A CONDITION BY PAUL OF VENICE (1369-1429) SOLVES RUSSELL'S PARADOX, BLOCKS CANTOR'S DIAGONAL ARGUMENT, AND PROVIDES A CHALLENGE TO ZFC

Thomas Colignatus

thomascool.eu

November 14 2014 - May 20 2015, June 4 2015

Abstract

Paul of Venice (1369-1429) provides a consistency condition that resolves Russell's Paradox in naive set theory without using a theory of types. It allows a set of all sets. It also blocks the (diagonal) general proof of Cantor's Theorem (in Russell's form, for the power set). It is not unlikely that the Zermelo-Fraenkel (ZFC) axioms for set theory are still too lax on the notion of a 'well-defined set'. The transfinites of ZFC may be a mirage, and a consequence of still imperfect axiomatics in ZFC for the foundations for set theory.

For amendment of ZFC two alternatives are mentioned: ZFC-PV (amendment of de Axiom of Separation) or BST (Basic Set Theory).

Lemma 3.2 shows that a Cantorian reading of ZFC implies the possibility of the weaker Pauline reading. Theorem 3.3 gives the existence of a Pauline set. It has a fundamental constructive proof and a compact non-constructive proof. Theorem 1.1.6 shows that ZFC has an anomaly. Corollary 3.3 turns that anomaly into an inconsistency.

Keywords: Paul of Venice • Russell's Paradox • Cantor's Theorem • ZFC • naive set theory • well-defined set • set of all sets • diagonal argument • transfinites

MSC2010: 03E30 Axiomatics of classical set theory and its fragments

03E70 Nonclassical and second-order set theories

97E60 Mathematics education: Sets, relations, set theory

Contents

Introduction

From Epimenides to Paul of Venice, to Cantor, to Russell

Russell devised his paradox after studying Cantor. In naive set theory, Russell's set is R = $\{x|\ x \notin x\}$. Subsequently $R \in R \Leftrightarrow R \notin R$ and naive set theory collapses. Define however S = $\{x|\ x \notin x\ \&\ x \in S\}$ with the small consistency condition inspired by the discussion by Bochenski (1956, 1970:250) of Paul of Venice (1369-1429) (Paulus Venetus). Without contradiction we find $S \notin S$. Paul of Venice developed his condition for the Liar Paradox. The translation to set theory is a bit involved, for example w.r.t. infinite regress, but such issues can be resolved.

There is no reason for a crisis in the foundations of logic and mathematics and there is no need for a theory of types. Observe: (1) a theory of types forbids the set of all sets while it is a useful concept, (2) a theory of types has R in the category *'may not be formed'* and thus implies a *third category* next to truth and falsehood. It would be illogical to reject that third category. It is logical instead to generalize that third category into the general notion of 'nonsense'. Thus we do well to formally develop a three-valued logic to determine that R is nonsense - though it has meaning (that allows us to see that it is nonsense). It remains an issue that three-valued logic is not without its paradoxes, but Colignatus (1981, 2007, 2011) (**ALOE**) holds that these can be solved too.

ALOE in 1981 applied the Paul of Venice consistency condition to the Russell set (p129), and applied it in 2007 (p239) also to Cantor's (diagonal) argument (in Russell's version for the power set). A set with $x \notin f[x]$ in its definition is called Cantorian. ALOE does not develop the formal ZFC system of axioms for set theory (from Zermelo, Fraenkel and the Axiom of Choice). ALOE's discussion may be seen as intermediate between naive set theory and this present paper. **Appendix A** discusses the versions of ALOE, for proper reference.

The new issue in this paper is the challenge to the ZFC axioms. The ZFC system may still be too lax on the notion of a *'well-defined set'.* The transfinites of ZFC may be a mirage, and a consequence of still imperfect axiomatics of ZFC w.r.t. the foundations for set theory.

The following sections will make the argument formal. Section 1 discusses singleton and infinity. Section 2 reviews the diagonal argument. Section 3 gives the challenge to ZFC. Section 4 concludes.

We take our definitions from a matricola course at Leiden and Delft.

Definition (Coplakova et al. (2011:18), I.4.7): Let *A* be a set. The power set of *A* is the set of all subsets of *A.* Notation: *P*[*A*]. Another notation is 2^A, whence its name.

Definition (Coplakova et al. (2011:144-145)): ZFC. This includes the axiom that each set has a power set.

Though this paper presents new results, it is still intended to be read by first year students. The general readership is served with an overview in the main body, and details in the appendices. Since we are dealing with self-referential constructs, understanding requires all (constructive) steps. Thus this paper should be read as a road map leading to, and inducement to finally study, the constructive proof in **Appendix F.3** of Theorem 3.3.

The appendices

Appendix B discusses recent Hart (2015) who considers Cantor's Theorem in traditional manner. **Appendix C** deproves the weakest form of the theorem, communicated by Hart in 2012. **Appendix D** is an 'initial review' by an editor of a peer-reviewed journal for the December 31 2014 version of this paper. **Appendix E** illuminates 'Cantorian' versus 'Pauline' strategies on ZFC that are relevant for a key point in the reasoning. **Appendix F** does a marginal analysis, when an arbitrary function is extended with one element mapped to the Cantorian set of the new situation. Studying this case gave the nutshell in Section 1.1, the theorem in Section 3, and subsequently also the self-referential Conjecture 1.2.B (and check the distinction in its first comment). **Appendix G** shows how ZFC blocks Russell's paradox.

This paper originally had six pages. It assumed a willingness of readers to study the references and fill in some details themselves. This drew comments that were replied to in subsequent versions. The fifth version had gotten cluttered and uninviting. This version returns to the original and puts supporting analyses in the appendices. Thus, when this Introduction or Section 2 cause questions, first consider Appendices H and J (there is no I).

This version extended with: the singleton in Section 1, formal deductions in Section 3, and the marginal analysis in **Appendix F**. Thus the number of pages rose again. The singleton however appears to be crucially enlightening, also for marginal analysis, and the formal deductions naturally help to see the logical framework.

Appendix H supports this Introduction. W.r.t. above definition of S one might hold that there would be infinite regress, if a test on S on the left causes a test on S on the right, which causes a test on the left again, and so on. The truthtable of $p \Leftrightarrow (\neg p \;\&\; p)$ allows a formal decision however. The appendix restates the exception switch from ALOE for when $x \neq S$. A new proposed notation with asymmetric '&&' allows for a shorthand of the exception switch. In the body of the paper '&&' is used, but the structure of the argument can be followed when taking this as '&' - with a backup in the appendix.

Appendix J supports Section 2 (the review of the ***common*** proof of Cantor's theorem that uses a bijection). There are: (a) A table with the various forms of the theorems, proofs and their refutations. Remember that Cantor presented his theorems before ZFC. (b) The semantics of *deproving a theorem*: refuting an argument by showing that a proof is invalid. (c) The ***standard*** theorem from a matricola course at Leiden and Delft: the form that uses only a surjection. One might compare with the weakest form in **Appendix C**.

It must also be observed that this author is no expert on Cantor's Theorem, see for background Colignatus (2013). We may reject the various proofs - see the table in **Appendix J** - but perhaps there are other proofs. A tentative check shows that the common proof is given at various locations that seem to matter but this may only mean that it is a popular proof. Hart (2013) & (2015) also gives Cantor's original forms of 1874 and 1890/91, but then see both **Appendix B** and their rejections in Colignatus (2012, 2013) (CCPO-PCWA).

1. The singleton and the infinite

1.1. A nutshell link between Russell and Cantor

1.1.1. The singleton

Let A be a set with a single element, $A = \{\alpha\}$. Thus $P[A] = \{\emptyset, A\}$. Let $f: A \rightarrow P[A]$.

If $f[\alpha] = \emptyset$ then $\alpha \notin f[\alpha]$. If $f[\alpha] = A$ then $\alpha \in f[\alpha]$. Thus $f[\alpha] = \emptyset \Leftrightarrow \alpha \notin f[\alpha]$. Consider:

(1) In steps: define $\Psi = \{x = \alpha \mid x \notin f[x]\}$, find $\Psi \in P[A]$, then try $f[\alpha] = \Psi$.

(2) Directly: $f[\alpha] = \{x = \alpha \mid x \notin f[x]\}$

(3) Either directly or indirectly via (1) or (2): $\Psi = \{x = \alpha \mid x \notin \Psi]\}$. This is *Russell* !

Thus $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin \Psi)$.

Thus (1) - (3) are only consistent when $\Psi \neq f[\alpha]$. This is an instance of **Appendix C.1**.

Formula (3) is an instance of Russell's paradox. Choosing $f[\alpha] = \Psi$ in (1) assumes freedom that conflicts with $f[\alpha] = \varnothing \Leftrightarrow \alpha \notin f[\alpha]$.

In general we have liberty to choose $f[\alpha] = \varnothing$ or $f[\alpha] = A$. Since this choice defines f, we actually should write $\Psi = \Psi[f]$.

This clarifies why the choice in (2) of $f[\alpha] = \Psi[f] = \{x = \alpha \mid x \notin f[x]\}$ is tricky. If (2) is an implicit definition of f then it doesn't exist. If it exists then this $f[\alpha]$ will not be in its definition.

Appendix J.3 indicates the properties in ZFC that are being used.

The singleton is interesting also for extending a function g using one single element in the domain at the margin, i.e. marginal analysis. This is important for key **Appendix F**.

1.1.2 A table for the singleton - not yet marginal analysis

Rewrite $x = \alpha$: $\Psi = \{x \in A \mid x \notin f[x]\}$. Checking all possibilities gives Table 1. To record the kind of margin, the cells are labeled with a Δ-*case-number*.

Table 1. Test of the singleton: $\alpha \in \Psi$? **via** $\alpha \in \Psi \Leftrightarrow \alpha \notin f[\alpha]$

For all cases: $\alpha \in A$	$\Psi = \varnothing$, $\alpha \notin \Psi$	$\Psi = A$, $\alpha \in \Psi$
$f[\alpha] = \varnothing$ $\alpha \notin f[\alpha]$	**Δ1**: $\alpha \in \varnothing \Leftrightarrow \alpha \notin \varnothing$ $f[\alpha] = \Psi$, **impossible**	**Δ2**: $\alpha \in A \Leftrightarrow \alpha \notin \varnothing$ $f[\alpha] \neq \Psi$, possible
$f[\alpha] = A$ $\alpha \in f[\alpha]$	**Δ3**: $\alpha \in \varnothing \Leftrightarrow \alpha \notin A$ $f[\alpha] \neq \Psi$, possible	**Δ4**: $\alpha \in A \Leftrightarrow \alpha \notin A$ $f[\alpha] = \Psi$, **impossible**

Table 1 summarizes as follows. The arrows indicate links and no equivalences: in an equivalence we would also test the possibility that $\alpha \in \varnothing$, and that is not relevant here.

$$(f[\alpha] = \varnothing) \leftrightarrow (\alpha \in \Psi) \leftrightarrow (\alpha \notin f[\alpha]) \leftrightarrow (\alpha \notin \varnothing) \leftrightarrow (\alpha \in A) \leftrightarrow (\Psi = A)$$

$$(f[\alpha] = A) \leftrightarrow (\alpha \notin \Psi) \leftrightarrow (\alpha \in f[\alpha]) \leftrightarrow (\alpha \in A) \leftrightarrow (\Psi = \varnothing)$$

1.1.3. A table for the singleton - Paul of Venice

How does the Paul of Venice consistency condition affect the singleton ? For the singleton we can still use '&'. (To prevent infinite regress in the general case, **Appendix H.3** introduces the notation with asymmetric '&&'.) The following Cantorian set is called *Pauline*.

Using $\Phi = \{x \in A \mid x \notin f[x] \;\&\&\; x \in \Phi\}$ gives Table 2. The Δ# are the same.

Table 2. Test of the singleton: $\alpha \in \Phi$? **via** $\alpha \in \Phi \Leftrightarrow (\alpha \notin f[\alpha] \;\&\; \alpha \in \Phi)$

For all cases: $\alpha \in A$	$\Phi = \varnothing$, $\alpha \notin \Phi$	$\Phi = A$, $\alpha \in \Phi$
$f[\alpha] = \varnothing$ $\alpha \notin f[\alpha]$	$\alpha \in \varnothing \Leftrightarrow (\alpha \notin \varnothing \;\&\; \alpha \in \varnothing)$ $f[\alpha] = \Phi$, **possible**: $\alpha \notin \varnothing$	$\alpha \in A \Leftrightarrow (\alpha \notin \varnothing \;\&\; \alpha \in A)$ $f[\alpha] \neq \Phi$, possible
$f[\alpha] = A$ $\alpha \in f[\alpha]$	$\alpha \in \varnothing \Leftrightarrow (\alpha \notin A \;\&\; \alpha \in \varnothing)$ $f[\alpha] \neq \Phi$, possible	$\alpha \in A \Leftrightarrow (\alpha \notin A \;\&\; \alpha \in A)$ $f[\alpha] = \Phi$, **impossible**: $\alpha \notin A$

Now Δ1 is allowed too: a possible $f[\alpha] = \Phi$ rather than an impossible $f[\alpha] = \Psi$. The possibility is relevant when we want to construct a bijection in the infinite.

If φ runs over A and we require $f[\varphi] = \Phi$ - which is the basis for a bijection in infinity - then we find $\varphi \notin \Phi$. In the singleton the only set in $P[A]$ that has non-membership is Φ = Ø. Choosing $f[\varphi] = \Phi$ reduces our freedom for choosing f: For the singleton $f[\varphi]$ = Ø.

1.1.4. The crucial observations

The following observations are crucial:

(a) The idea that Ψ would cover all instances of $\alpha \notin f[\alpha]$ is false: it doesn't cover Δ1.

(b) We can look at the tables in horizontal or vertical direction. This reflects the schism in philosophy between *nominalism* and *realism*. (See William of Ockham.)

(c) The horizontal view gives the **realists** who take predicates as 'real': $\alpha \notin f[\alpha]$ versus $\alpha \in f[\alpha]$. They are also sequentialist: Δ1 & Δ2 versus Δ3 & Δ4.

(d) The vertical view gives the **nominalists** who regard the horizontal properties as mere stickers, and who more realistically look at $\Psi = \Phi$ = Ø versus $\Psi = \Phi = A$. They see the table in *even* versus *uneven* fashion: Δ2 & Δ4 versus Δ1 & Δ3.

(e) The nominalist reasoning is: The sets Ø and A exist, as above tables show. We are merely discussing how they are referred to. The expressions for the sets Ψ and Φ are not *defining* statements but *derivative* observations. Once the functions have been mapped out, the criteria can be used to see whether the underlying sets may get also another sticker Ψ or Φ. We are discussing *'consistent referring'* and not existence. If $\Psi \neq \Phi$ then this may arise as a property of the logical phrases but this doesn't change anything about the properties and existence of the underlying sets and the the ability to define functions.

(f) At issue is now whether ZFC has sufficient logical strength to block nonsensical situations. ZFC has a realist bend. It translates predicates into sets (their extensions). Instead it can be better to *only test* whether a predicate is useful. Merely cataloguing differently what already exists should not be confused with existence itself. The freedom of definition can be a mere illusion (see above $f[\varphi] = \Phi$) and then should not be abused to create nonsense.

(g) Lemma 3.2 (see Table 3) shows that the allocations for Ψ imply those for Φ. Thus not the other way around. Thus the cases in which Ψ exists (Δ2 and Δ3) also occur in Φ. Lemma 3.2 and the tables show that Ψ as generator of stickers is too restrictive.

(h) Corollary 3.2 is that $\Psi = \Phi$ or $\Psi \neq \Phi$. This is the vertical view, not the horizontal one.

(i) There is the tricky question (TQ) how to deal with the case when $f[\alpha] = \Psi$ is impossible and $f[\alpha] = \Phi$ is possible. Notably $f[\alpha]$ = Ø = Φ (Δ1), while this is impossible for Ψ. A realist deduces from $f[\alpha]$ = Ø that $\Psi = A$ (Δ2). A nominalist will respond that if one prefers the sticker $\Psi = \Phi$ = Ø then a small redefinition to some g gives $\Psi[g]$ = Ø (Δ3). It is just a sticker.

1.1.5. The truthtable

It is amazingly enlightening again to simply consider the truthtable, see Table 3.

Table 3. Truthtable for the tautology that $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \,\&\, p))$

Case	$(\alpha \in \Psi$	$\Leftrightarrow$	$\alpha \notin f[\alpha])$	$\Rightarrow$	$(\alpha \in \Psi$	$\Leftrightarrow$	$(\alpha \notin f[\alpha]$	&	$\alpha \in \Psi))$
Δ2	1	**1**	1	1	1	**1**	1	1	1
Δ4	1	**0**	0	1	1	**0**	0	0	1
Δ1	0	**0**	1	1	0	**1**	1	0	0
Δ3	0	**1**	0	1	0	**1**	0	0	0

Thus when α $\notin \Psi$, there still exists a case of α $\notin f$[α]. Now, isn't Ψ *supposed* to cover *all* such cases ? The conclusion is: ZFC does not cover the *intended interpretation*.

Row Δ1 indicates a case of α $\notin f$[α] that is not covered by α $\in \Psi$. Since α $\notin \Psi$ is ambiguous, i.e. allows two possibilities, a new name Φ can be used for the relaxed condition on the RHS that also catches this obscured case that α $\notin f$[α]. This new name Φ generates the two possibilities that $\Psi = \Phi$ and $\Psi \neq \Phi$.

Table 3 is an example of lemma 3.2 (Section 3). It shows (in the columns in bold) that the LHS (Table 1) only allows Δ2 or Δ3, while the RHS (Table 2) also allows Δ1. The change of tables involved, for us, a switch of the name Ψ to Φ. The name change is justified by that the LHS allows two cases and the RHS allows three cases: and one should know what one is discussing. That John is known as Charley in Amsterdam should not be too confusing. The different names Ψ to Φ should not confuse us either. A nominalist will point to the phenomenon that there are only two sets that we are discussing here: Ø and *A*.

Again, consider f[α] = Ø = Φ (Δ1). This is consistent, but impossible to see for Ψ, even though it is covered in Table 3 by the falsehood of α $\in \Psi$. In a realist mode of thought, we deduce from f[α] = Ø that Ψ = *A,* which is the only possibility on the LHS for α $\notin f$[α] that Ψ recognises (row Δ2). This is not necessarily the proper response. The problem with ZFC is that it sees only the LHS and doesn't see the RHS. We can also switch to a better axiomatic system that covers the *intended interpretation* and that blocks the paradoxical Ψ. The better system, ZFC adapted to Paul of Venice (ZFC-PV), blocks the LHS and allows only the RHS.

1.1.6. ZFC has an anomaly

Above discussion shows that ZFC doesn't fit the intended interpretation and has an anomaly.

Definition: There is an *anomaly* when the independent application of an important implied relaxation causes a contradiction.

Theorem 1.1.6: ZFC has an anomaly.

Proof: Δ1 is an inconsistency: both possible and impossible. The inconsistency however only results from independent application of the Paul of Venice condition.

In steps:

Let *A* be a set with a single element, *A* = {α}. Thus *P*[*A*] = {Ø, *A*}. Let $f : A \rightarrow P[A]$.

Let Ψ = $\{x = \alpha \mid x \notin f[x]\}$, find $\Psi \in$ *P*[*A*]. This is allowed in ZFC.

(i) Derive f[α] ≠ Ψ.

(ii) Write $\Psi = \Phi$, for convenience of Table 1 and Table 2.

Use tautology $(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \,\&\, p))$ (Table 3) and apply modus ponens to derive

$\alpha \in \Phi \Leftrightarrow (\alpha \notin f[\alpha] \;\; \& \;\alpha \in \Phi)$.

The latter allows f[α] = Φ = Ø (Δ1).

We only wrote $\Psi = \Phi$, thus f[α] = Ψ = Ø.

(iii) With dependence, (ii) is in contradiction of the earlier (i) and thus falls away.

(iv) An independent application however gives (i) f[α] ≠ Ψ and (ii) f[α] = Ψ .

Q.E.D.

Comment: Here, independent application assumes the same name. The singleton doesn't need quantifiers. Lemma 3.2 allows for different names. See the discussion there.

1.2. The problem lies in infinity

Sets *A* and *B* have 'the same size' when there is a bijection or one-to-one function between them. Cantor's Theorem holds that a set is always 'smaller' than its power set. For finite sets of arbitrary size - the potential infinite - this can be proven by *numerical succession*. (A.k.a. *mathematical induction:* but this is no *induction*, see Colignatus (2015f).) The method of numerical succession does not work for the *actual* infinite. Cantor's 'diagonal method' is supposed to work here. In terms of didactics: once there is an elegant method that also works for the infinite, then numerical succession might be dropped as a method of proof, since then there would be this elegant general proof. But does the 'diagonal method' really work ?

The reason to look into Cantor's Theorem thus lies only in infinity. The infinite is special and requires special care. The common theorem and proof, and in particular for infinite sets, uses a self-reference that strongly reminds of Russell's paradox (deconstructed in the Introduction). Hart (2015:42) recalls that this version was actually created by Russell in 1907.

ALOE 2007 showed that Russell's paradox and Cantor's Theorem were blocked by the condition of Paul of Venice. Colignatus (2012, 2013) (CCPO-PCWA) subsequently developed the notion of ***bijection by abstraction****,* that generates an abstract bijection between the natural and real numbers, without requiring more than the denumerable infinite ($\aleph \sim \Re$). This is a counterexample to Cantor's Theorem in ZFC, and thus causes questions about ZFC.

While ZFC blocks Russell's paradox (**Appendix G**), how does it deal with $x \notin f[x]$? The manipulations in 1.1.1 may seem innocuous in plain logic but are still not without problem, see the analysis at the margin (**Appendix F**). How does ZFC relate to Paul of Venice ? Getting clarity on this is the purpose of this paper.

The infinite generates an inconsistency for ZFC in the following manner. In a first reading, the next deductions may seem too simple, but there is a logical framework around them. (See Theorem 3.3 and its constructive proof in **Appendix F.3**.)

Theorem 1.2.A: ZFC is inconsistent.

Proof: Let *A* be denumerable infinite, *P*[*A*] the power set.

CCPO-PCWA shows that there is a bijection by abstraction. Let this be $f: A \to P[A]$.

ZFC allows the creation of $\Psi = \{x \in A \mid x \notin f[x]\}$.

Because of the bijection there is a ψ such that $f[\psi] = \Psi$.

The direct check on consistency gives: $(\psi \in \Psi) \Leftrightarrow (\psi \notin \Psi)$. Q.E.D.

For ZFC, the problem however is not really caused by infinity. The real question is how ZFC deals with self-reference. The steps are looked at more closely in **Appendix F**. Let us here just formulate a conjecture to test the properties of ZFC.

Conjecture 1.2.B: Let *A* be denumerable infinite, *P*[*A*] the power set. (i) For any arbitrary non-trivial $h: A \to P[A]$ there are a $f: A \to P[A]$ and a $\psi \in A$ with $f[\psi] = \Psi = \{x \in A \mid x \notin f[x]\}$. (ii) There is a contradiction. (Note that *h* does not have to be a bijection.)

Proof: Consider an ordering of *A* = {*a*[1], ...} and let be ψ = *a*[1].

(i) Let $B = A \setminus \{ \psi \}$ and $g: B \to P[A]$ as in Hilbert's Hotel $g[a[n]] = h[a[n-1]]$ for $n > 1$.

Define $f: A \to P[A]$ as:

(a) $x \in B$: $f[x] = g[x]$

(b) $x = \psi$: $f[\psi] = \Psi = \{x \in A \mid x \notin f[x]\}$.

(ii) The direct check on consistency gives: $(\psi \in \Psi) \Leftrightarrow (\psi \notin \Psi)$. Q.E.D.

Discussion:

(1) Distinguish two very different issues:

(1a) Test whether ZFC allows the definition of this self-referential f.

(1b) There is a proper definition.

Given the contradiction it is obvious that f is not well-defined. At issue in this paper however is whether ZFC allows the construction (if so, it becomes inconsistent), and how this relates to the Paul of Venice consistency condition (since we might like self-reference for fixed points, even in constructivism).

(2) **Appendix C.1** assumes an arbitrary function h and then holds that $\Psi[h]$ exists. Conjecture 1.2.B then only makes the first element available for mapping to the relevant $\Psi[f]$. The re-map may cause that $\Psi[h] \neq \Psi[f]$, of course. There is a problematic self-reference that must be dealt with. **Appendix J.3** shows the relevant ZFC conditions for Ψ (that don't seem to block problematic self-reference). One way to read Conjecture 1.2.B is as criticism: **Appendix C.1** assumes that ZFC would block above self-reference but does not show that it does.

(3) This paper considers above steps. Section 3 has a non-constructive flavour, to better show the line of reasoning. **Appendix F** is constructive with a marginal approach.

(4) Some constructive methods still allow for fixed points (Brouwer). The proposition *"This proposition is true"* is self-referential without much problem.

Hodges (1998) also reviewed criticism w.r.t. Cantor's Theorem but confirmed in a personal communication August 10 2012 that he allows me to quote from (though w.r.t. a different paper): "You are coming at Cantor's proof from a constructivist point of view. That's something that I didn't consider in my paper, because all of the critics that I was reviewing there seemed to be attacking Cantor from the point of view of classical mathematics; I don't think they knew about constructivist approaches. Since then some other people have written to me with constructivist criticisms of Cantor. There is not much I can say in general about this kind of approach, because constructivist mathematicians don't always agree with each other about what is constructivist and what isn't."

(5) The point may also be rephrased in this manner: the logical construction $x \notin f[x]$ and only a single problematic element, in badly understood self-reference, should not be abused to draw conclusions on the infinite.

2. Review of the common proof of Cantor's Theorem

The following essentially copies the discussion in ALOE p129 & 239. See **Appendix A** for accurate reference. Section 2.1 reproduces the common proof and Section 2.2 reproduces its refutation using the Paul of Venice consistency condition.

2.1. Cantor's Theorem on the bijection (common)

Cantor's Theorem (for the power set, Russell's version, with the bijection): Let A be a set. There is no bijective function $f: A \to P[A]$.

Proof: Regard an arbitrary set A. Let $f: A \to P[A]$ be the hypothetical bijection. Let $\Psi = \{x \in A \mid x \notin f[x]\}$. Clearly Ψ is a subset of A and thus there is a $\psi = f^{-1}[\Psi]$ so that $f[\psi] = \Psi$. The question now arises whether $\psi \in \Psi$ itself. We find that $\psi \in \Psi \Leftrightarrow \psi \notin f[\psi] \Leftrightarrow \psi \notin \Psi$ which is a contradiction. Ergo, there is no such f. Q.E.D.

Comments:

(1) The bijection is sufficient and the surjection is necessary, see **Appendix B** point 1 and **Appendix J.3** for the surjection version. **Appendix C** has an even weaker version.

(2) It is *not shown* in above common proof that the creation of Ψ is acceptable in ZFC. From the contradiction derived above, the proper conclusion is not that Cantor's Theorem is proven, but only that it is proven in ZFC. Either Cantor's Theorem is true *or* ZFC doesn't yet provide for well-defined sets. See **Appendix J.3** in [*NB*] how ZFC is part of the proof. (Those formal conditions are also to be applied to the nutshell 1.1 and **Appendix F**.)

We cannot base mathematical conclusions upon an improper way of expressing our statements. E.g. if our form-conventions allow a substitution of "a" and "oo" so that our conclusions about "man" and "moon" are the same, then we can create art, but not necessarily something that we would want to teach as serious mathematics. Discovering what a good notation for well-defined sets is, has been studied even before the discovery of Russell's paradox.

(3) The ZFC *Axiom of Separation* blocks Russell's set, but at the price of a universal set, see **Appendix G.** It doesn't block Cantor's Ψ yet.

The subsequent discussion intends to show that the common proof cannot be accepted.

2.2. Rejection of this proof (ALOE)

We might hold that above Ψ is badly defined since it is self-contradictory under the hypothesis of a bijection or surjection. A badly defined 'something' may just be a weird expression and need not represent a true set. A test on this line of reasoning is to insert a small consistency condition, giving us $\Phi = \{x \in A \mid x \notin f[x] \;\&\&\; x \in \Phi\}$. See the Introduction and **Appendix H.3** for the notation on '&&'. Reading it as '&' still gives an idea of the argument. The surjection gives that there is a φ such that $f[\varphi] = \Phi$. Now we get:

$$\varphi \in \Phi \Leftrightarrow (\varphi \notin f[\varphi] \;\&\; \varphi \in \Phi)) \Leftrightarrow (\varphi \notin \Phi \;\&\; \varphi \in \Phi) \Leftrightarrow \textit{ falsum.}$$

We find $\varphi \notin \Phi$ without contradiction. This closes the argument against the proof.

Comments:

(1) ALOE:239 holds, for infinite sets: Puristically speaking, the Ψ defined in 2.1 differs lexically from the Φ defined here, with the first expression being nonsensical and the present one consistent. Ψ is part of a lexical description but does not meaningfully refer to a set. Using this, define $\Phi^* = \Phi \cup \{\varphi\}$ and we can express consistently that $\varphi \in \Phi^*$. So the 'proof' in 2.1 can be seen as using a confused mixture of Φ and Φ^*. (The confusion affects infinity and is obscured for finite sets. For example, the $x \notin f[x]$ rule works in Section 1.1 (nutshell).)

(2) In Section 1.1 (nutshell) there is no surjection and thus the above does not quite compare. With only one element, $\alpha \notin \Phi$ implies $\Phi = \varnothing$ and $\Phi^* = A$. We however have freedom to take $f[\alpha] = \Phi$ or $f[\alpha] \neq \Phi$. Instead we have always $f[\alpha] \neq \Psi$, so it is no useful building block.

(3) The additional condition in Φ only enhances consistency, but Ψ and Φ still have a different effect. Explain this.

(4) This paper uses *'well-defined'* rather than *'well-formed'.* The context of this paper is ALOE that presents three-valued logic, to the effect that logic allows to determine whether an expression reduces to nonsense. This allows leisure on form, so that the Russell set can be said to be of acceptable form - so that it is meaningful for deductive steps - but it turns out to be nonsense and thus not-well-defined. In the same way the Cantorian set would not be rejected merely because of form, i.e. in three-valued logic. In two-valued logic, both Russell and Cantor do not satisfy criteria on well-formed-ness.

(5) In writing CCPO-PCWA in 2012 I considered using the more general &&-construction, but still preferred $\Phi = \{(x \in A)$ & $(x \neq f^{-1}[\Phi]) \mid x \notin f[x]\}$ to avoid the infinite regress. Now, looking at the challenge to ZFC, it seems better not to linger in *ad hoc* solutions but to emphasize the general idea. If one feels uncomfortable with the &&-switch then it is useful to know that there is this *ad hoc* definition for Φ. See **Appendix C** for another ad hoc form.

(6) ALOE deals with logic and inference and thus keeps some distance from number theory and issues of the infinite. Historically, logic developed parallel to geometry and theories of the infinite (Zeno's paradoxes). Aristotle's syllogisms with *none, some* and *all* helped to discuss the infinite. Yet, to develop logic and inference proper, it appeared that ALOE could skip the tricky bits of number theory, non-Euclidean geometry, the development of limits, and Cantor's development of the transfinite. Though it is close to impossible to discuss logic without mentioning the subject matter that logic is applied to, ALOE originally kept and keeps some distance from those subjects themselves. But, if logic uses the notion of *all,* it seems fair to ask whether there are limitations to the use of this *all*. Thus it is explained why ALOE 2007 said something about Cantor's Theorem and why this present paper came about. Later CCPO-PCWA developed the notion of *bijection by abstraction* which is a counterexample to the idea that there would be no bijection between infinite sets.

(7) It may be added here that I did not enjoy the idea of rejecting the proof for Cantor's Theorem, that I originally accepted in 1981-2006, and that has such an acceptance in mathematics since Hilbert, even in constructivism. But, when one studies logic, one learns to respect necessity.

3. The challenge to ZFC

3.1. The Axiom of Separation in ZFC

The proof in 2.1 relies on the following axiom.

Definition of the *Axiom of Separation* (Coplakova et al. (2011:145), inserting here a by-line on freedom): If A is a set and $\gamma[x]$ is a formula with variable x, then there exists a set B that consists of the elements of A that satisfy $\gamma[x]$, while B is not free in $\gamma[x]$:

$$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \iff ((x \in A)\ \&\ \gamma[x])) \qquad \text{(SEP)}$$

The discussion will now focus on the condition "B is not free in $\gamma[x]$". This will allow us to pose the problem more acutely.

The *Pauline* reading of SEP is (supported by Lemma 3.2 below): the definition of Φ in 2.2 with the Paul of Venice consistency condition uses $\gamma'[x] = (\gamma[x]$ && $(x \in \Phi))$, in which $B = \Phi$ is not free since it is bound by the existential quantifier $(\exists B)$. Thus the formation of Φ in 2.2 is allowed in ZFC.

Appendix E discusses both this *Pauline* interpretation and the *Cantorian* interpretation of ZFC that rejects this reading of the axiom. There exist not only different forms of ZFC but also different readings. Lemma 3.2 however proves that the Cantorian reading implies the possibility of the Pauline reading.

3.2. Pauline and Cantorian readings, with lemma and corollary

Definition: The **Pauline** reading of Axiom of Separation takes the axiomatic formula, $(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \iff ((x \in A)\ \&\ \gamma[x]))$, as basic, so that $\gamma'[x] = (\gamma[x]$ && $(x \in \Phi))$ is allowed since $B = \Phi$ is bound by the existential quantifier. The expression $\gamma'[x]$ is created within the formula and can be taken out for inspection, but one should be careful about conclusions once it has been taken out. The explanatory text in the axioma around the formula is about the whole formula, and not necessarily about expressions once they have been taken out.

Definition: The **Cantorian** reading is that the expression $\gamma[x]$ is created outside of the axiomatic formula, is judged on its own properties (though unstated *why* and *how*), and only afterwards substituted into the formula. It holds: $\gamma'[x]$ = ($\gamma[x]$ && ($x \in \Phi$)) is not allowed bedause $B = \Phi$ would be a free variable (and not a constant).

See **Appendix E** for more angles on this.

The following gives a key relation between these readings.

Lemma 3.2: For "($x \in B$)": (The Cantorian reading) $\Rightarrow$ (A possible Pauline reading).

Proof: There is the tautology in propositional logic (see Table 3):

$$(p \Leftrightarrow q) \Rightarrow (p \Leftrightarrow (q \,\&\, p))$$

With $p = (x \in B)$ and $q = (x \in A \,\&\, \gamma[x])$ we get:

$$\big((x \in B) \Leftrightarrow (x \in A \,\&\, \gamma[x])\big) \Rightarrow \big((x \in B) \Leftrightarrow (x \in A \,\&\, \gamma[x] \,\&\, (x \in B))\big)$$

Take the Cantorian reading that $(\forall A)\ (\exists B)\ (\forall x)\ ((x \in B) \Leftrightarrow (x \in A \,\&\, \gamma[x]))$ and eliminate the existential quantifier by some constant set, say $C = C[A, \gamma]$:

$$(\forall A)\ (\forall x)\ ((x \in C) \Leftrightarrow (x \in A \,\&\, \gamma[x])).$$

Given that the tautology holds for all p and q, then apply Modus Ponens.

$$(\forall A)\ (\forall x)\ \big((x \in C) \Leftrightarrow (x \in A \,\&\, \gamma[x] \,\&\, (x \in C))\big).$$

Then abstract to an existential quantifier again. There are two ways to do this. The first way is set-preserving, in which the constant C is kept on the RHS.

$$(\forall A)\ (\exists B)\ (\forall x)\ \big((x \in B) \Leftrightarrow (x \in A \,\&\, \gamma[x] \,\&\, (x \in C))\big).$$

The other abstraction considers the whole expression and gives the Pauline reading:

$$(\forall A)\ (\exists B)\ (\forall x)\ \big((x \in B) \Leftrightarrow (x \in A \,\&\, \gamma[x] \,\&\, (x \in B))\big).$$

Propositional logic also accomodates the exception switch '&&'. (**Appendix H.3**.)

Q.E.D.

The lemma shows that *there is no necessity* to reject the Pauline interpretation. The proof also generates the following consequence:

Corollary 3.2: The set created by the Pauline reading is not necessarily equal to the set created by the Cantorian reading.

Proof: The proof of lemma 3.2 used the method of eliminating the existential quantifier by subsitution of a constant set $C = C[A, \gamma]$. Thus the constant set C that satisfies the Cantorian reading also satisfies a Pauline reading. This conclusion is not affected by the later step of abstraction to ($\exists B$). However, abstraction over the whole expression may introduce new solutions. Q.E.D.

Comments:

(1) The above holds for any set and expression, not just the paradoxical ones.

(2) An example is given in Section 1.1, with Table 1 for Ψ and Table 2 for Φ. While we have the option to select $\Psi = \Phi$ there is also the option to differ. While it was not clear there under what circumstances an independent application of Φ was allowed, see theorem 1.1.6, this clarity has now been given. If ZFC goes only from existential abstraction to sets, then the Cantorian interpretation would block the Pauline interpretation. Normally, however, one would allow the flexible abstraction from a proposition.

(3) It might be useful - e.g. for the expansion to '&&' - to write out some relations for the paradoxical Φ and Ψ. Given the deduction in 3.2 we find for 2.1:

$$\big((x \in \Psi) \Leftrightarrow (x \in A \,\&\, (x \notin f[x]))\big) \;\Rightarrow\; \big((x \in \Psi) \Leftrightarrow (x \in A \,\&\, (x \notin f[x]) \,\&\, (x \in \Psi))\big)$$

Given that the antecedens holds (∀*A*) (∀*x*), also the consequence holds (∀*A*) (∀*x*).

For 2.2 there is the consequence term only:

$$(\forall A)\ (\forall x)\ \big((x \in \Phi) \Leftrightarrow (x \in A \,\&\, (x \notin f[x]) \,\&\, (x \in \Phi))\big)$$

Thus we have the same expression for both now:

For $K = \Phi, \Psi$: $(\forall A)\ (\forall x)\ \big((x \in K) \Leftrightarrow (x \in A \,\&\, (x \notin f[x]) \,\&\, (x \in K))\big)$

With only the latter information it is doubtful whether ZFC is strong enough to derive whether these in ZFC are just different names for the same set $\Phi = \Psi$ or not. However, lemma 3.2 and corollary 3.2 make it certain, via another route, that there is ambiguity indeed.

The following question then becomes more acute.

3.3. What is the difference between Ψ in 2.1 and Φ in 2.2 ?

The deduction in Section 2 poses a challenge to ZFC. Sets R and S in the Introduction were in naive set theory, so it has relatively little meaning - for now - to ask about the difference between R and S. However, Ψ in 2.1 and Φ in 2.2 belong to ZFC - see 3.1 & 3.2 and **Appendix E** - and thus the question is (more) meaningful. Users of ZFC will have a hard time trying to clarify:

(a) that the consistency condition should have no effect,

(b) but actually can have an effect.

I have not really pursued this question further since I have no vested interest in ZFC. I leave it to users of ZFC to clarify this issue.

The following are useful clarifications based upon the little that I could do.

(A) Co-author of Coplakova et al. (2011) is Edixhoven (Leiden) who agrees with (a). I have asked him to explain (b), and to describe the relation between Ψ in 2.1 and Φ in 2.2. Since November 2014, see Colignatus (2014ab), I have not received a response even though the question was clear and articulate. Hart (Delft) simply rejects that Φ in 2.2 belongs to ZFC, but without answering to the question, see also the ramifications in **Appendix E**. Overall, see Colignatus (2015e).

(B) My solution of this issue is that Ψ in 2.1 is badly defined and that Φ in 2.2 is well-defined. Accepting that Ψ is ill-defined has the effect of the collapse of the standard proof to Cantor's theorem (in the version of Russell for the power set). I am interested in an argument to the contrary but haven't seen it yet. Note that Ψ seems to work for finite sets, see for example the nutshell in Section 1.1. However, we there identified that Ψ does not fit the intended interpretation and causes a serious anomaly.

(C) The lemma and corollary in 3.2 are new to the June 4 2015 version of this paper. The question about Ψ in 2.1 and Φ in 2.2 can be posed a bit more acutely with those.

Lemma 3.3: The surjective f also causes that $\Phi \neq \Psi$.

Proof: There is the *Axiom of Extensionality*, see Coplakova et al. (2011:145):

$$(A = B) \Leftrightarrow ((\forall x)\ (x \in A \Leftrightarrow x \in B)) \qquad \text{(EXT)}$$

Given that we found $\varphi \notin f[\varphi]$ in Section 2.2 we conclude that $\varphi \notin \Phi$ and $\varphi \in \Psi$, so that $\Phi \neq \Psi$. Q.E.D.

Comments:

(1) The lemma's reference to informal section 2.2 is repaired in theorem 3.3 below, for the formal existence of a consistent Pauline Cantorian set. The lemma only serves the connection to corollary 3.2, of possibly finding new solutions. There would be an inconsistency if also $\Phi = \Psi$. Section 2.1 and **Appendix J.3** already derived a contradiction, assuming a surjection, for Ψ. In that case anything can be derived. Importantly, however, the result derived in lemma 3.3 is different. The deduction does not quite rely on problematic self-reference but on corollary 3.2. The deduction on $\Phi \neq \Psi$ relies on a consistent φ too. This suggests that our problems are caused by Ψ, and by that ZFC allows it. The sets should be the same since the condition enhances consistency.

(2) Rejection of the surjection for infinity is no good option since there is the *bijection by abstraction*, see Colignatus (2012, 2013). Rather: ZFC needs amendment.

(3) Of course it is still possible to reject the surjection for infinity. Thus there is reason to see whether the assumption of a surjection can be weakened. This consideration caused the marginal analysis in **Appendix F**. This gave the following result.

Lemma 3.3 refers to Section 2.2 that is somewhat informal. Let us give formal expression of $\varphi \notin \Phi$. Let us revisit Conjecture 1.2.B and see how it can be adapted with what we learned about the Paul of Venice consistency condition. The finite case is not relevant, since there we can avoid the paradoxical construct via the proof via numerical succession.

Theorem 3.3 (existence): Let A be denumerable infinite, $P[A]$ the power set. (i) For any arbitrary non-trivial h: $A \to P[A]$ there are a f: $A \to P[A]$ and a $\varphi \in A$ with $f[\varphi] = \Phi = \{x \in A \mid x \notin f[x] \;\&\&\; x \in \Phi\} \in P[A]$. (ii) The direct test has $\varphi \notin \Phi$ without direct contradiction.

Proof: Consider an ordering of $A = \{a[1], ...\}$ and let be φ the first element $a[1]$.

(i) Let $B = A \setminus \{ \varphi \}$ and g: $B \to P[A]$ as in Hilbert's Hotel $g[a[n]] = h[a[n-1]]$ for $n > 1$.

Define f: $A \to P[A]$ as:

(a) $x \in B$: $f[x] = g[x]$

(b) $x = \varphi$: $f[\varphi] = \Phi = \{x \in A \mid x \notin f[x] \;\&\&\; x \in \Phi\}$

(ii) The direct test on consistency is:

$$(\varphi \in \Phi) \Leftrightarrow (\varphi \notin f[\varphi] \;\&\; \varphi \in \Phi)$$

Whence it follows without direct contradiction that $\varphi \notin \Phi$. Q.E.D.

Comments:

(1) **Appendix F.3** has a more constructive proof of (i). There: $f[\varphi]$ is set to $\Phi^*[B] = \{x \in B \mid x \notin g[x] \;\&\&\; (x \in \Phi^*[B])\}$. Note that g's domain is B and its range is $P[A]$. Then it is proven that $\Phi = \Phi^*[B]$. The steps are not difficult but only distractive for an overview. The core is the restriction of self-reference. It is advisable to consider the whole appendix.

(2) Theorem 3.3 suits ZFC, see **Appendix J.3** (part (iii) [*NB*]): (a) Note that g can be regarded as a subset of B x $P[A]$. Then g exists because of the Axiom of Pairing. (b) This Φ exists because of the Axiom of Separation applied to the part without '&&', and then applying Lemma 3.2. A non-constructivist cannot object to this definition since, for each f: $A \to P[A]$ there is a $\Phi[A] = \Phi = \{x \in A \mid x \notin f[x] \;\&\&\; x \in \Phi\}$. For example, the theorem in **Appendix C.1** also uses the existence of the associated set Ψ in the Cantorian reading *for any function.*

(3) A line of defence for ZFC would be to hold that above definition of f would not be allowed. In the Cantorian reading, (b) would actually use: $\Psi = \{x \in A \mid x \notin f[x]\}$. The consistency check gives $(\varphi \in \Psi) \Leftrightarrow (\varphi \notin \Psi)$. Then there would be no difference between

Conjecture 1.2.B and Theorem 3.3. Thus, a Cantorian might hold that the definition of the function would not be appropriate. However, this confuses the issue. The issue w.r.t. this theorem and its proof is not a general notion of well-defined-ness but whether the axioms of ZFC allow definition (b) or not. Thus distinguish:

(3a) The axioms of ZFC allow the definition in step (b) of Theorem 3.3 (if not non-constructively as done here, then at least constructively as in **Appendix F.3**).

(3b) Theorem 3.3 has been written such that it also applies in ZFC-PV (below).

(3c) Theorem 3.3 has been written to show that ZFC is inconsistent *not because of step (b)* but because of corollary 3.3.

Corollary 3.3: ZFC is inconsistent.

Proof: (i) According to Theorem 1.1.6 ZFC has an anomaly.

(ii) Lemma 3.2: the Cantorian reading generates the possibility of a Pauline reading.

(iii) The latter allows an independent application of the Pauline Axiom of Separation.

(iv) The independent application turns the anomaly into a contradiction.

For the singleton we can derive $\alpha \in \Psi \Leftrightarrow (\alpha \notin f[\alpha] \ \& \ \alpha \in \Psi)$ dependently from the truthtable and we can derive $\alpha \in \Phi \Leftrightarrow (\alpha \notin f[\alpha] \ \& \ \alpha \in \Phi)$ from the Pauline Axiom of Separation. The two formulas are however the same in their self-referential structure, and thus we are dealing with mere names. Thus we must have $\Psi = \Phi$. The independent formula allows the independent conclusion of the possibility of $f[\alpha] = \Phi = \varnothing$ (Δ1). Thus also $f[\alpha] = \Psi = \varnothing$. This however conflicts with the earlier finding that $f[\alpha] \neq \Psi$.

Q.E.D.

Comments:

(1) This is not the major insight in this paper. It are the considerations that count. No inconsistency is likely to cure the infatuation with the transfinites.

(2) Table 1 and Table 2 can stand each by themselves, and only create a contradiction when it is shown that $\Psi = \Phi$.

(3) In the Cantorian reading: $\Psi = \Phi$ for all cases, and Table 2 does not exist. Theorem 3.3 generates $\Phi \neq \Psi$ for new cases but in principle with independent Table 2. There need not be a proper contradiction when one cannot show how the sets and/or names are linked. The singleton shows the conflict in concreteness, but only if an independent path has been given.

3.4. Amendments to the Axiom of Separation in ZFC

To meet the challenge in 3.3 we would require the PV-condition in general.

Possibility 3.4.1: Amendment by Paul of Venice to the Axiom of Separation:

$$(\forall A)\ (\exists B)\ (\forall x)\ ((x \in B) \Leftrightarrow ((x \in A)\ \&\ \gamma[x]\ \&\&\ (x \in B))) \qquad \text{(SEP-PV)}$$

Lemma 3.2 proves that SEP implies SEP-PV, but there arises a new system when SEP is replaced by SEP-PV. In this case, 2.1 is no longer possible, the proof for Cantor's theorem collapses, and question 3.3 disappears since Ψ becomes ill-formed and nonsensical. My suggestion is to call this the *neat* solution, and use the abbreviation **ZFC-PV**.

Another possibility is to move from ZFC closer to naive set theory, discard the axiom of separation, and adopt an axiom that allows greater freedom to create sets from formulas.

Possibility 3.4.2: Discard the separation axiom and have extensionality of formula's, a.k.a. comprehension:

$$(\forall\varphi)\ (\exists B)\ (\forall x)\ ((x \in B)\ \Leftrightarrow\ (\gamma[x]\ \&\&\ (x \in B))) \qquad \text{(EFC-PV)}$$

This axiom protects against Russell's paradox and destroys the standard proof of Cantor's theorem. This resulting system might be called ZFC-S+PV.

The Axiom of Regularity (REG) forbids that sets are member of themselves. Instead, it is useful to be able to speak about the set of all sets. Though it is another discussion, my suggestion is to drop this axiom too, then to call this the 'basic' solution, and use the abbreviation BST (basic set theory), thus **BST** = ZFC-S+PV-R. I would also propose a rule that the PV-condition could be dropped in particular applications if it could be shown to be superfluous. However, for paradoxical γ[*x*] it would not be superfluous.

I am not aware of a contradiction yet. I have not looked intensively for such a contradiction, since my presumption is that others are better versed in (axiomatic) set theory and that the problem only is that those authors aren't aware of the potential relevance of the consistency condition by Paul of Venice. A question for historians is: Zermelo (1871-1953) and Fraenkel (1891-1965) might have embraced the Paul of Venice's condition if they had been aware of it.

4. Conclusion

We can restate some earlier conclusions and supplement these from looking at ZFC:

1. The common cq. standard proof for Cantor's Theorem on any set is based upon a badly defined and problematic self-referential construct. The proof and the variants evaporate once the problematic construct is outlawed and a sound construct is used.

2. The theorem is proven for finite sets by means of *numerical succession* ('mathematical induction') but is still unproven for (vaguely defined) infinite sets: that is, this author is not aware of other proofs. We would better speak about *'Cantor's Impression'* or *'Cantor's Supposed Theorem'*. It is not quite a conjecture since Cantor might not have done such a conjecture (without proof) if he would have known about above refutation.

3. It becomes feasible to speak again about the *'set of all sets'.* This has the advantage that we do not need to distinguish (i) sets versus classes, (ii) *all* versus *any*.

4. The transfinites that are defined by using 'Cantor's Theorem' evaporate with it.

5. The distinction w.r.t. the natural and the real numbers now rests (only) upon the specific interval (Cantor 1874) or specific diagonal argument (Cantor 1890/91) - that differ from the common cq. standard proof. See CCPO-PCWA for the conclusion that these original proofs by Cantor for the natural and real numbers evaporate too, specifically for a convenient *'degree of constructivism'*. See also **Appendix B** on Hart (2015) on these original proofs.

6. Users of ZFC who not accept these points should give an answer to Section 3.3, and clarify why they accept 2.1 and not 2.2 that has a better definition of a well-defined set. Theorem 3.3 and F.3 would require deproofs. If one holds that ZFC is consistent, then one accepts the construction of a 'proof' for 'Cantor's Impression' that generates the transfinites, which makes one wonder what this system is a model for. We can agree with Cantor that the essence of mathematics lies in its freedom, but the freedom to create nonsense somehow would no longer be mathematics proper. Useful alternatives are in ZFC-PV or BST (Section 3.4).

7. The prime importance of this discussion lies in education, see Colignatus (2011). Mathematics education should respect that education itself is an empirical issue. In teaching, there is the logic that students can grasp and the idea to challenge them with more; and there is the wish for good history and and still not burden students with the confusions of the past. My suggestion is that Cantor's transfinites can hardly be grasped, are not challenging, and are burdening rather than enlightening. CCPO-PCWA clarifies that highschool education and matricola for non-math majors could be served well with a theory of the infinite that consistently develops both the natural and real numbers, without requiring more than the denumerable infinite ($\aleph \sim \Re$), using the notion of *bijection by abstraction*. See Colignatus (2015af) for a discussion on abstraction. A major problem would be when mathematics teachers think that 'Cantor's Theorem' and its transfinites would be a great result and that they would feel frustrated when they would not be in a position to explain it properly - while such frustration would only be based upon a mirage and still show up in behaviour.

It was Cantor himself who emphasized the freedom in mathematics, but that freedom is limited when alternatives are not mentioned. Even a university course like Coplakova et al. (2011) currently presents matricola students only with 'Cantor's Theorem' without mentioning the alternative analysis in ALOE, potentially seducing some students to waste their lives on transfinites.

Acknowledgements

Let me repeat my gratitude stated in the other paper CCPO-PCWA. For this paper, I thank Richard Gill (Leiden) for various discussions, and Klaas Pieter Hart (Delft) over 2011-2015 and Bas Edixhoven (Leiden) in 2014 for some comments and for causing me to look closer at ZFC. Hart and Edixhoven apparently have missed the full argument of this paper and take the Cantorian position in Appendix E. I am sorry to have to report a breach in scientific integrity, see Colignatus (2015e). All errors remain mine.

Appendix A: Versions of ALOE

The following comments are relevant for accurate reference.

(1) Colignatus (1981, 2007, 2011) (**ALOE**) existed first unpublished in 1981 as *In memoriam Philetas of Cos,* then in 2007 rebaptised and self-published. It was both retyped and programmed in the computer-algebra environment of *Mathematica* to allow ease of use of three-valued logic. In 2011 it was marginally adapted with a new version of *Mathematica.* At that moment it could also refer to a new rejection of Cantor's particular argument for the natural and real numbers, using the notion of *bijection by abstraction* - in 2011 still called *bijection in the limit* but now developed in Colignatus (2012, 2013).

(2) Gill (2008) reviewed the 1st edition of ALOE of 2007. That edition refers to Cantor's standard set-theoretic argument and rejects it, as in the above. ALOE refers to Wallace (2003) as the book that caused me to look into the issue again. Wallace's book is critically reviewed by Harris (2004). It will be useful to mention that ALOE does not rely on Wallace's book but indeed only mentions it as a source of inspiration to look into the issue again.

(3) Gill (2008) did not review the 2nd edition of ALOE of 2011. That edition also refers to Cantor's original argument on the natural and real numbers in particular. That edition of ALOE mentions the suggestion that $\aleph \sim \Re$. The discussion itself is not in ALOE but is now in Colignatus (2012, 2013) (CCPO-PCWA), using the notion of *bijection by abstraction.*

(4) A visit to a restaurant and subsequent e-mail exchange led to the memo Colignatus (2014a), and the inspiration to write this present article on the challenge to ZFC. Edixhoven also refers to Coplakova et al. (2011), theorem I.4.9, pp. 18-19, that gives the standard theorem and proof, also reproduced and challenged in **Appendix J.3**.

(5) ALOE is a book on logic and not a book on set theory. It presents the standard notions of naive set theory (membership, intersection, union) and the standard axioms for first order predicate logic that of course are relevant for set theory. But I have always felt that discussing *axiomatic* set theory (with ZFC) was beyond the scope of the book and my actual interest and developed expertise. This present paper is in my sentiment rather exploratory, by discussing axiomatic set theory in Section 3 and actually presenting some possible alternatives.

Appendix B: The traditional view, in Hart (2015)

This article on Paul of Venice and ZFC was basically written in November 2014 and has been slightly updated with some clarifications following some comments from others.

It so happens that Hart (2015) recently reviews the issue too, giving the traditional view after Hilbert. He starts with Cantor's orginal arguments on non-denumerability and the diagonal. For foreign readers a discussion of this Dutch article will be difficult to follow, but let me still give my comments now that I am dealing with the subject.

Hart replied to these comments and the text below has been adapted a bit for further clarity. Hart also looked at Appendix C and some earlier arguments, and this is discussed in point (6) below and Colignatus (2015de).

(1) Hart (2015:43) holds correctly that a bijection doesn't have to be used, but only the surjection (i.e. in the mode of thought that the proof would be valid). He however holds incorrectly that the common short proof with the bijection would rely on a 'spurious

contradiction' - referring here to Gillman 1987. This would be incorrect if we rely on the common meaning of 'spurious': (a) there is a real contradiction: the assumption of the bijection implies the assumption of the surjection, which causes the contradiction, (b) the context of discussion is infinity, for which we use isomorphisms, and thus injections, and in that case the properties of surjection and bijection are equivalent: and then the shortness of the proof must be appreciated. Indeed Hart (2015:41) explains that Cantor himself also used 'eindeutig' (column 1) and injection (column 3 - below the photograph of 'Georde Cantor'). Overall: the open 'reductio ad absurdum' form and the 'direct' form that Hart suggests are equivalent, and the reference to 'spurious contradiction' is incorrect. PM. Hart (2015:42 first column) suggests that the power set version of Cantor's Theorem was given by Bertrand Russell 1907, also using a 'supposition' and basically using a bijection.

(2) On page 42, third column, Hart agrees that Cantor's distinction between proper sets and improper sets ('classes'), or the distinction between *all* and *any*, still is used informally. Thus mathematics uses both a formal ZFC and an informal naive set system. It is useful to see this confirmed. It remains curious that Hart as a mathematician is happy to live with this incongruity. Hart then discusses the axiom of separation, but it gives a wrong impression, because its main weaknesses and alternatives are not discussed. One may write a book or syllabus on 'set theory' but if this only discusses ZFC and its ZFC-sets then this is a biased presentation.

(3) On page 43 Hart mentions the argument concerning $\aleph \sim \Re$ that uses decimal expressions. He states that this particular form does not occur in Cantor's work. This is not quite true. Cantor's proof of 1890/91 uses a binary representation - see Hart (2015:41) - which, for these purposes, is equivalent to using decimals. Hart traces the proof with decimals to Young & Young in 1906, who explicitly refer to Cantor 1890/91, and who explicitly call it his 'second proof'. Thus mathematicians were aware already in 1906 that binaries and decimals are equivalent here. It is curious that Hart in 2015 does not express that awareness. His review of what Cantor originally did thus is biased. (3a) For this proof structure, binaries and decimals are equivalent. (3b) The binaries are mathematically more elegant, since changing an element has only one alternative. The decimals are didactically more useful, since students are more used to decimal expression of the real numbers - which is the representation of the continuum. It would be improper to criticize the decimal form of the proof for being didactic. (3c) It is correct that Cantor claimed that the proof structure was "independent from looking at irrationals" but the proof does *implicitly use* irrationals.

(4) We may wonder why Hart's paper might be biased. It is a good hypothesis that he wants to emphasize that some authors still have questions about Cantor's argument.

(4a) On page 43 Hart refers to Wilfrid Hodges (1998) who discusses "hopeless papers". Hart does not mention Hodges's email to me that I cited in CCPO-PCWA that I informed him about.

(4b) Hart accuses those "hopeless papers" of that they don't check what Cantor did himself originally. This is an improper accusation since such authors discuss a particular argument, that so happens to go by the name of "Cantor's diagonal argument", while it is not always at issue what Cantor himself did - who indeed wrote before ZFC.

(4c) Just to be sure: My own first contact with Hart - in 2011 - was about Cantor 1874. CCPO-PCWA wanted to know whether there were more proofs, and thus also looked at Cantor 1874, and found it inadequate. Hart's page 40 with Cantor 1874 finds a refutation in the appendix of CCPO-PCWA - but he knows about the latter and does not refer to that refutation.

(4d) Hart suggests that the proof with decimals causes most "hopeless papers", but that this proof can be "thrown in the trash can", because Cantor's original proof from 1874 and his second and more general proof of 1890/91 would be more attractive.

(4d1) This is improper, since it evades the question whether the argument with the decimals is a good deduction or not. Mathematics should not ditch arguments because they cause questions but should answer the questions.

(4d2) It also is an inconsistent argument, see (3): the proofs are equivalent, differ only in binaries versus decimals. Thus Hart suggests to throw Cantor's own proof into the trash can - but doesn't do so.

(4d3) In a personal communication, Hart acknowledges that the binary and decimal proofs are equivalent (without drawing the inference on (3)) but that he only expressed his preference for the aesthetics of the binary form. He is free to state his preference, but the decimal form is the most didactic one, and thus the form *cannot* be ditched.

(4e) Hart holds that such "hopeless papers" and/or internet discussions quickly replace mathematics by ad hominem fallacies. An ad hominem would be: "You have no mathematics degree and hence I will not listen to your arguments." Obviously Hart presents himself as not falling into that trap. My problem however is that he applies an 'ad gentem fallacy', by reducing critique on Cantor's Theorem into "hopeless papers" and/or internet ad hominem fallacies. This is a racket or ballyhoo to induce a sentiment amongst his readership to no longer look at critique on Cantor's Theorem, and to join in the putting down of such critics. We thus may understand why Hart (2015) is a biased presentation, unworthy of mathematics that wants to claim to be scientific.

(5) Hart (2015:42, last column): "The best known impossibility theorems in mathematical logic all use a version of Cantor's idea to flip all elements on a diagonal" - and then he refers to Gödel's first incompleteness theorem. This is not quite true. Gödel's theorem uses self-reference. This property was already known in antiquity in the Liar Paradox. Gödel's use of number-coding has historical explanations, like the trust in arithmetic in a period of a foundations crisis in mathematical logic. Gödel's numerical listing is not crucial to the argument. The influence of Cantor should not be made greater than it is. Hart could have known about this, reading both ALOE and Gill (2008) in the same Dutch journal for mathematics, with my refutation of Gödels two theorems.

(6) Hart does not refer to ALOE or CCPO-PCWA that he knows about, thus misinforms his readership. He reproduces Cantor's 'proofs' of 1874 and 1890/91 without mentioning their refutations. He states the common misconceptions and adds some new ones.

In a personal communication, Hart now has looked at my criticism. It leads too far to look into this here. Colignatus (2015b) reviews the email exchange with K.P. Hart (TU Delft) in 2011 - May 2015. Colignatus (2015c) reviews Hart's response on **Appendix B** as above. The reader can check that the criticism still stands. Colignatus (2015d) reviews Hart's new combined criticism of May 18 2015 on that version of the paper, Appendix C, and earlier refutations in CCPO-PCWA - which should cover point (6). There is now also the issue on scientific integrity, see Colignatus (2015e).

Appendix C: Deproving the weakest theorem

In a personal communication in 2012 K.P. Hart (TU Delft) presented this theorem and proof. If no one else presented this theorem earlier it may be called the Cantor-Hart Theorem but for now I label it for what it does. In 2012 my reply was asking Hart whether he understood the refutation in ALOE of the proof of Cantor's Theorem that uses the power set and bijection, but I did not receive a response on that. If he had understood, he could have given below refutation of the proof himself.

C.1. Weakest Theorem underlying Cantor's Theorem (for the power set, Hart 2012): Let A be a set. For every $f : A \to P[A]$ there is a subset $\Psi \subseteq A$ - thus $\Psi \in P[A]$ - such that for all $\alpha \in A$ it holds that $\Psi \neq f[\alpha]$.

Proof: Define $\Psi = \{x \in A \mid x \notin f[x]\}$. Take $\alpha \in A$. Check the two possibilities.

Case 1: $\alpha \in \Psi$. In that case $\alpha \notin f[\alpha]$. Thus $\Psi \neq f[\alpha]$. (We have $\alpha \in \Psi \setminus f[\alpha]$.)

Case 2: $\alpha \notin \Psi$. In that case $\alpha \in f[\alpha]$. Thus $\Psi \neq f[\alpha]$. (We have $\alpha \in f[\alpha] \setminus \Psi$.)

Q.E.D.

C.2. Discussion:

Positive is: This is essentially a rewrite of the definition $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin f[\alpha])$. This proof would hold for any set and function. Obviously, once the theorem is accepted, it follows that there can be no surjection and hence no bijection. The strength of the theorem and proof is that (1) it avoids using concepts like surjection, injection and bijection, (2) it would be constructive and avoids the *reductio ad absurdum*.

Negative is: For finite sets the proper constructive method uses numerical succession (mathematical induction), and this method is beyond doubt. The problem lies with infinity, for which I have proposed the notion of *'bijection by abstraction'.* If such a bijection would exist for the natural and real numbers, then there is something wrong with above 'proof'. Indeed, the refutation of the *reductio ad absurdum* proof of Cantor's Theorem shows what is the problem with above 'proof' too. There is a 'spurious non-contradiction': the 'proof' looks without contradiction but in fact relies on a hidden assumption that causes a contradiction.

It may be mentioned that 'proof' should mention the [*NB*]-addendum in **Appendix J.3**. When a contradiction is reached also the possibility that ZFC fails must be mentioned. This appendix also shows that the necessary condition is a surjection (rather than a bijection).

Refutation: While the 'proof' in cases 1 and 2 assumes any f, it ought to distinguish between kinds of functions: the surjections versus the non-surjections.

C.3. Proper proof structure - that does not prove above theorem

If f is not surjective, then like the above.

If f is surjective, then from the discussion above we know that above definition of Ψ causes a contradiction, so it is no useful Ψ. The closest analogue requires the condition to prevent reliance on hidden contradictions. This gives us Φ. We also know that there is a φ such that $f[\varphi] = \Phi$. Let us split the subcases on the risk of infinite regress (ad hoc, no '&&'):

Define $\Phi = \{x \in A \ \& \ f[x] \neq \Phi \mid x \notin f[x]\} \cup \{x \in A \ \& \ f[x] = \Phi \mid x \notin f[x] \ \& \ x \in \Phi\}$.

We subsequently distinguish cases $\alpha = \varphi$ and $\alpha \neq \varphi$.

Case A. $\alpha \neq \varphi$. All is like the above.

Case 1: $\alpha \in \Phi$. In that case $\alpha \notin f[\alpha]$. Thus $\Phi \neq f[\alpha]$.

Case 2: $\alpha \notin \Phi$. In that case $\alpha \in f[\alpha]$. Thus $\Phi \neq f[\alpha]$.

Case B. $\alpha = \varphi$.

Case 3: $\varphi \in \Phi$. Then ($\varphi \notin \Phi$ & $\varphi \in \Phi$): contradiction. This case cannot occur.

Case 4: $\varphi \notin \Phi$. Then ($\varphi \notin \Phi$ or $\varphi \in \Phi$): no contradiction. It is false that $\Phi \neq f[\alpha]$ however since we have $\alpha = \varphi$ such that $f[\varphi] = \Phi$.

Ergo: The 'proof' fails.

C.4. Discussion:

(1) The proof in C.1 thus holds for ZFC that allows Ψ. One must explain why one uses a system that allows ill-defined-ness.

(2) For well-defined systems in C.3, the theorem cannot stand with the proof that uses Ψ. When we look for something that matches Ψ then we find Φ. For some f, namely surjections, this Φ has an element in *A,* namely $\alpha = \varphi$ such that $f[\varphi] = \Phi$. This holds without contradiction, so that one cannot hold that f cannot exist.

(3) This particular refutation does not mean that there might be other cases such that the theorem still stands. This may be doubted however.

(4) This does not mean that we have shown how such a surjection can be created. We have merely shown that the assumption of a surjection invalidates the 'proof' in C.1. It is a fair question: can one actually show a surjection between infinite sets, like the natural numbers and the reals ? For this I refer to CCPO-PCWA and the *bijection by abstraction.*

Appendix D: An inadequate 'initial review'

The December 31 2014 version of this paper, see Colignatus (2014b), got this response from a peer-reviewed journal:

"An initial review of "A condition by Paul of Venice (1369-1429) solves Russell's Paradox, blocks Cantor's Diagonal Argument, and provides a challenge to ZFC" has made it clear that this submission does not meet the minimal requirements for publication in [our journal]. It is not sufficiently clear what the goal of the paper is, and (most importantly) it is not at all shown that the two possibilities listed on p.5 [i.e. in Section 3.4] have the intended consequences."

This June 4 2015 version has only made small changes. The major difference is to reduce the confusion on the shorthand form, now also with the &&-construction and notation in Appendix J, but this has indeed been relocated to the appendix because of clutter. New are appendices B, C, D and E but those are not really material to the main challenge. New are also the lemmata en corollaries in Section 3 that only hightlight the challenge, and that are only there to refute a rather artificial defence by a 'Cantorian reading'. However, one can argue that Appendix F with the constructive proof F.3 and subsequent non-constructive version of Theorem 3.3 do provide a serious addition. Corollary 3.3 is less relevant again.

Thus, you can check that it is curious that the editor holds that the goal of the paper would not be clear. Also, the two possibilities listed in Section 3.4 directly modify the application of ZFC in the [*NB*]-addendum in the proof for Cantor's theorem (see Appendix J.3), and block that proof, as indeed has been shown in the discussion in Section 2. There is no indication that the changes to the Axiom of Separation would be inconsistent.

This 'initial review' by the editor is inadequate. There should have been a full review with decent reports.

PM. Another reader wrote: "You seem to be saying that ZFC makes Cantor's theorem true which you find paradoxical and therefore you feel that ZFC needs amendment. But I think Cantor's theorem is cool so I am happy with ZFC." This reader got the main idea, contrary to the editor who suggested that it was not clear. However, observe that the word 'paradox' means 'seeming contradiction' (which is no real contradiction), and that the paper identifies these problems:

(a) the inclusion of a *consistency*-criterion causes the proofs to collapse,

(b) the question about Ψ in 2.1 and Φ in 2.2,

(c) the rejection of the other proofs, see CCPO-PCWA,

(d) the lack of a *'set of all sets'* and the schizophrenia of the formal use of proper sets and the informal use of improper sets ('classes');

(e) It is not irrelevant that we lack an empirical example for the transfinites. (But with this caveat: see Colignatus (2015af) for abstraction versus empirics, and on Wigner. Mathematical modeling indeed creates ideas for which there might be no 'reality'.)

A reader: "You say *"the proper conclusion is not that Cantor's Theorem is proven, but only that it is proven in ZFC".* Of course, whether or not Cantor's theorem is true, depends on your axioms of set theory, especially axioms pertaining to the infinite." Indeed, well understood again. One presumes however that some fundamental notions must be selected for their foundational values, not for their cool results when neglecting problems like (a) - (e).

Appendix E: Pauline versus Cantorian ZFC

This paper discusses the choice of various possible axiomatic systems for set theory. The chosen system defines what is well-defined. ZFC provides for ZFC-sets. BST provides for BST-sets. In **Appendix J.3**, the [*NB*] Addendum point (iv) "ZFC provides for well-defined sets" gives the hypothesis that developers of ZFC have succeeded in capturing that notion. This paper challenges that hypothesis. It may be that ZFC-PV or BST capture it better. They would not have the transfinites. Let us now consider ***Pauline*** versus ***Cantorian*** approaches to ZFC.

Section 3.2 in the paper defines the Pauline versus Cantorian readings of the Axiom of Separation. Lemma 3.2 shows that the Cantorian reading implies the possibility of the Pauline one (for membership to the *B* that is defined). We take account of two cases:

(α) Lemma 3.2 is known and accepted. What are the consequences ? (α1) Does one hold that the Pauline reading does not affect ZFC ? Or is some effect accepted as well, and which one ? (α2) Does one also accept non-constructive Theorem 3.3 and/or constructive Theorem F.3 ? (α3) Does one accept the *bijection by abstraction* ?

(β) Lemma 3.2 is not known or not accepted. In this case the different positions are regarded as deriving from a difference in personal view only. How does this affect the perception of the challenge to ZFC ? The Pauline view will be ***tolerant,*** the Cantorian view will be ***defensive.*** What are the reasons for tolerance or defence ?

E.1. The Pauline cq. tolerant interpretation

Section 3.1 presents the view that the condition "*B* is not free in $\gamma[x]$" is satisfied when *B* is bound by the existential quantifier. This is a **Pauline** interpretation and approach. It causes that Ψ in 2.1 and Φ in 2.2 belong to ZFC, so that the question about their differences can be discussed as an anomaly *within* ZFC: which causes a rejection of ZFC as a proper axiomatic development for set theory.

This Pauline approach acknowledges the existence of the condition "*B* is not free in $\gamma[x]$" since this highlights the challenge to ZFC. However, ZFC is ambiguous. There are versions available without this condition. We rely on Coplakova et al. (2011:145), but we had to insert the by-line in Section 3.1. A notable example is also Weisstein (2015) of MathWorld.

3. Axiom of Subsets: If φ is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : \varphi(u, p)\}$ that contains all those $u \in X$ that have the property φ. (also called Axiom of Separation or Axiom of Comprehension)

$$\forall X \, \forall p \, \exists Y \, \forall u \, (u \in Y \equiv (u \in X \wedge \varphi(u, p))).$$

In Hart (2013:29) we find the following formulation that can be judged to be at least *ambiguous*. Its formulation allows the *Pauline* interpretation, i.e. that the test on the free variables of $\gamma[x]$ happens under the existential quantifier. *B,* or in this case *y,* can be regarded as a given or constant, and not a free variable for $\gamma[x]$. It would be an additional assumption ('clarification of how to read the axiom') to adopt the Cantorian defence below.

Axioma 3. Het Afscheidingsschema. Als ϕ een (welgevormde) formule is met zijn vrije variabelen in de rij x, z, w_1, $\ldots$, w_n (allen ongelijk aan y) dan bestaat bij elke verzameling x een verzameling die bestaat uit precies die elementen van x die aan ϕ voldoen:

$$(\forall x)(\forall w_1)\cdots(\forall w_n)(\exists y)(\forall z)\big(z \in y \leftrightarrow (z \in x \wedge \phi)\big)$$

E.2. The Cantorian cq. defensive interpretation

A **Cantorian** approach to ZFC would be to maintain that authors in the world are free, but that only the version or interpretation of ZFC is acceptable that blocks Φ in 2.2. This approach is to reject the Pauline interpretation, and deny the challenge.

The defence is: $\gamma[x]$ *is lifted out from the Axiom of Separation and its existential quantifier, and judged as an independent expression.*

The condition "*B* is not free in $\gamma[x]$" then allows the creation of Ψ in 2.1 but blocks Φ in 2.2. This also presumes that Φ cannot be a constant. (Some systems allow that 'variable' might indicate symbols: constants and proper variables.)

This Cantorian approach seems to have the appeal of solidity, i.e. that ZFC exists now for some time, that some researchers find fransfinites attractive and a work of art, and that the view can be maintained even in the face of the Paul of Venice anomaly. But it comes at the price of some questions that are not answered except by dogma.

E.3. Questions for the Cantorian interpretation

(1) Why would one lift $\gamma[x]$ out from the Axiom of Separation and its existential quantifier ? Why judge it separately ? Axioms are not posed out of thin air, but we generally look for a rationale.

(1a) The transfinites are no reason, when they can be diagnosed as an illusion based upon not-well-defined sets. They don't exist in reality, only in fancy because of ZFC. Sticking to ZFC-Cantorian merely because of the transfinites is begging the question.

(1b) A defence is that $\gamma[x]$ is not lifted out from the formula, but already existed as an independent expression, say as a predicate in predicate logic. This merely shifts the problem to predicate logic, and without looking into that problem there. One idea for set theory was to harness predicate logic, but if one drops this idea then how is one to establish well-definedness for predicates ?

(2) Why deny the freedom for researchers to adopt the Pauline interpretation ? Was it not Cantor himself who argued that mathematics allows for freedom ? Why could ZFC-PV or

BST not be fine axiomatic systems, that deserve mention in an introduction course on set theory ?

(3) Beware of theology and the dispute between Gomarus (predestination) and Arminius (freedom of choice). A former version of this paper met with criticism that the Pauline approach was based upon a 'misconception' and 'elementary error', and that the use of the consistency condition 'was not allowed'. While the paper only posed a problem, clearly formulated so that others could understand it, if only they opened their minds to it: but the reader did not see the problem but only error and sin. This reader apparently was so married to ZFC in its Cantorian interpretation, that he did not see alternatives, and he was no longer aware that set theory was about studying axioms for sets and not just ZFC. Instead, in reality, there are alternatives to ZFC, also alternatives in interpreting ZFC. (It is fortunate that there now is lemma 3.2, so that the power of the *rational argument in mathematics* can be relied upon to help resolve an issue. In a way though this is too simple, since arguments (1) and (2) were already convincing mathematically. This also holds for the next points.)

(4) While (2) emphasizes freedom, there is also necessity. While the Cantorian approach blocks the question on "Ψ in 2.1 and Φ in 2.2" *within* ZFC-Cantorian, ZFC-defenders must acknowledge that the question exists *within* ZFC-Pauline.

(4a) Thus, instead of answering the question now *within* ZFC-Pauline, they must answer it *across* variants of ZFC.

(4b) Thus, please, explain why Ψ in 2.1 in ZFC-Cantorian generates transfinites and Φ in 2.2 in ZFC-Pauline does not ? What causes the difference, while the cause is a consistency condition that should have no effect ?

(4c) Rather than neglecting the issue, and getting lost in the illusion of the transfinites, ZFC-defenders might feel obliged to explain why that difference arises. It is not only the consequence of the evaluation $\gamma[x]$, but also the impact of the consistency condition. Is it a *consistency* condition or not ? How can it be that the insertion of consistency can cause the collapse of Cantor's Theorem and the transfinites ?

(4d) While questions (4a) - (4c) allow for that lemmata 3.2 and 3.3 and theorems 3.3 and F.3 and their corollaries are not known, they become more acute when those and their proofs are accepted. One must now answer such questions across systems, say between ZFC and ZFC-PV (Section 3.4).

(5) It would help to establish whether this challenge to ZFC based upon the Paul of Venice condition is *new* to researchers of set theory or not. I have no knowledge on this.

(5a) If it is new, then perhaps the tradition of ZFC has been based upon an illusion.

(5b) If it is old, then perhaps it was not properly evaluated in the past.

(5c) There is also the variety of formulations of ZFC that needs explanation, compare e.g. Coplakova et al. (2011), Hart (2013) and Weisstein (2015).

(6) How is it with naive notions like the *'set of all sets'* ? Hart (2015) describes the incongruity of using formal ZFC-sets and informal 'classes'. Would it not be mathematically attractive when these could be brought in line within one consistent system ? The objective is not to limit the freedom in mathematics but to find an adequate system for education, see CCPO-PCWA and Colignatus (2011) and the conclusion above (Section 4).

(7) There is Section 1.1.6 on the intended interpretation and the anomaly.

E.4. Diagram of the views when the propositions in Section 3 are known

Let us assume that the propositions in Section 3 are known and accepted.

Figure 1 shows how the views on ZFC relate to each other and to the overall *'Intended Interpretation'.* The figure indicates only the expressions that are allowed. (In the Cantorian reading of ZFC, the bijection by abstraction is rejected, and its Ψ disappears in terms of 'content', leaving only its grin. But a non-surjective *f* would generate its $\Psi[f]$.) The advantage of this situation & figure is that it shows that the issues can be discussed *across* systems.

Notions are:

(1) The roman letters indicate subareas, and the text labels their unions.

(2) ZFC-Pauline = $M \cup P \cup R$. ZFC-Cantorian = M. ZFC-PV = $R \cup S$.

(3) The Intented Interpretation might be $K \cup M \cup P \cup R \cup S$ but perhaps parts drop out. Ideally, ZFC and ZFC-PV are a model for the Intended Interpretation. Then at least $K = \emptyset$.

(4) Ideally ZFC and ZFC-PV have the same Intended Interpretation, so researchers have to determine which has to give way.

(5) Lemma 3.2 is that ZFC-Cantorian implies ZFC-Pauline, and thus the former is a subset of the latter.

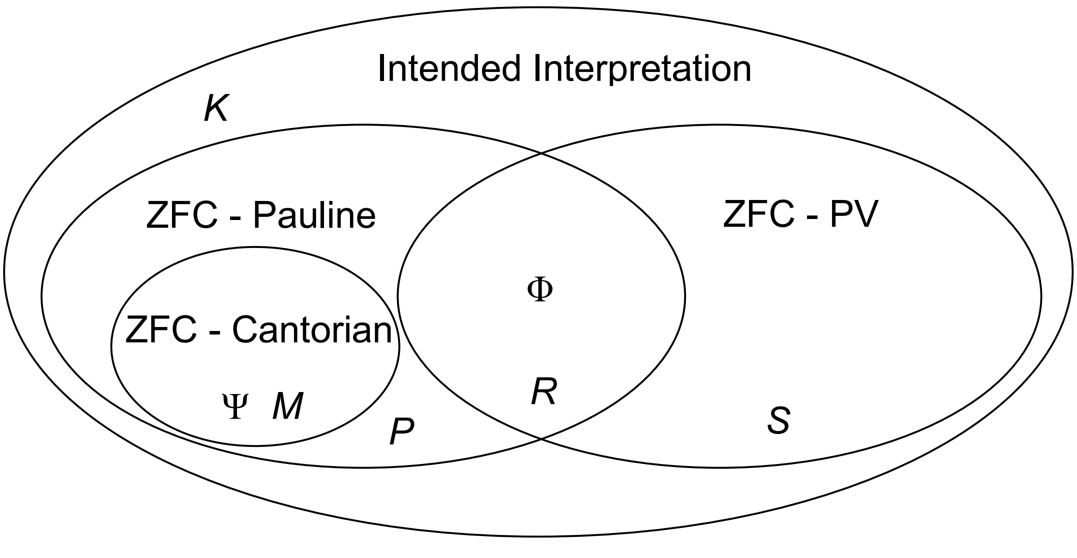
(6) Corollary 3.2. holds that Φ may differ from Ψ. The symbols are distinct in the figure because it looks at the formulas. The sets might disappear e.g. if the bijection by abstraction is rejected. ZFC-Cantorian might hold that on content $\Phi = \Psi$.

(7) Transfinites do not exist out of ZFC-Cantorian (unless there are *really* valid proofs), and thus would not be part of the Intended Interpretation. Thus we should cut out a part of *M* that depends upon Ψ. Thus ZFC-Cantorian has to explain why it includes a part that would not belong to the Intended Interpretation.

(8) ZFC-Cantorian still must explain the difference between Sections 2.1 and 2.2. One would tend to hold, as in (7), that 2.2 falls under the Intended Interpretation, so that 2.1 has a problem.

(9) This paper only looked at *R* and *M*, and didn't look at other areas. Since ZFC is restrictive, it is not unlikely that $S = \emptyset$ and $P = \emptyset$. The relevant question is: what can ZFC-Cantorian achieve in *M* that falls under the Intended Interpretation, but which cannot be achieved by ZFC-PV in *R* ? (The transfinites are excluded because of (7).)

Figure 1: Venn-diagram of ZFC versus ZFC-PV, when the propositions in Section 3 are known and accepted: allowed formulas only (e.g. neglecting that Ψ does not exist when its bijection (by abstraction) doesn't exist)



E.5. Diagram of the views when the propositions in Section 3 aren't known

Let us assume that the propositions in Section 3 are not known or accepted. This is at least the situation before the June 4 2015 version of this paper.

Figure 2 shows how the views on ZFC relate to each other and to the overall *'Intended Interpretation'.* The figure indicates only the expressions that are allowed. (In the Cantorian reading of ZFC, the bijection by abstraction is rejected, and Ψ disappears in terms of 'content', leaving only its grin. But a non-surjective *f* would generate its $\Psi[f]$.) The advantage of this situation & figure is that it shows that the issues can be discussed *within* ZFC-Pauline.

Notions are:

(1) The roman letters indicate subareas, and the text labels their unions.

(2) ZFC-Pauline = $R \cup M$ (assuming above $P = Ø$). ZFC-Cantorian = $L \cup M$. (Lemma 3.2 found that $L = Ø$, but one does not know or accept this.)

(3) The Intented Interpretation might be $K \cup L \cup M \cup R$ but perhaps parts drop out. Ideally, ZFC is a model for the Intended Interpretation, and at least $K = Ø$.

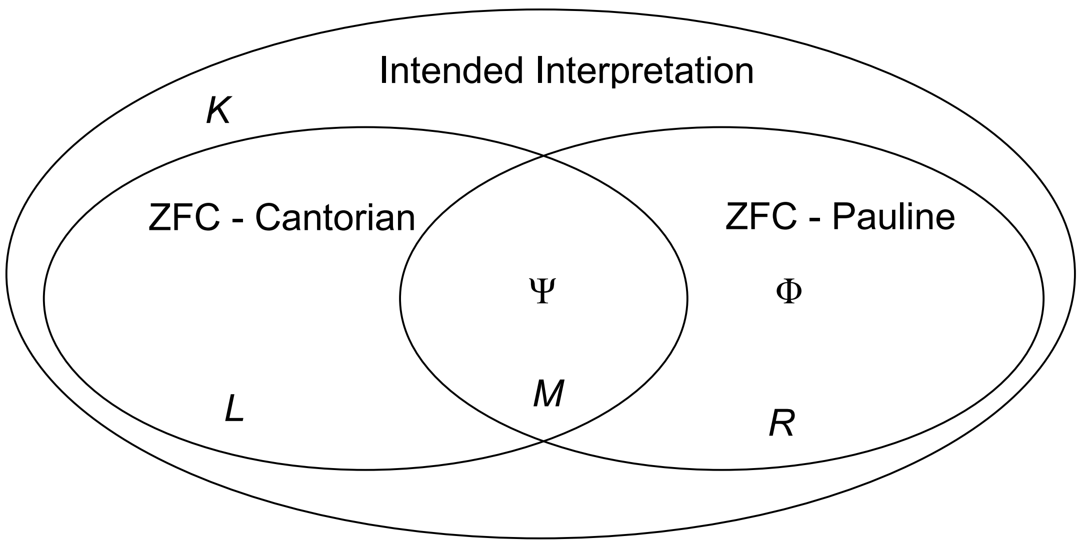
(4) Ideally the two ZFC readings have the same Intended Interpretation, so researchers have to determine which has to give way.

(5) Transfinites do not exist out of ZFC-Cantorian (unless there are *really* valid proofs), and thus would not be part of the Intended Interpretation. Thus we should cut out a part of *M* that depends upon Ψ. Thus ZFC-Cantorian has to explain why it includes a part that would not belong to the Intended Interpretation.

(6) ZFC-Cantorian holds that *R* would be nonsense: since lemma 3.2 is unknown. But then ZFC-Cantorian must first explain the difference between Sections 2.1 and 2.2. One would tend to hold that 2.2 falls under the intended interpretation, so that 2.1 has a problem.

(7) This paper only looked at *M* and *R*, and didn't look at *L*. (Lemma 3.2 found that $L = Ø$.) The relevant question here is: what can ZFC-Cantorian achieve in *L* that falls under the Intended Interpretation, but which cannot be achieved by ZFC-Pauline ?

Figure 2: Venn-diagram of the Pauline and Cantorian readings of ZFC, when the propositions in Section 3 are not known or accepted: allowed formulas only (e.g. neglecting that Ψ does not exist when its bijection (by abstraction) doesn't exist)



Appendix F: Marginal analysis and the re-occurrence of Russell's paradox

ZFC seems to block the use of a significant set of normal functions. The failure on a single element $\varphi \in A$ such that $f[\varphi] = \Phi$ seems to have too many consequences. The restriction only concerns a single element, and this suggests the approach of a marginal analysis in constructive manner. This would show more about the mechanism. It seems that one can indeed take arbitrary functions and reserve a single element for Φ.

Section 3 presents lemma 3.3 and theorem 3.3. Lemma 3.3 still relies on a surjection. It still allows the conclusion that surjections don't exist between a set and its power set. Looking for a way to weaken this assumption caused the consideration of an analysis at the margin. This analysis generated the nutshell 1.1 and Conjecture 1.2.B in the Introduction.

Constructive methods would allow for the notion of a fixed point, i.e. $p = f[p]$. Thus we are not afraid of some aspects of self-reference or circularity. At issue is only how much ZFC allows. Perhaps ZFC allows the wrong kind and disallows the useful kind.

Let us first set up a marginal analysis that fails, so that we can better understand the subsequent marginal analysis that generates the proofs that we desire.

Up to now such a marginal analysis was not very well possible, apparently, because there arises a contradiction that reminds of Russell's Paradox. If the steps satisfy the formal conditions of ZFC then we have actually a contradiction in ZFC. This is reflected in corollary 3.3. Such a contradiction is now resolved by the use of the Paul of Venice consistency condition - but then in system ZFC-PV.

This appendix can be read in two ways. Experienced readers might jump directly to F.3. Other readers might well prefer to read the subsections sequentially, to see the build-up of arguments and the reasoning that led to F.3 and Theorem 3.3.

See F.4 for some elementary properties that we use here.

F.1. A marginal analysis with failure because of Russell's paradox

In principle we should be able to take an arbitrary function and reserve a single element, to map this element to a single new set. Let us see what happens when we extend g on the margin with α, which gives function f: $A \to P[A]$, with $f[\alpha]$ defined as well. Let us distinguish the disputable core (i) and the discussion (ii).

Let us look in more detail at the steps in conjecture 1.2.B.

Conjecture F.1: Let A have at least one element α. $P[A]$ is the power set.

Let $B = A \setminus \{ \alpha \}$ not be empty. Consider arbitrary g: $B \to P[A]$.

(i) There are f: $A \to P[A]$ and a $\psi \in A$ with $f[\psi] = \Psi = \{x \in A \mid x \notin f[x]\}$.

(ii) There is a contradiction. (Note that h does not have to be a bijection.)

Proof: (i) Steps that seem acceptable in ZFC:

$\Psi^*[B] = \{x \in B \mid x \notin g[x]\}$. (Because of $P[A]$, instead of $P[B]$, there is $\Psi^*[B]$.)

Define: f as: (a) $x \in B$: $f[x] = g[x]$

(b) $x = \alpha$: $f[\alpha] = y$, for some $y \in P[A]$ (e.g. in fixed point form).

Note: $\Psi^*[B] = \{x \in B \mid x \notin f[x]\} = \{x \in B \mid x \notin g[x]\}$.

Let: $M = \{x = \alpha \mid x \notin f[x]\}$ (Margin)

Ergo: $\Psi^*[B] \cup M = \{x \in A \mid x \notin f[x]\} = \Psi[A]$ (See F.4.1.)

Observe that $\Psi^*[B] \in P[A]$ & $M \in P[A]$, so that $\Psi^*[B] \cup M \in P[A]$. Take $y = \Psi[A]$.

(ii) Take ψ = α. The direct check on consistency gives: $(\alpha \in \Psi) \Leftrightarrow (\alpha \notin \Psi)$.

Q.E.D.

(ii) **Discussion 1:** If ZFC would formally allow this deduction, then ZFC is inconsistent. See **Appendix C.1** that precisely concludes that $f[\alpha] \neq \Psi[A]$. I am no expert on ZFC and would like to hear the views. My guess is that some would hold that *M* would be a necessary step for the creation of a surjective function, and that such would not exist in ZFC. This is not the right answer. The question is which axiom has been applied wrongly.

The definition of *M* at the Margin seems an acceptable application of the Axiom of Separation, namely to set {α} and predicate $\gamma[x] = (x \notin f[x])$. See **Appendix J.3** [*NB*]. Doesn't the general restriction $y \in P[A]$ allow us to treat f as *any* f ?

Discussion 2: From (i) we find:

Ergo: $f[\alpha] = \Psi[A] = \{x \in A \mid x \notin f[x]\}$.

Even though these steps seem innocuous, we derive:

$$(\alpha \in \Psi[A]) \Leftrightarrow (\alpha \notin f[\alpha]) \Leftrightarrow (\alpha \notin \Psi[A])$$

The reason lies in the definition of *M.* Here $(\alpha \notin \Psi^*[B])$ is true and can be dropped:

$$(\alpha \in M) \Leftrightarrow (\alpha \notin f[\alpha]) \Leftrightarrow (\alpha \notin \Psi^*[B] \cup M) \Leftrightarrow (\alpha \notin M) \qquad \text{(key step, F.4.2)}$$

$$M = \{x = \alpha \mid x \notin M\} \qquad \text{(Russell !)}$$

We thus have an instance of Russell's paradox - see the nutshell in Section 1.1. A singleton, a set with one element, obviously limits our freedom of definition. Now in F.1 we have a large *A*, possibly infinite. What is crooked here ?

While ZFC blocks Russell's paradox (**Appendix G**), its occurrence here is not direct but indirect by the (circular) steps $M = M[f]$ and $f = f[M]$ in (i), giving a fixed point $M^* = M[f[M^*]]$. One question is whether ZFC allows the break up into steps as shown. Creating a ZFC* that blocks such steps and fixed points is not the answer. We might like some beneficial fixed point. Now there is a contradiction, but only because of the paradoxical Ψ.

Discussion 3, now on the more general issue:

(1) It is not an adequate conclusion that this g does not exist - it is arbitrary - or that it is impossible to reserve an element for a newly created set.

(2) It is not so useful to look into the issue for finite sets, since such sets are smaller than their power sets. The issue remains relevant for infinity, for which the room to select a single element is infinitely larger.

(3) The adequate response is to eliminate this occurrence of contradiction anyhow. Whatever the formal properties of ZFC, we would want to be able to extend a function with a single element while using elementary logic.

F.2. A marginal analysis with the required proof

The following lemma only helps to identify a logical structure.

Lemma F.2: Let *A* be a nonempty set, *P*[*A*] the power set. For a significant set of arbitrary $f: A \rightarrow P[A]$ the sets $\Psi = \{x \in A \mid x \notin f[x]\}$ and $\Phi = \{x \in A \mid x \notin f[x] \;\&\&\; x \in \Phi\}$ give $\Phi \neq \Psi$, namely those f for which there is a $\varphi \in A$ such that $f[\varphi] = \Phi$.

Proof via Appendix C: Appendix C.1 holds for arbitrary f that $\forall x \in A$: $\Psi \neq f[x]$. Thus $\Psi \neq f[\varphi] = \Phi$. Q.E.D. (This neglects the deconstruction of that particular proof.)

Proof in constructive manner:

Let $B = A \setminus \{ \varphi \}$. Let g: $B \to P[A]$. The decomposition of f is:

(a) $x \in B$: $f[x] = g[x]$

(b) $x = \varphi$: $f[\varphi] = \Phi$ (It is now given what was a problem in F.1.)

A direct test on consistency:

$$(\varphi \in \Phi) \Leftrightarrow (\varphi \notin f[\varphi] \ \& \ \varphi \in \Phi)$$

Whence it follows without direct contradiction that $\varphi \notin \Phi$.

With $\varphi \notin \Phi$ we have $\varphi \notin f[\varphi]$ and thus $\varphi \in \Psi$. Thus $\Phi \neq \Psi$. Q.E.D.

Comment: Lemma F.2 assumes a property for f that rather requires proof.

F.3. The original more constructive proof of Theorem 3.3

Theorem F.3 is Theorem 3.3. The proof below applies the marginal analysis of F.1, but now selects a particular *y* without circularity. It observes the Paul of Venice consistency, and proves the property that is assumed in Lemma F.2.

Theorem F.3 (Theorem 3.3): Let *A* be denumerable infinite, *P*[*A*] the power set. (i) For any arbitrary non-trivial *h*: $A \to P[A]$ there are a f: $A \to P[A]$ and a $\varphi \in A$ with $f[\varphi] = \Phi = \{x \in A \mid x \notin f[x] \ \&\& \ x \in \Phi\} \in P[A]$. (ii) The direct test has $\varphi \notin \Phi$ without direct contradiction.

Proof in constructive manner:

(i) Consider an ordering of *A* = {*a*[1], ...} and let be φ the first element *a*[1].

Let $B = A \setminus \{ \varphi \}$ and g: $B \to P[A]$ as in Hilbert's Hotel $g[a[n]] = h[a[n\text{-}1]]$ for $n > 1$.

$\Phi^*[B] = \{x \in B \mid x \notin g[x] \ \&\& \ (x \in \Phi^*[B])\}$. When Φ stands alone: $\Phi = \Phi[A]$.

(PM 1. $\Phi^*[B]$ suits ZFC, see **Appendix J.3** (part (iii) [*NB*]): (a) Note that g can be regarded as a subset of *B* x *P*[*A*]. Then g exists because of the Axiom of Pairing. (b) Because of *P*[*A*], instead of *P*[*B*], there is a different $\Phi^*[B]$. This $\Phi^*[B]$ exists because of the Axiom of Separation applied to the part without '&&', and then applying Lemma 3.2.)

Define f: $A \to P[A]$ as:

(a) $x \in B$: $f[x] = g[x]$

(b) $x = \varphi$: $f[\varphi] = \Phi^*[B]$

We need to prove that $\Phi = \Phi[A] = \Phi^*[B]$.

Since $(\varphi \notin B)$ also $(\varphi \notin \Phi^*[B])$.

Define: $M = \{x = \varphi \mid x \notin f[x] \ \& (x \in \Phi^*[B])\} = \varnothing$. (Margin, not '&&')

(PM 2. *M* exists in ZFC. Apply Separation to {φ} with $\gamma^*[x] = (x \notin f[x]) \& (x \in \Phi^*[B])$, in which $\Phi^*[B]$ is not a free variable but a constant given from the above.)

Rewrite g as f, using $(x \in B)$:

$\Phi^*[B] = \{x \in B \mid x \notin f[x] \ \&\& \ (x \in \Phi^*[B])\}$.

$\Phi^*[B] = \Phi^*[B] \cup M = \{x \in A \mid x \notin f[x] \ \&\& \ (x \in \Phi^*[B])\}$ (F.4.1)

Rewrite: $K = \{x \in A \mid x \notin f[x] \ \&\& \ (x \in K)\}$ for $K = \Phi^*[B]$.

We recognise the self-referential definition of Φ so that $K = \Phi^*[B] = \Phi$.

(ii) The direct test on consistency is:

$$(\varphi \in \Phi) \Leftrightarrow (\varphi \notin f[\varphi] \ \& \ \varphi \in \Phi)$$

Whence it follows without direct contradiction that $\varphi \notin \Phi$.

Q.E.D.

Comments:

(1) In F.3 we can make sure that the margin *M* = Ø. In F.1 the condition on the margin causes a problematic self-reference.

(2) Joining the two sets into *K* and directly introducing the '&&' notation may be tricky. However, work in the opposite direction. Test $(\varphi \in K) \Leftrightarrow ((\varphi \notin f[\varphi]) \;\&\; (\varphi \in \Phi^*[B]))$ since the contradiction takes &&-precedence: whence we conclude to non-membership. Then *K* reduces to $(x \in B)$ so that the original definition of $\Phi^*[B]$ is retrieved.

(3) Let $Y = \{x \in A \mid x \notin h[x] \;\&\&\; x \in Y\}$. With the theorem in **Appendix C.1** there is no *x* in *A* such that *h*[*x*] = *Y.* Thus *h*[φ] is unequal to *Y.* It is not precluded that one can construct a bijection (by abstraction).

F.4. Some elementary properties

The above uses some elementary properties.

F.4.1. $(\{x \in A \mid \gamma[x]\} \cup \{x \in B \mid \gamma[x]\}) \Leftrightarrow \{x \in A \cup B \mid \gamma[x]\}$

For the elements of *A* and *B,* also allowing for infinity, LHS and RHS mean, also when those subsets reduce to the empty set:

$\{a_1 \mid \gamma[a_1]\} \cup ... \cup \{b_1 \mid \gamma[b_1]\} \cup ...$

F.4.2. When $\alpha \notin X$ then:

$$\neg(\alpha \in X \cup M)$$

$$\neg(\alpha \in X \vee \alpha \in M)$$

$$\alpha \notin X \;\&\; \alpha \notin M$$

$$\alpha \notin M$$

Appendix G: How does ZFC block Russell's paradox ?

G.1. Truthtable

For ZFC, take the axiom of separation, substitute Russell's idea that the set of teaspoons is not a teaspoon, denote that result as *R* = *R*[*A*]*,* then test it.

$$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \Leftrightarrow ((x \in A)\ \&\ \gamma[x])) \qquad \text{(SEP)}$$

$$(\forall A)\ (\forall x)\ (x \in R \Leftrightarrow ((x \in A)\ \&\ (x \notin x)) \qquad \text{(RUS)}$$

$$(\forall A)\ \ (R \in R \Leftrightarrow ((R \in A)\ \&\ (R \notin R)) \qquad \text{(test } R\text{)}$$

The truthtable allows only for $R \notin A$ & $R \notin R$.

$R \in R$	$\Leftrightarrow$	$(R \in A$	&	$R \notin R)$
1	0	1	0	0
1	0	0	0	0
0	0	1	1	1
0	1	0	0	1

Thus RUS itself is not blocked. Note that $R = R[A]$, and not the unqualified total in the original Russell paradox. The order is $(\forall A)$ $(\exists B)$ and not $(\exists B)$ $(\forall A)$.

RUS means that $R = R[A]$ can exist as a subset of A, i.e. $(R \subset A)$. Thus $R \in P[A]$.

Select $B = P[A]$. The Russell set for this is different now:

$$(\forall x)\ (x \in R[B] \ \Leftrightarrow\ ((x \in B)\ \&\ (x \notin x)) \qquad \text{(RUS on } B = P[A])$$

Since $R \in P[A]$ and $R \notin R$, we find $R \in R[B]$ without contradiction.

Similarly when $R[B]$ is applied to itself again:

$$(R[B] \in R[B] \Leftrightarrow\ ((R[B] \in B)\ \&\ (R[B] \notin R[B])) \qquad (R[B] \text{ with } B = P[A])$$

The truthtable allows consistently for $(R[B] \notin B)$ & $(R[B] \notin R[B])$. Thus $R[B] \subset P[A]$.

PM. Observe that there is also the ZFC Axiom of Regularity - no formula now (REG).

REG implies that sets cannot be a member of themselves.

On this account already $R \notin R$.

REG also implies that the universe U would not be a ZFC-set.

G.2. Wikipedia

The portal (no source) Wikipedia on Russell's paradox (retrieved May 30 2015) gives the traditional view. They sometimes clearly refer to ZFC-sets, but not always.

> "ZFC does not assume that, for every property, there is a set of all things satisfying that property. Rather, it asserts that given any set X, any subset of X definable using first-order logic exists. The object R discussed above [in naive set theory] cannot be constructed in this fashion, and is therefore not a ZFC set. In some extensions of ZFC, objects like R are called proper classes."
>
> "In ZFC, given a set A, it is possible to define a set B that consists of exactly the sets in A that are not members of themselves. B cannot be in A by the same reasoning in Russell's Paradox. This variation of Russell's paradox shows that no set contains everything."

The latter should rather be "no ZFC-set contains everything". The latter is a bit implicit. It is better to show: assume the possibility that there is an universal set U, then $R[U] \notin U$ would generate a contradiction. (In ALOE it would be mere nonsense.)

G.3. Stanford Encyclopedia of Philosophy

Irvine & Deutsch (2014) seem to take the Cantorian position, see **Appendix E.** In these quotes their notation has been adapted to the present one. Comments are in square brackets:

> "Again, to avoid circularity [really ?], B cannot be free in γ. [Presumably the Cantorian reading ?] This demands that in order to gain entry into B, x must be a member of an existing set A. [This is a different property than the Cantorian reading.] As one might imagine, this requires a host of additional set-existence axioms, none of which would be required if [naive set theory] had held up.
>
> How does SEP avoid Russell's paradox? One might think at first that it doesn't. After all, if we let A be U – the whole universe of sets – and γ[x] be $x \notin x$, a contradiction again appears to arise. [Indeed, for in that case $((R \in U)\ \&\ (R \notin U))$.] But in this case, all the contradiction shows is that U is not a set. All the contradiction shows is that "U" is an empty name (i.e., that it has no reference, that U does not exist), since the ontology of Zermelo's system consists solely of sets."

[Observe that REG causes that the universe U would not be a ZFC-set anyhow. It is a pity that the authors give the word "set" to ZFC, instead of speaking about ZFC-sets.]

> "This same point can be made in yet another way, involving a relativized form of Russell's argument. [Meaning: not U but any A.] Let A be any set. By SEP, the set RB = $\{x \in A: x \notin x\}$ exists, but it cannot be an element of A. For if it is an element of A, then we can ask whether or not it is an element of RB; and it is if and only if it is not. Thus something, namely RB, is “missing” from each set A. So again, U is not a set, since nothing can be missing from U. But notice the following subtlety: unlike the previous argument involving the direct application of *Aussonderungs* to U, the present argument hints at the idea that, while U is not a set, “U” is not an empty name. The next strategy for dealing with Russell's paradox capitalizes on this hint. [Referring to Von Neumann's approach that they discuss next]"

Appendix H: Support on the Introduction

H.1. History and dynamic if-switch

Aristotle gave the first formalisation of the notions of *none*, *some* and *all*, of which an origin can be found in the Greek language. This developed into modern set theory, in which the notion of a *set* provides for the *all*. There is a parallel between constants in propositional logic and set theory: *and* giving *intersection*, *or* giving *union, implication* giving *subset.* Still, different axioms give different systems. A common contrast is between the formal ZFC system (from Zermelo, Fraenkel and the Axiom of Choice) and *naive set theory* (not quite defined, but perhaps Frege's system, and not to be confused with Halmos's verbal description of ZFC). There is a plethora - perhaps an infinity - of models for properties of sets.

In naive set theory, Russell's set is $R = \{x \mid x \notin x\}$. Subsequently $R \in R \Leftrightarrow R \notin R$ and naive set theory collapses. Russell's problem was a blow to Frege's system, and researchers spoke about a crisis in the foundations of logic and mathematics. The idea of a crisis was eventually put to rest by the ZFC system. A consequence of ZFC is a *'theory of types',* so that a set cannot be member of itself, and with the impossibility of a *'set of all sets'.*

Define however $S = \{x \mid (x \notin x) \;\&\; (\text{If}(x = S)\,\text{then}\,(x \in S))\}$ i.e. with the small consistency condition inspired by the discussion by Bochenski (1956, 1970:250) of Paulus Venetus or Paul of Venice (1368-1428). The consistency condition with the exception switch was presented in Colignatus *"A Logic of Exceptions"* (1981, 2007, 2011:129) (ALOE).

The *If*-switch gives a dynamic process of going through the steps, and it is not a mere static implication. We find $S \in S \Leftrightarrow (S \notin S \;\&\; S \in S)$, which reduces to $S \notin S$ without contradiction. One might hold that there would be infinite regress, if a test on S on the left causes a test on S on the right, which causes a test on the left again, and so on; but the truthtable of $A \Leftrightarrow (\neg A \;\&\; A)$ allows a formal decision.

It is not clear what Russell's set would be, since it is inconsistent; but who wants to work sensibly with a related notion can use S without problem. There is no reason for a crisis in the foundations of logic and mathematics and there is no need for a theory of types - though you can use them if needed.

H.2. Some aspects of the exception switch

PM 1. The dynamic *If*-switch may be replaced by static $S = \{x \mid (x \notin x) \;\&\; ((x = S) \Rightarrow (x \in S))\}$ but then the truthtable is a bit more involved.

PM 2. Obviously $S = \{x \neq S \mid x \notin x\}$ has the same effect, but this has the suggestion of choice, while the point is that one must show that the property $x \neq S$ is necessary.

PM 3. In some texts like the Introduction above I have used the shorthand form $S = \{x \mid x \notin x \ \& \ x \in S\}$, as shorthand only. This allows students an introductory focus on S. Experts however do not regard themselves as students who need education; they quickly recognise that this shorthand form causes infinite regress when $x \neq S$, and then they put this analysis aside, disappointed that it contains such an elementary confusion. However, the shorthand only indicates the intuition by Paul of Venice on the Liar paradox, that must be developed into modern consistency for sets. It is rather curious that this intuition doesn't inspire the experts on set theory.

H.3. A shorthand notation with asymmetric '&&'

The use of a shorthand form remains useful, and thus I propose the following notation.

Notation: $V = \{x \mid f[x] \ \&\& \ x \in V\}$, with non-symmetric '&&', stands for the longer $V = \{x \mid f[x]$ unless $(f[x] \ \& \ x \in V)$ is contradictory (also formally, preventing infinite regress)$\}$. Alternatively $V = \{x \mid \mathrm{If}(f[x] \ \& \ x \in V) \Leftrightarrow \mathit{falsum} \text{ then } \mathit{falsum} \text{ else } f[x]\}$ in which the first test can be formal again without infinite regress. In static logic this reduces to $V = \{x \mid f[x] \ \& \ x \in V\}$ but the idea is the dynamic switch, in which it is tested first whether the *Unless*-condition reduces to a falsehood, formally without infinite regress, and if not, then the unprotected original rule *f*[*x*] is applied.

Also: $V = \{\backslash x \mid f[x] \,/\}$ means $V = \{x \mid f[x] \ \&\& \ x \in V\}$.

Example: In the above we could write $S = \{\backslash x \mid (x \notin x) /\}$ - and compare this with R.

H.4. More on the purpose of the paper

An objection to ZFC is that a theory of types forbids the set of all sets while it is a useful concept. For formalisation of an alternative to ZFC there are at least two approaches. One approach is to forbid the formation of R by always requiring the Paul of Venice consistency condition. Alternatively we can allow that R is formally acceptable: then we need a three-valued logic to determine that R is nonsense. (It has meaning, that allows us to see that it is nonsense.) Observe that a theory of types has R in the category 'may not be formed' and thus already implies a 'third category' next to truth and falsehood. It would be illogical to reject such a third category. It is logical instead to generalise that third category to the general notion of 'nonsense'. This gives a three-valued logic with values *true, false, nonsense.* It remains an issue that three-valued logic is not without its paradoxes, but Colignatus (1981, 2007, 2011) holds that these can be solved too.

A closely related issue is what *infinity* actually means. When set theory (with perhaps infinite models) is used to help to explain *infinity* then there might be an infinite number of possible meanings for *infinity.* The real question becomes what would be consistent systems, and what systems might be used for what practical purposes. A critical property of ZFC is that it also allows for *transfinites*, and without models in reality those might be a mere product of nonsense.

The notion of infinity brings us to Cantor's Theorem, in this paper in the form which Bertrand Russell created for the power set (Hart (2015:42 first column)). This theorem would hold in ZFC. It need not hold if we amend ZFC.

Appendix J: Support on Section 2. Review

J.1. Overview of various theorems and the refutations of their proofs

Since this paper refers to various forms of 'Cantor's Theorem' it will be useful to collect them in a table, and see also Hart (2015) and Appendices B & C for a discussion.

Author & date	*Theorem*	*Refutation*
Cantor 1874	Reals are nondenumerable, via intervals	CCPO-PCWA 2012
Cantor 1890/91	Diagonal argument, binary, bijection	CCPO version 2007j
Russell 1907	Power set theorem, using bijection ("common")	ALOE & Section 2
Coplakova et al. 2011	Power set theorem, using surjection ("standard")	Here, Appendix J.3
Hart 2012	Weakest theorem underlying Cantor's Theorem	Here, Appendix C

J.2. Semantics of deproving a theorem

There is an issue on terminology w.r.t. the term 'refutation' in the table. A theorem is refuted by a counterexample. For the natural numbers and the reals we can find $\aleph \sim \Re$ via the notion of *'bijection by abstraction'*, Colignatus (2012, 2013). This uses constructive methods that some might not agree with, and this is not the topic of the present discussion. What is refuted in this paper is Cantor's 'diagonal argument' in the power set form. Perhaps a better term might be 'deproven', in the sense that the theorem is stripped from its proof and no longer can count as a theorem. It may be that the theorem would still hold, but via a different proof. The refutation of the diagonal argument is done by showing that the proof relies on logically improper constructs so that the proof can be rejected as invalid. Saying that 'the proof is rejected' would be too simple because in this realm of discussion - axiomatics - this might suggest that it is a mere act of volition to reject one of the axioms. One might say that the proof is 'invalidated' but this seems uncommon. A proper phrase is that *'it is refuted that the 'proof' would be valid'.* The latter becomes short: *'the proof is refuted'.*

J.3. Cantor's Theorem on the surjection (standard)

The following is from a matricola course in mathematics. When a paper challenges a widely accepted theorem then the reader may require a substantial argument and a detailed reconstruction of the proof. Conventionally it would be necessary to go to the source too. In this case Cantor presented his theorem before ZFC existed, and our focus is rather on the challenge to ZFC. It appears useful to restate the matricola material to show how we arrive at that challenge for ZFC. We take the course that is in use at the universities of Leiden and Delft for students majoring in mathematics. The online syllabus is by Coplakova et al. (2011), and the issue concerns theorem I.4.9, pages 18-19. We translate Dutch into English, also using the proof addendum by Edixhoven in Colignatus (2014a).

Definition (Coplakova et al. (2011:144-145)): ZFC.

Definition (Coplakova et al. (2011:18), I.4.7): Let A be a set. The power set of A is the set of all subsets of A. Notation: $P[A]$. (Another notation is 2^A.)

Cantor's Theorem (for the power set, Hart (2015:42)) (Coplakova et al. (2011:18), I.4.9): Let A be a set. There is no surjective function $f: A \to P[A]$.

Proof (Coplakova et al. (2011:19), replacing their *B* by Ψ, and inserting a [*NB*]): Assume that there is a surjective function $f : A \rightarrow P[A]$. Now consider the set Ψ = $\{x \in A \mid x \notin f[x]\}$. [***NB*** (nota bene): Prove (iii) and (iv) below.]

Since $\Psi \subseteq A$ we also have $\Psi \in P[A]$. Because of the assumption that f is surjective, there is a $\psi \in A$ with $f[\psi] = \Psi$. There are two possibilities: (i) $\psi \in \Psi$ or (ii) $\psi \notin \Psi$.

If (i) then $\psi \in \Psi$. From the definition of Ψ it follows $\psi \notin f[\psi]$ or $\psi \notin \Psi$. Thus (i) gives a contradiction.

If (ii) then we know $\psi \notin \Psi$ and thus also $\psi \notin f[\psi]$. With the definition of Ψ it follows that $\psi \in \Psi$. Thus (ii) gives a contradiction too.

Both cases (i) and (ii) cannot apply, and hence we find a contradiction. Q.E.D.

[*NB*] Addendum for above Proof: (iii) Ψ is in ZFC, (iv) ZFC provides for well-defined sets.

Proof for (iii) (Edixhoven in Colignatus (2014a), appendix D): (a) *P*[*A*] exists because of the Axiom of the Powerset. (b) Note that f can be regarded as a subset of *A* x *P*[*A*]. Then f exists because of Axiom of Pairing. (c) Ψ exists because of the Axiom of Separation. Q.E.D.

Proof for (iv): Not available. This is not proven but remains an assumption.

(Finding an example in reality would be sufficient but might not be necessary. DeLong (1971) explains that an axiomatic system tends to have an 'intended interpretation', so that it is a model for that interpretation. Overall, with an axiomatic system *AS,* the system defines well-defined-ness in its realm. When there is an anomaly *a* for *AS*, so that *AS* & *a* cause a contradiction, then adherents to *AS* will reject *a*, but one must always keep in mind that it is also possible to reject *AS.*)

Comments:

(1) The insertion of [*NB*] is relevant here. Ψ belongs to ZFC because of the Axiom of Separation. (As stated by P.K. Hart in 2012 and B. Edixhoven in 2014, see Colignatus (2014a)(2015b).) Given this addendum, it now should be clearer that above standard proof actually provides a challenge to ZFC. If ZFC allows a paradoxical construct then one may feel that ZFC needs amendment.

(2) From the contradiction derived above, the proper conclusion is not that Cantor's Theorem is proven, but only that it is proven in ZFC. Either Cantor's Theorem is true *or* ZFC doesn't yet provide for well-defined sets.

Rejection of this proof:

See the body of the text Section 2 for the rejection (in ALOE) on the bijection and implied surjection, and see **Appendix C** for its hidden assumption on surjections.

References

Bochenski, I.M. (1956, 1970), *A history of formal logic*, 2nd edition, Chelsea, New York

Colignatus, Th. (1981 unpublished, 2007, 2011), *A logic of exceptions,* (ALOE) 2nd edition, Thomas Cool Consultancy & Econometrics, Scheveningen (PDF of the book online at http://thomascool.eu/Papers/ALOE/Index.html)

Colignatus, Th. (2011), *Neoclassical mathematics for the schools,* http://thomascool.eu/Papers/Math/2011-09-06-NeoclassicalMathematics.pdf

Colignatus, Th. (2012, 2013), *Contra Cantor Pro Occam - Proper constructivism with abstraction*, paper, http://thomascool.eu/Papers/ALOE/2012-03-26-CCPO-PCWA.pdf

Colignatus, Th. (2013), *What a mathematician might wish to know about my work*, http://thomascool.eu/Papers/Math/2013-03-26-WAMMWTKAMW.pdf

Colignatus, Th. (2014a), *Logical errors in the standard "diagonal argument" proof of Cantor for the power set",* memo, http://thomascool.eu/Papers/ALOE/2014-10-29-Cantor-Edixhoven-02.pdf

Colignatus, Th. (2014b), *A condition by Paul of Venice (1369-1429) solves Russell's Paradox, blocks Cantor's Diagonal Argument, and provides a challenge to ZFC,* First versions, http://vixra.org/abs/1412.0235

Colignatus, Th. (2015a), *An explanation for Wigner's "Unreasonable effectiveness of mathematics in the natural sciences*, January 9, http://thomascool.eu/Papers/Math/2015-01-09-Explanation-Wigner.pdf

Colignatus, Th. (2015b), *Review of the email exchange between Colignatus and K.P. Hart (TU Delft) in 2011-2015 on Cantor's diagonal argument and his original argument of 1874,* May 6 (thus limited to up to then), http://thomascool.eu/Papers/ALOE/KPHart/2015-05-06-Review-emails-Colignatus-KPHart-2011-2015.pdf

Colignatus, Th. (2015c), *Reaction to Hart (2015) about Cantor's diagonal argument,* http://thomascool.eu/Papers/ALOE/KPHart/2015-05-08-DerdeWet-KPHart-with-comments-KPH-and-TC.pdf

Colignatus, Th. (2015d), *Background supporting documentation for Appendix B in the paper on Paul of Venice,* http://thomascool.eu/Papers/ALOE/KPHart/2015-05-20-Paul-of-Venice-more-on-Appendix-B.pdf

Colignatus, Th. (2015e), *A breach of scientific integrity since 1980 on the common logical paradoxes*, http://thomascool.eu/Papers/ALOE/2015-05-21-A-breach-of-integrity-on-paradoxes.pdf

Colignatus, Th. (2015f), *Abstraction & numerical succession versus 'mathematical induction',* https://boycottholland.wordpress.com/2015/05/26/abstraction-numerical-succession-versus-mathematical-induction

Coplakova, E., B. Edixhoven, L. Taelman, M. Veraar (2011), *Wiskundige Structuren,* dictaat 2011/2012, Universiteit van Leiden and TU Delft, http://ocw.tudelft.nl/courses/technische-wiskunde/wiskundige-structuren/literatuur

DeLong, H. (1971), *A profile of mathematical logic*, Addison-Wesley

Gill, R.D. (2008), 'Book reviews. Thomas Colignatus. A Logic of Exceptions: Using the Economics Pack Applications of Mathematica for Elementary Logic", *Nieuw Archief voor Wiskunde,* 5/9 nr. 3, pp. 217-219, http://www.nieuwarchief.nl/serie5/pdf/naw5-2008-09-3-217.pdf

Harris, M. (2008). 'Book Review. A Sometimes Funny Book Supposedly about Infinity. A Review of *Everything and More*', *Notices of the AMS*, Vol. 51 (6), pp. 632-638

Hart, K.P. (2013), *Verzamelingenleer,* http://fa.its.tudelft.nl/~hart/37/onderwijs/verzamelingenleer/dictaat/dictaat-A4.pdf

Hart, K.P. (2015), 'Cantors diagonaalargument', *Nieuw Archief voor Wiskunde* 5/16, nr 1, March, pp. 40-43, http://www.nieuwarchief.nl/serie5/pdf/naw5-2015-16-1-040.pdf

Hodges, W. (1998), 'An editor recalls some hopeless papers', *Bulletin of Symbolic Logic*, 4 pp. 1-16

Irvine, A.D. & Deutsch, H. (2014), *Russell's Paradox*, The Stanford Encyclopedia of Philosophy (Winter 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2014/entries/russell-paradox/>. http://plato.stanford.edu/entries/russell-paradox/#RPCL

Wallace, D.F. (2003), *Everything and more. A compact history of* ∞, Norton, New York

Weisstein, E. W. (2015), *Zermelo-Fraenkel Axioms*, From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Zermelo-FraenkelAxioms.html, retrieved May 19

THOMAS COLIGNATUS

Thomas Colignatus is the name in science of Thomas Cool, econometrician (Groningen 1982) and teacher of mathematics (Leiden 2008).

cool a t dataweb.nl

thomascool.eu