A CONDITION BY PAUL OF VENICE (1369-1429) SOLVES RUSSELL'S PARADOX, BLOCKS CANTOR'S DIAGONAL ARGUMENT, AND PROVIDES A CHALLENGE TO ZFC

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Abstract

Paul of Venice (1369-1429) provides a consistency enhancer that resolves Russell's Paradox in naive set theory without using a theory of types. It allows a set of all sets. It also blocks the (diagonal) general proof of Cantor's Theorem on the power set. It is not unlikely that the Zermelo-Fraenkel (ZFC) axioms for set theory are still too lax on the notion of a "well-defined set". The transfinites of ZFC may be a mirage of still imperfect axiomatics w.r.t. the proper foundations for set theory.

Keywords: Paul of Venice • Russell's Paradox • Cantor's Theorem • ZFC • naive set theory • well-defined set • set of all sets • diagonal argument • transfinites

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1. Introduction

In naive set theory, Russell's set is $R = \{x \mid x \notin x\}$. Subsequently $R \in R \Leftrightarrow R \notin R$ and naïve set theory collapses. Define hoewever $S = \{x \mid x \in S \ \& \ x \notin x\}$ with the small consistency enhancer inspired by the discussion by Bochenski (1956, 1970:250) of Paulus Venetus or Paul of Venice (1368-1428). Without contradiction we find $S \notin S$. There is no reason for a crisis in the foundations of logic and mathematics and there is no need for a theory of types, but we merely need a three-valued logic to determine that R is nonsense - though it has meaning (that allows us to see that it is nonsense). Observe: (1) a theory of types forbids the set of all sets while it is a useful concept, (2) a theory of types has R in the category "may not be formed" and thus already implies a "third category" next to truth and falsehood. It would be illogical to reject that third category. It is logical instead to generalize that third category to the general notion of "nonsense". It remains an issue that three-valued logic is not without its paradoxes, but Colignatus (1981, 2007, 2011) holds that these can be solved too.

Colignatus (1981, 2007, 2011:239) already presented the Paul of Venice consistency enhancer, and applied it to Cantor's diagonal argument for the power set. The new issue in this paper is the challenge to the ZFC axioms, i.e. of Zermelo, Fraenkel and the axiom of choice. These may be still too lax on the notion of a "well-defined set". The transfinites of ZFC may be a mirage of still imperfect axiomatics w.r.t. the foundations for set theory.

Section 2 reviews that diagonal argument, section 3 gives the challenge to ZFC.

2. Review of the standard proof of Cantor's Theorem

The following essentially copies the discussion in Colignatus (1981, 2007, 2011:239). See the Appendix for some comments for accurate reference.

Sets A and B have "the same size" when there is a bijection or one-to-one function between them. The power set $P[A]$ or $\mathbf{2}^{\mathrm{A}}$ of set A is the set of all subsets of A. Cantor's Theorem holds that a set is always "smaller" than its power set. For finite sets this can be proven by mathematical induction too. The standard proof, and in particular for infinite sets, uses a construction that strongly reminds of Russell's paradox (deconstructed in section 1 above).

Colignatus (1981, 2007, 2011) (ALOE) is on logic and inference and thus keeps some distance from number theory and issues of the infinite. Historically, logic developed parallel to geometry and theories of the infinite (Zeno's paradoxes). Aristotle's syllogisms with *All, Some* and *None* helped to discuss the infinite. Yet, to develop logic and inference proper, it appeared that ALOE could skip the tricky bits of number theory, non-Euclidean geometry, the development of limits, and Cantor's development of the transinfinite. Though it is close to impossible to discuss logic without mentioning the subject matter that logic is applied to, ALOE originally kept and keeps some distance from those subjects themselves. But, if logic uses the notion of "all", it seems fair to ask whether there are limits to the use of this "all".

It must also be observed that this author is no expert on Cantor's Theorem. We may reject the standard proof but perhaps there are other proofs. A marginal check shows that this proof is the only one given at various locations that seem to matter but this may only mean that it is a popular proof. For now, we reproduce that standard proof and the refutation using the Paul of Venice consistency criterion, following Colignatus (1981, 2007, 2011:239).

2.1. Cantor's theorem and the standard proof (in ALOE)

Regard an arbitrary set A. Let f: $A \rightarrow \mathbf{2}^{\mathrm{A}}$ be the hypothetical bijection between A and its power set. Let $\Phi = \{x \in A \mid x \notin f[x]\}$. Clearly Φ is a subset of A and thus there is a $\varphi = f^{-1}[\Phi]$ so that $f[\varphi] = \Phi$. The question now arises whether $\varphi \in \Phi$ itself. We find that $\varphi \in \Phi \Leftrightarrow \varphi \notin f[\varphi] \Leftrightarrow \varphi \notin \Phi$ which is a contradiction. Ergo, there is no such f. This concludes the standard proof of Cantor's theorem. For the discussion below, relabel Φ in this paragraph into Φ'.

Colignatus (2014) records a supplement to this proof provided by professor Edixhoven of Leiden that Φ belongs to ZFC because of the axiom of separation (see below). As Colignatus asked Edixhoven to provide this supplement, it now should be clearer that above standard proof actually provides a challenge to ZFC. If ZFC allows such a paradoxical construct then ZFC needs amendment.

The subsequent discussion intends to show that this proof cannot be accepted.

2.2. Rejection of this proof (in ALOE)

We might hold that above Φ is badly defined since it is self-contradictory under the hypothesis. A badly defined "something" may just be a weird expression and need not represent a true set. A test on this line of reasoning is to insert a small consistency condition, giving us $\Phi = \{x \in A \mid x \notin f[x] \;\&\; x \in \Phi\}$. Now we conclude that $\varphi \notin \Phi$ since it cannot satisfy the condition for membership, i.e. we get $\varphi \in \Phi \Leftrightarrow (\varphi \notin f[\varphi] \;\&\; \varphi \in \Phi) \Leftrightarrow (\varphi \notin \Phi \;\&\; \varphi \in \Phi) \Leftrightarrow$ *falsum*. Puristically speaking, the Φ defined in 2.1 differs lexically from the Φ defined here, with the first expression being nonsensical and the present one consistent. It will be useful to reserve the term Φ for the proper definition in 2.2, and use Φ' for the expression in 2.1. The latter symbol is part of the lexical description but does not meaningfully refer to a set. Using this, we can also use $\Phi^* = \Phi \cup \{\varphi\}$ and we can express consistently that $\varphi \in \Phi^*$. So the "proof" above can be seen as using a confused mixture of Φ and Φ^*.

3. The challenge to ZFC

3.1. What is the difference between Φ' in 2.1 and Φ in 2.2 ?

Above deduction in section 2 poses a challenge to ZFC. Sets R and S above were in naive set theory, so it has little meaning - for now - to ask about the difference between R and S. However, Φ' in 2.1 and Φ in 2.2 are in ZFC, and thus the question is meaningful. Users of ZFC will have a hard time trying to clarify (a) that the consistency enhancer should have no effect but (b) actually does have an effect. To answer the question we would use the axiom of extensionality, see Coplakova et al. (2011:145):

$(A = B) \Leftrightarrow ((\forall x)\ (x \in A \Leftrightarrow x \in B))$

My solution of this issue is that 2.1 is badly defined and that 2.2 is well-defined. This comes with the benefit of the collapse of the standard proof to Cantor's theorem. I am interested in an argument to the contrary but haven't seen it yet.

3.2. Amendments to the Axiom of Separation in ZFC

The proof relies on the separation axiom in ZFC, see Coplakova et al. (2011:145):

If A is a set and $\varphi[x]$ is a formula with variable x, then there exists a set B that consists of the elements of A that satisfy $\varphi[x]$, while B is not free in $\varphi[x]$:

$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \Leftrightarrow ((x \in A)\ \&\ \varphi[x]))$

Note the condition "B is not free in $\varphi[x]$". The consistency enhancer by Paul of Venice in the definition of Φ in 2.2 uses $\varphi'[x] = (\varphi[x]\ \&\ (x \in \Phi))$, in which $B = \Phi$ is not free since it is bound by the existential quantifier $(\exists B)$. Thus the formation of Φ in 2.2 is allowed in ZFC.

To meet the challenge in 3.1 we would require the enhancer for all sets.

Possibility 3.2.1: Amendment by Paul of Venice to the Axiom of Separation:

$(\forall A)\ (\exists B)\ (\forall x)\ (x \in B \Leftrightarrow ((x \in A)\ \&\ \varphi[x]\ \&\ (x \in B)))$

In this case, the proof for Cantor's theorem collapses, and question 3.1 disappears since Φ' becomes non-well-formed and nonsensical. My suggestion is to call this the "neat" solution, and use the abbreviation ZFC-PV.

Another possibility is to move from ZFC closer to naive set theory, discard the axiom of separation, and adopt an axiom that allows greater freedom to create sets from formulas.

Possibility 3.2.2: Discard the separation axiom and have extensionality of formula's:

$(\forall \varphi)\ (\exists B)\ (\forall x)\ (x \in B \Leftrightarrow ((x \in B)\ \&\ \varphi[x]))$

This axiom also protects against Russell's paradox. It destroys the standard proof of Cantor's theorem. It would also allow for a set of sets - which for ZF (without C) would also require to drop the Axiom of Regularity, that forbids that sets are member of themselves. My suggestion is to call this the "basic" solution, and use the abbreviation BST (basic set theory). I would also propose a rule that the enhancer could be dropped in particular applications if it could be shown to be superfluous. However, for paradoxical $\varphi[x]$ it would not be superfluous.

I am not aware of a contradiction yet. I have not looked intensively for such a contradiction, since my presumption is that others are better versed in set theory and that the problem only is that those authors aren't aware of the potential relevance of the consistency enhancer by Paul of Venice. A question for historians is: Zermelo (1871-1953) and Fraenkel (1891-1965) might have embraced Paul of Venice's enhancer if they had been aware of it.

4. Conclusion

Colignatus (1981, 2007, 2011) concludes, and we now supplement with the questions on ZFC:

1. The standard proof for Cantor's Theorem (given above) is based upon a badly defined and inherently paradoxical construct. This proof evaporates once a sound construct is used.

2. The theorem is still unproven for (vaguely defined) infinite sets: that is, this author is not aware of other proofs. We would better speak about "Cantor's Impression" or "Cantor's Supposed Theorem". It is not quite a conjecture since Cantor might not have done such a conjecture (without proof) if he would have known about above refutation.

3. It becomes feasible to speak again about the "set of all sets". This has the advantage that we do not need to distinguish (i) sets versus classes, (ii) "all" versus "any".

4. The transfinites that are defined by using "Cantor's Theorem" evaporate with it.

5. The distinction between the natural and the real numbers now rests (only) upon the specific diagonal argument (that differs from the standard proof). See Colignatus (2012, 2013) for the conclusion that Cantor's original proof for the natural and real numbers evaporates too, specifically for a convenient level of constructivity.

6. Users of ZFC should give an answer to 3.1, and clarify why they accept 2.1 and not 2.2 that has a better definition of a well-defined set. ZFC might be consistent but allows the construction of a "proof" for "Cantor's Impression" that generates the transfinites, which makes one wonder what this system is a model for. We can agree with Cantor that the essence of mathematics lies in its freedom, but the freedom to create nonsense somehow would no longer be mathematics proper. Alternatives are in ZFC-PV or BST.

7. The prime importance of this discussion lies in education. Mathematics education should respect that education is an empirical issue. Teachers should respect the logic that students can grasp and not burden mathematics education with the confusions of the past. Colignatus (2012, 2013) clarifies that highschool education could be served well with a theory of the infinite that consistently develops both the natural and real numbers, without requiring more than the denumerable infinite ($\aleph \sim \mathfrak{R}$). It was Cantor himself who emphasised the freedom in mathematics, but that freedom is limited if alternatives are not mentioned. Even a university course like Coplakova et al. (2011) currently presents students only with "Cantor's Theorem" without mentioning the alternative analysis in Colignatus (1981, 2007, 2011).

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Appendix

The following comments are relevant for accurate reference.

(1) Colignatus (1981, 2007, 2011) existed first unpublished in 1981 as *In memoriam Philetas of Cos,* then in 2007 rebaptised and self-published both retyped and programmed in the computer-algebra environment of *Mathematica* to allow ease of use of three-valued logic. In 2011 it was marginally adapted with a new version of *Mathematica*. At that moment it could also refer to a new rejection of Cantor's particular argument for the natural and real numbers, using the notion of *bijection by abstraction* - in 2011 still called *bijection in the limit*.

(2) Gill (2008) reviewed the 1st edition of ALOE of 2007. That edition refers to Cantor's standard set-theoretic argument and rejects it, as in the above. ALOE refers to Wallace (2003) as the book that caused me to look into the issue again. Wallace's book is critically reviewed by Harris (2004). It will be useful to mention that ALOE does not rely on Wallace's book but indeed only mentions it as a source of inspiration to look into the issue again.

(3) Gill (2008) did not review the 2nd edition of ALOE of 2011. That edition also refers to Cantor's original argument on the natural and real numbers in particular. That edition of ALOE mentions the suggestion that $\aleph \sim \Re$. The discussion itself is not in ALOE but is now in Colignatus (2012, 2013), using the notion of *bijection by abstraction*.

(4) A visit to a restaurant and subsequent e-mail exchange led to the memo Colignatus (2014), and the inspiration to write this present article on the challenge to ZFC. Edixhoven also refers to Coplakova et al. (2011), theorem 1.4.9, pp. 18-19, that gives the standard theorem and proof, also reproduced and challenged in above section 2.

(5) Colignatus (1981, 2007, 2011) is a book on logic and not a book on set theory. It presents the standard notions of naive set theory (membership, union, intersection) and the standard axioms for first order predicate logic that of course are relevant for set theory. But I have always felt that discussing *axiomatic* set theory (with ZFC) was beyond the scope of the book and my actual interest and developed expertise. This present paper is in my sentiment rather audacious by discussing axiomatic set theory in section 3 and actually presenting two possible revisions for the axiom of separation.

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