 &= \partial_1
\end{aligned}
\tag{9.5}
$$

By using (9.5) we obtain

$$
\partial_2\sigma_2 + \partial_3\sigma_3 = (\sigma_2\cos\varphi + \sigma_3\sin\varphi)\,\partial_\rho + \frac{1}{\rho}\,(-\sigma_2\sin\varphi + \sigma_3\cos\varphi)\,\partial_\varphi \tag{9.6}
$$

with consequent expressions for matter field equations (9.2) in the new coordinate system.

To complete the transition to the polar coordinate system, we need to account for *change of spinor components* due to change of basis vectors in each point: $\{\mathbf{e}_2,\ \mathbf{e}_3\} \to \{\ \mathbf{e}_\rho,\ \mathbf{e}_\varphi\ \}$.

The pair of vectors $\{\mathbf{e}_\rho,\ \mathbf{e}_\varphi\}$ can be obtained by rotating the pair of Cartesian basis vectors $\{\mathbf{e}_2,\ \mathbf{e}_3\}$ by the angle φ around the axis $\mathbf{e}_1$ at every point with coordinates $(\rho,\ \varphi,\ x^1)$. This rotation results in the following transformation of the spinor components:

$$
\begin{aligned}
\xi^{'} &= \exp\left[i\tfrac{\varphi}{2}\sigma_1\right]\xi = \left[\cos\tfrac{\varphi}{2} + i\sigma_1\sin\tfrac{\varphi}{2}\right]\xi = S\xi \\
\dot\eta^{'} &= \exp\left[i\tfrac{\varphi}{2}\sigma_1\right]\dot\eta = \left[\cos\tfrac{\varphi}{2} + i\sigma_1\sin\tfrac{\varphi}{2}\right]\dot\eta = S\dot\eta
\end{aligned}
\tag{9.7}
$$

with the following transition operators

$$
\begin{aligned}
S &= \exp\left[i\tfrac{\varphi}{2}\sigma_1\right] \\
S^{-1} &= \exp\left[-i\tfrac{\varphi}{2}\sigma_1\right]
\end{aligned}
\tag{9.8}
$$

By applying operators S and S^{-1} to the equations (9.2) we obtain

$$
\begin{aligned}
S\ (\partial_0 + \partial_1\sigma_1 + \partial_2\sigma_2 + \partial_3\sigma_3)\,S^{-1}S\dot\eta &= -\ i\ \lambda\ S\xi \\
S\,(\partial_0 - \partial_1\sigma_1 - \partial_2\sigma_2 - \partial_3\sigma_3)\,S^{-1}S\xi &= -\ i\ \bar\lambda\ S\dot\eta
\end{aligned}
\tag{9.9}
$$

or

$$
\begin{aligned}
S\ (\partial_0 + \partial_1\sigma_1 + \partial_2\sigma_2 + \partial_3\sigma_3)\,S^{-1}\dot\eta^{'} &= -\ i\ \lambda\ \xi^{'} \\
S\,(\partial_0 - \partial_1\sigma_1 - \partial_2\sigma_2 - \partial_3\sigma_3)\,S^{-1}\xi^{'} &= -\ i\ \bar\lambda\ \dot\eta^{'}
\end{aligned}
\tag{9.10}
$$

Since operator S commutes with σ_0 and σ_1, we will have

$$\begin{aligned} S\ (\partial_0 + \partial_1\sigma_1 + \partial_2\sigma_2 + \partial_3\sigma_3)\, S^{-1} &= \partial_0 + \partial_1\sigma_1 + S\,(\partial_2\sigma_2 + \partial_3\sigma_3)\, S^{-1} \\ S\,(\partial_0 - \partial_1\sigma_1 - \partial_2\sigma_2 - \partial_3\sigma_3)\, S^{-1} &= \partial_0 - \partial_1\sigma_1 - S\,(\partial_2\sigma_2 + \partial_3\sigma_3)\, S^{-1} \end{aligned} \tag{9.11}$$

It is now easy to check that

$$S\,(\partial_2\sigma_2 + \partial_3\sigma_3)\, S^{-1} = \sigma_2\partial_\rho + \frac{1}{\rho}\sigma_3\partial_\varphi + \frac{1}{2\rho}\sigma_2 \tag{9.12}$$

and we complete transition of the matter field equations (9.2) to polar cylindrical coordinates:

$$\begin{aligned} \left[\partial_0 + \sigma_1\partial_1 + \sigma_2\partial_\rho + \tfrac{1}{\rho}\sigma_3\partial_\varphi + \tfrac{1}{2\rho}\sigma_2\right]\dot{\eta}' &= -\ i\ \lambda\ \xi' \\ \left[\partial_0 - \sigma_1\partial_1 - \sigma_2\partial_\rho - \tfrac{1}{\rho}\sigma_3\partial_\varphi - \tfrac{1}{2\rho}\sigma_2\right]\xi' &= -\ i\ \bar{\lambda}\ \dot{\eta}' \end{aligned} \tag{9.13}$$

Now we can use (2.17) and (2.19) to express components of the spinors ξ' and $\dot{\eta}'$ *via* components of the fields $\mathbf{E}$, $\mathbf{B}$ in *cylindrical coordinates* (i.e. in the basis) $\{\mathbf{e}_\rho,\ \mathbf{e}_\varphi,\ \mathbf{e}_1\}$:

$$\begin{aligned} \xi' &= \begin{bmatrix} \xi'^1 \\ \xi'^2 \end{bmatrix} = \begin{bmatrix} \sqrt{F^1 - iF^\rho} \\ \sqrt{F^1 + iF^\rho} \end{bmatrix} \\ \dot{\eta}' &= \begin{bmatrix} \eta'_{\dot{1}} \\ \eta'_{\dot{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{\bar{F}^1 - i\bar{F}^\rho} \\ \sqrt{\bar{F}^1 + i\bar{F}^\rho} \end{bmatrix} \end{aligned} \tag{9.14}$$

and write matter field equations as follows:

$$\begin{aligned} \begin{bmatrix} \partial_0 + \frac{1}{\rho}\partial_\varphi & \partial_1 - i\partial_\rho - i\frac{1}{2\rho} \\ \partial_1 + i\partial_\rho + i\frac{1}{2\rho} & \partial_0 - \frac{1}{\rho}\partial_\varphi \end{bmatrix} \begin{bmatrix} \sqrt{\bar{F}^1 - i\bar{F}^\rho} \\ \sqrt{\bar{F}^1 + i\bar{F}^\rho} \end{bmatrix} &= -\ i\ \lambda \begin{bmatrix} \sqrt{F^1 - iF^\rho} \\ \sqrt{F^1 + iF^\rho} \end{bmatrix} \\ \begin{bmatrix} \partial_0 - \frac{1}{\rho}\partial_\varphi & -\partial_1 + i\partial_\rho + i\frac{1}{2\rho} \\ -\partial_1 - i\partial_\rho - i\frac{1}{2\rho} & \partial_0 + \frac{1}{\rho}\partial_\varphi \end{bmatrix} \begin{bmatrix} \sqrt{F^1 - iF^\rho} \\ \sqrt{F^1 + iF^\rho} \end{bmatrix} &= -\ i\ \bar{\lambda} \begin{bmatrix} \sqrt{\bar{F}^1 - i\bar{F}^\rho} \\ \sqrt{\bar{F}^1 + i\bar{F}^\rho} \end{bmatrix} \end{aligned} \tag{9.15}$$

After expanding expressions (9.15) and applying complex conjugation to the first two equations, we obtain

$$\begin{aligned}
&\left(\partial_0 + \tfrac{1}{\rho}\partial_\varphi\right)\sqrt{F^1 + iF^\rho} + \left(\partial_1 + i\partial_\rho + i\tfrac{1}{2\rho}\right)\sqrt{F^1 - iF^\rho} = +\, i\ \bar{\lambda}\sqrt{\bar{F}^1 + i\bar{F}^\rho} \quad (i)\\
&\left(\partial_1 - i\partial_\rho - i\tfrac{1}{2\rho}\right)\sqrt{F^1 + iF^\rho} + \left(\partial_0 - \tfrac{1}{\rho}\partial_\varphi\right)\sqrt{F^1 - iF^\rho} = +\, i\ \bar{\lambda}\sqrt{\bar{F}^1 - i\bar{F}^\rho} \quad (ii)\\
&\left(\partial_0 - \tfrac{1}{\rho}\partial_\varphi\right)\sqrt{F^1 - iF^\rho} + \left(-\partial_1 + i\partial_\rho + i\tfrac{1}{2\rho}\right)\sqrt{F^1 + iF^\rho} = -\, i\ \bar{\lambda}\sqrt{\bar{F}^1 - i\bar{F}^\rho} \quad (iii)\\
&\left(-\partial_1 - i\partial_\rho - i\tfrac{1}{2\rho}\right)\sqrt{F^1 - iF^\rho} + \left(\partial_0 + \tfrac{1}{\rho}\partial_\varphi\right)\sqrt{F^1 + iF^\rho} = -\, i\ \bar{\lambda}\sqrt{\bar{F}^1 + i\bar{F}^\rho} \quad (iv)
\end{aligned} \tag{9.16}$$

By adding (i) and (iv) we obtain

$$\left(\partial_0 + \frac{1}{\rho}\partial_\varphi\right)\left(F^1 + iF^\rho\right) = 0 \tag{9.17}$$

Similarly, by adding (ii) and (iii) we obtain

$$\left(\partial_0 - \frac{1}{\rho}\partial_\varphi\right)\left(F^1 - iF^\rho\right) = 0 \tag{9.18}$$

Naturally, (9.17) and (9.18) are consistent with assumed stationarity and axial symmetry of the field configuration:

$$\begin{aligned}
&\partial_0 \equiv 0 \quad \textit{stationarity}\\
&\partial_\varphi \equiv 0 \quad \textit{axial symmetry}
\end{aligned} \tag{9.19}$$

With the two remaining equations the whole system can be written as

$$\begin{gathered}
\left(\partial_1 - i\partial_\rho\right)\left(F^1 + iF^\rho\right) - i\tfrac{1}{\rho}\left(F^1 + iF^\rho\right) = +\, 2\ i\ \bar{\lambda}\ \sqrt{\bar{F}^1 - i\bar{F}^\rho}\ \sqrt{F^1 + iF^\rho}\\
\left(\partial_0 - \tfrac{1}{\rho}\partial_\varphi\right)\left(F^1 - iF^\rho\right) = 0\\
\left(\partial_0 + \tfrac{1}{\rho}\partial_\varphi\right)\left(F^1 + iF^\rho\right) = 0\\
\left(\partial_1 + i\partial_\rho\right)\left(F^1 - iF^\rho\right) + i\tfrac{1}{\rho}\left(F^1 - iF^\rho\right) = +\, 2\ i\ \bar{\lambda}\ \sqrt{\bar{F}^1 + i\bar{F}^\rho}\ \sqrt{F^1 - iF^\rho}
\end{gathered} \tag{9.20}$$

or

$$
\begin{aligned}
\frac{1}{\rho}F^\rho + (\partial_1 - i\partial_\rho)\,(F^1 + iF^\rho) &= +\ 2\ i\ \bar{\lambda}\ \sqrt{\bar{F}^1 - i\bar{F}^\rho}\ \sqrt{F^1 + iF^\rho} + \frac{i}{\rho}F^1 \\
\left(\partial_0 - \frac{1}{\rho}\partial_\varphi\right)(F^1 - iF^\rho) &= 0 \\
\left(\partial_0 + \frac{1}{\rho}\partial_\varphi\right)(F^1 + iF^\rho) &= 0 \\
\frac{1}{\rho}F^\rho + (\partial_1 + i\partial_\rho)\,(F^1 - iF^\rho) &= +\ 2\ i\ \bar{\lambda}\ \sqrt{\bar{F}^1 + i\bar{F}^\rho}\ \sqrt{F^1 - iF^\rho} - \frac{i}{\rho}F^1
\end{aligned}
\tag{9.21}
$$

From comparison with (5.18) we conclude that (9.21) will coincide with Maxwell equations if we define:

$$
\begin{aligned}
J^0 - J^\varphi &= i\ \bar{\lambda}\,(P^0 - P^\varphi) - i\frac{1}{\rho}F^1 \\
J^0 + J^\varphi &= i\ \bar{\lambda}\,(P^0 + P^\varphi) + i\frac{1}{\rho}F^1
\end{aligned}
\tag{9.22}
$$

where J^0, J^φ and P^0, P^φ are non-zero components of charge density and momentum density correspondingly.

It is interesting that in axially symmetric case

$$
\begin{aligned}
J^0 &= i\ \bar{\lambda}\ P^0 \\
J^\varphi &= i\ \bar{\lambda}\ P^\varphi + 2i\ \frac{1}{\rho}F^1
\end{aligned}
\tag{9.23}
$$

and hence the ratio $\frac{J^\varphi}{J^0}$ is not equal to the ratio $\frac{P^\varphi}{P^0}$. Consequently, the "velocity of charge" is not the same as "velocity of mass" any more.

References

[1] F. Rohrlich, *31st Joseph Henry Lecture,* State University of Iowa (1962), `http://www.philsoc.org/1962Spring/1526transcript.html`

[2] Laporte, O. and G. E. Uhlenbeck, *Phys. Rev. 37, 1380* (1931)

[3] Rumer, Yu.B. and A.I. Fet. *Group theory and quantum fields.* Moscow, USSR: Nauka publisher, 1977

[4] Landau and Lifshitz. *Course of Theoretical Physics, vol. IV, Quantum Electrodynamics.* 3rd edition. Moscow, USSR: Nauka publisher, 1989

[5] Hans de Vries. Understanding Relativistic Quantum Field Theory, `http://www.physics-quest.org/`

[6] Dubrovin, B.A., Novikov S.P. and A.T. Fomenko. *The Modern Geometry.* Moscow, USSR: Nauka publisher, 1979

[7] Hans C. Ohanian, *Am. J. Phys. 54 (6)* (1986)