Jay R. Yablon
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Dirac-Wu-Yang Monopoles, Gauge Symmetry, Orientation-Entanglement & Twist, Quantum Topology, and the Fractional Quantum Hall Effect

Jay R. Yablon
910 Northumberland Drive
Schenectady, New York 12309-2814
jyablon@nycap.rr.com

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Abstract: The purpose of this paper is to explain the pattern of fill factors observed in the Fractional Quantum Hall Effect (FQHE), which appears to be restricted to odd-integer denominators as well as the sole even-integer denominator of 2. The method is to use the mathematics of gauge theory to develop Dirac monopoles without strings as originally taught by Wu and Yang, while accounting for orientation / entanglement and related "twistor" relationships between spinors and their environment in the physical space of spacetime. We find that the odd-integer denominators are included and the even-integer denominators are excluded if we regard two fermions as equivalent only if both their orientation and their entanglement are the same, i.e., only if they are separated by 4π not 2π. We also find that the even integer denominator of 2 is permitted because unit charges can pair into boson states which do not have the same entanglement considerations as fermions, and that all other even-integer denominators are excluded because only integer charges, and not fractional charges, can be so-paired. We conclude that the observed FQHE fill factor pattern can be fundamentally explained using nothing other than the mathematics of gauge theory in view of how orientation / entanglement / twist applies to fermions but not to bosons, while restricting all but unfractionalized fermions from pairing into bosons.

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Contents

1. Introduction: Wu and Yang and the Dirac Monopole without Strings

The Fractional Quantum Hall Effect (FQHE) observed in two-dimensional systems of electrons at low temperatures in superconducting materials subjected to large magnetic fields is characterized by observed filling factors $\nu = n/m$, where *n* and *m* are each integers, but where *m* is an *odd integer only*, with the exception that *m may also be the even integer 2*. In other words, the apparent pattern, widely reported and studied in the literature, is $\nu = n/m$ with $n = 0, \pm 1, \pm 2, \pm 3...$ and $m = 1, 2, 3, 5, 7, 9, 11...$, see, e.g., [1], [2], [3], [4], [5], [6] generally, and for the even denominator 2, see, e.g., [6], [7] for $\nu = 1/2$, [8] for $\nu = 3/2$, [9] for $\nu = 5/2$ and [10] for $\nu = 7/2$. Two questions arise from this effect: why are the denominators in the filling factor odd but not even (including the quantization of whole unit charges with *m*=1), and why is the even denominator *m*=2 an apparent exception? We show that this pattern of filling factor denominators has a fundamental explanation based on using the mathematics of gauge theory to develop the Dirac Quantization Condition (DQC) for Dirac-Wu-Yang monopoles, in view of how orientation-entanglement applies to fermion spinors but not to bosons, and also in view of a "twisting" associated with orientation-entanglement which appears to have been underreported in the literature.

In 1931 Dirac discovered that the existence of magnetic monopoles implies that the electric charge must be quantized [11]. While charge quantization had been known for several decades based on the experimental work of Thompson [12] and Millikan [13], Dirac was apparently the first to lay out a possible theoretical imperative for this quantization. Using a hypothesized solenoid of singularly-thin width known as the Dirac string to shunt magnetic field lines out to mathematical infinity, Dirac established that a magnetic charge strength μ would be related to the electric charge strength *e* according to $e\mu = 2\pi n$, where *n* is an integer, which became known as the Dirac Quantization Condition (DQC). Subsequently, Wu and Yang used gauge potentials, which are locally- but not globally-exact, to obtain the exact same DQC without strings [14], [15]. Their approach is concisely summarized by Zee on pages 220-221 of [16] and will be briefly reviewed here, because it provides the methodological basis for understanding the pattern of filling factors observed for the FQHE. Throughout we use the natural units of $\hbar = c = 1$.

Using the differential one form $A = A_\mu dx^\mu$ for the electromagnetic gauge field a.k.a. potential and the differential two-form $F = \frac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu = dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu$, a magnetic charge μ may be *defined* as the total net magnetic flux $\mu \equiv \oiint F$ passing through a closed two-dimensional surface S^2 which for convenience and symmetry we may take to be a sphere. Differential exterior calculus in spacetime geometry teaches that the exterior derivative of an exterior derivative is zero, *dd*=0, which means that the three-form equation $dF = ddA = 0$. Thus, via Gauss / Stokes, $\iiint 0 = \iiint dF = \oiint F = \mu$. In classical electrodynamics prior to Dirac, this was taken to mean that the magnetic charge μ=0. But a close consideration of gauge symmetry, which is locally but not globally exact, tells a different story.

When a spin ½ fermion wavefunction (which we may regard to be that of the electron) undergoes a local gauge (really, phase) transformation $\psi(x) \rightarrow \psi'(x) = e^{i\Lambda(x)}\psi(x)$, the gauge field one-form transforms as

$$A \rightarrow A' = A + e^{-i\Lambda} de^{i\Lambda} / ie \,. \tag{1.1}$$

If we represent F in polar coordinates (r, φ, θ) as $F = (\mu / 4\pi)\, d\cos\theta d\varphi$, then because $F = dA$ and dd=0, we can deduce that $A = (\mu / 4\pi)\cos\theta d\varphi$. However, $d\varphi$ is not defined on the north and south poles. So we may define a north coordinate patch over which $A_N = (\mu / 4\pi)(\cos\theta - 1)\, d\varphi$ and a south patch over which $A_S = (\mu / 4\pi)(\cos\theta + 1)\, d\varphi$. But at places where these patches overlap, these gauge potentials are not the same, and specifically, the difference is $A_S - A_N = (\mu / 2\pi)\, d\varphi$, or written slightly differently:

$$A_N \rightarrow A'_N \equiv A_S = A_N + (\mu / 2\pi)\, d\varphi. \tag{1.2}$$

So comparing this with (1.1), we may regard A_S as a gauge-transformed state A'_N of A_N, for which the gauge transformation is simply:

$$\frac{1}{ie} e^{-i\Lambda} de^{i\Lambda} = \frac{\mu}{2\pi} d\varphi \,. \tag{1.3}$$

This differential equation for Λ and φ in relation to e and μ is satisfied by:

$$\exp(i\Lambda) = \exp\left(ie\mu \frac{\varphi}{2\pi} \right), \tag{1.4}$$

as can be seen simply by plugging $e^{i\Lambda}$ from (1.4) into the left hand side of (1.3) and reducing. This relates the azimuth angle φ which is one of the three spacetime coordinates, to the local gauge (phase) angle Λ, and thereby connects rotations through φ in physical space to rotations through Λ in the gauge space in a manner that we shall now explore in detail.

In polar coordinates, $\varphi = 0$ and $\varphi = 2\pi$ in (1.4) describe exactly the same *orientation* (but not entanglement) on the surface of S^2. So to make sense of (1.4) at like-orientations, we must have:

$$\exp(i\Lambda) = \exp(ie\mu \cdot 0) = 1 = \exp(ie\mu \cdot 1), \tag{1.5}$$

Specifically, this means that $\exp(ie\mu) = 1$. Mathematically, the general solution for an equation of this form is $\exp(i2\pi n) = 1$ for any integer $n = 0, \pm 1, \pm 2, \pm 3...$, which is infinitely degenerate but quantized. As a result, the solution to (1.5) is:

$$\Lambda = e\mu = 2\pi n\,. \tag{1.6}$$

This, of course, is the Dirac Quantization Condition, which may also specified in relation to the gauge (phase) parameter Λ which is seen to be an quantized integer multiple of 2π. Specifically, (1.6) with simple rearrangement tells us that the electric charge is quantized according to:

$$e = n\frac{2\pi}{\mu} = ne_{\text{u}} = \frac{\Lambda}{\mu}, \tag{1.7}$$

where the n=1 "unit" (u) of electric charge is $e_{\text{u}} \equiv 2\pi/\mu$, defined as 2π times the inverse of the magnetic charge. The customary interpretation of $e = n\left(2\pi/\mu\right)$ in (1.7), ever since Dirac first found this relationship, is that if this magnetic charge "exists," then the electric charge is quantized in units of e_{u}. It is important to keep in mind that the converse is not true: the observed quantization of electric charge does *not* imply that the magnetic charges do exist. In fact, as best as is known, this magnetic charge μ has not been observed to date, and the quantization of electric charge is explained not on the basis of these Dirac-Wu-Yang magnetic charges, but on the basis of the charge generators $Q = Y/2 + I^3$ which emerge in Yang-Mills gauge theory following the electroweak symmetry breaking of $SU(2)_W \times U(1)_Y$ down to $U(1)_{em}$.

We may then go back to the original definition $\mu \equiv \oiint F$ and use (1.6) to write:

$$\oiint F = \mu = \frac{2\pi}{e}n = n\mu_{\text{u}} = \frac{\Lambda}{e}, \tag{1.8}$$

where we also define an n=1 unit of magnetic charge $\mu_{\text{u}} \equiv 2\pi/e$, similarly quantized. By appropriate local gauge transformation, and specifically by choosing n=0 which is the same as choosing the phase angle $\Lambda = 0$, the nonzero surface integral can be made to vanish, $\oiint F = 0$. But this does not invalidate (1.7) and (1.8) nor does it prevent us from seeking to draw physical conclusions from these. It simply means that n=0 with no monopoles and no electric charges is one of the permitted states. But again, the meaning of the whole range of charges $e = ne_{\text{u}}$ for $n \neq 0$ has been physically-interpreted since Dirac, as suggesting that the "existence" of a magnetic charge would imply electric charge quantization, with the further understanding that the converse is not true.

In the lowest positive non-zero n=1 state, from (1.6), we have $\Lambda = e\mu = 2\pi$. So if we define a reduced $\text{Ƛ} \equiv \Lambda/2\pi$, then by (1.6), the reduced gauge parameter $\text{Ƛ} = n$ is simply the charge quantum number n. So every gauge transformation adding an angle of 2π adds one unit of electric and magnetic charge. Thus, although $\Lambda = 2\pi, 4\pi, 6\pi...$ are *mathematically*-equivalent angles, they do appear (if these monopoles "exist") to be *physically*-distinguishable because of their connection $\text{Ƛ} = n$ to charge quantization. This is an important observation which will show itself in a number of ways throughout the forthcoming development. It should also be

noted that using the local angle $\varphi_0(x)=0$ and $\varphi_1=\varphi_0+2\pi$ in (1.4), $\varphi_0=0$ had no special physical significance. One could have used any other $0<\varphi_0<2\pi$ or indeed any φ_0 whatsoever and still ended up with the exact same DQC in (1.6); $\varphi_0=0$ was merely the easiest mathematical choice. This all means that the DQC (1.6) is invariant under local gauge symmetry, as it must be to have possible physical meaning.

This is how Wu and Yang obtain Dirac monopoles and the DQC without strings.

2. The Fractional Denominators Indicated by Dirac-Wu-Yang: are they Somehow Related to FQHE?

If we closely study this derivation by Wu and Yang, we see that there are some additional quantum states indicated that have not yet been considered. Referring to (1.5), not only do $\varphi=0$ and $\varphi=2\pi$ describe exactly the same *orientation* (sans entanglement), but so too do $\varphi=4\pi$, $\varphi=6\pi$, $\varphi=8\pi$, etc. So we may now extend (1.5) to:

$$\exp\left(i\Lambda\right)=1=\exp\left(ie\mu\cdot 1\right)=\exp\left(ie\mu\cdot 2\right)=\exp\left(ie\mu\cdot 3\right)=\exp\left(ie\mu\cdot 4\right)=\exp\left(ie\mu\cdot 5\right)=\exp\left(ie\mu\cdot 6\right)....(2.1)$$

Each of the above is a separate relationship of the general form $\exp\left(ie\mu\cdot l\right)=1$, where $l=1,2,3,4,5,6...=\varphi/2\pi$ is an integer not the same as the *n* already in use and corresponds to the number of azimuth windings. At the same time, as noted after (1.5), the general solution for an equation of this form is $\exp\left(i2\pi n\right)=1$ with this integer $n=0,\pm1,\pm2,\pm3....$ Comparing $\exp\left(ie\mu\cdot l\right)=1$ with $\exp\left(i2\pi n\right)=1$ means that more generally, $e\mu\cdot l=2\pi n$, or, restated (also using $\Lambda=2\pi n$ from (1.6)):

$$e=\frac{n}{l}\frac{2\pi}{\mu}=\frac{n}{l}e_{\text{u}}=\nu e_{\text{u}}=\frac{\Lambda}{l}\frac{1}{\mu},\tag{2.2}$$

where we define a "filling factor"

$$\nu\equiv\frac{n}{l};\quad n=0,\pm1,\pm2,\pm3...;\quad l\equiv\varphi/2\pi=1,2,3,4,5,6....\tag{2.3}$$

So this tells us that if these monopoles "exist," not only is the electric (and magnetic) charge quantized, but each unit of electric charge e_{u} (or magnetic charge μ_{u}) can be fractionalized into any $\nu=n\cdot\left(1/l\right)$ quantized *n* fraction $1/l$ of itself. As with (1.6) this relationship is locally gauge invariant.

Taking (1.7) together with (2.2) and (2.3), this means that if the Dirac-Wu-Yang monopole "exists," then all particles carrying electromagnetic charge must obey (1.7). But they will also obey the more permissive conditions of (2.2) which lead to fractionalized charges. We

then see, that (1.7), $e = ne_{\text{u}}$, is a special case of (2.2) and (2.3), $e = \left(n / l \right) e_{\text{u}}$ with $l = 1, 2, 3, 4, 5, 6...$, in the particular circumstance where l=1.

Now, one may take the view that (2.2) and (2.3) are just a "trivial" extension of (1.7), because mathematically, it is certainly trivial that the angles $\varphi = 2\pi$, 4π, 6π, 8π, etc. have the same trigonometric properties as the angle $\varphi = 0$, and rotational orientation is indistinguishable as between any set of angles $\varphi = \varphi_0 + 2\pi n$ with $n = 0, \pm 1, \pm 2, \pm 3...$ differing from some base angle φ_0 by an integer multiple of 2π. And this may explain why (2.2) and (2.3) do not appear to have been developed or studied in the literature to nearly the same degree as the DQC. But in the context of (2.2) and (2.3), the fact that this trivial trigonometric concurrence of $\varphi = 0$ with $\varphi = 2\pi$, 4π, 6π, 8π leads to fractional charges is anything but trivial: Just as the DQC motivates us to consider whether the Dirac monopoles exist and if so in what fashion and under what circumstances, the logical extension of the DQC via Wu and Yang motivates us to ask the same questions about fractional charges which we ask about Dirac monopoles, because (2.2) and (2.3) package all of these questions inseparably together: If these monopoles exist, then *gauge theory itself* inexorably implies that electric charge is quantized, *and also that electric charge is fractionalized*. This means that the DQC is really a DQFC, Dirac Quantization *and Fractionalization* Condition. And although this emanates from something that is mathematically trivial, namely the trigonometric indistinguishability of rotational angles $\varphi = 2\pi$, 4π, 6π, 8π etc. from one another, the fact that gauge theory tells us that the DQC is really a DQFC, is highly non-trivial and must be explored.

So, given that the Dirac-Wu-Yang arguments do lead not only to charge quantization, but inexorably to the *quantization of fractionalized charges*, i.e., to filling factors $\nu = n \cdot \left(1 / l \right)$, and given that fractionalized charges *are experimentally observed* in the FQHE, one is motivated to explore the question whether there is a connection between the two. So we now pose the question which will be the subject of the remainder of this paper: might the DQFC (2.2) and (2.3) actually be related in some way to the FQHE? And if so, how?

For there to be a valid connection between the fractionalized charge states of Dirac-Wu-Yang monopoles and the fractionalized quasiparticle states of FQHE, there are two main problems that must be overcome, one experimental and arithmetical, the other theoretical and physical. First, *experimentally and arithmetically*, while the fractional charge denominators permitted by Dirac-Wu-Yang in (2.3) may assume any integer value $l = 1, 2, 3, 4, 5, 6...$, the denominators *observed* in the FQHE are more restricted: they only take the values $l = 1, 2, 3, 5, 7, 9...$. That is, the observed denominators are always odd integers, with the exception that the even integer $l = 2$ is also observed, see again, [6], [7], [8], [9], [10]. So, the Dirac-Wu-Yang approach – if it applies at all to FQHE – must be able to explain this arithmetically-restricted experimental pattern of observed denominators. The result in (2.3) is simply too inclusive, i.e., it includes states which are not observed alongside states which are.

Second, *theoretically and physically*, even if the denominator pattern $l = 1, 2, 3, 5, 7, 9...$ can be explained, applying the Dirac-Wu-Yang arguments to the fractionally charged

quasiparticles in FQHE systems is still physically nontrivial. The Dirac-Wu-Yang theoretical argument is developed within the three-dimensional physical space of spacetime geometry, and is understood to apply to systems of electrons, protons, and neutrons for which no fractionally charged particles and no Dirac-Wu-Yang magnetic monopoles have ever been observed. But at the level of analysis where the quasiparticle language applies, the system is fundamentally two-dimensional, because the superconducting materials used together with the ultra-low temperatures and large magnetic fields applied to stimulate the observed FQHE, combine in some fashion to substantially remove one degree of spatial freedom from the electrons and so restrict the electrons to two space dimensions. And in some way that needs to be understood, these all synergistically coact to produce the $l = 1, 2, 3, 5, 7, 9...$ denominator pattern. Because of this difference between the three-dimensional space of Dirac-Wu-Yang and the two-dimensional restricted space of FQHE, one might take the *a priori* view that there is no connection between Dirac-Wu-Yang and FQHE. So for certain, at the very least, if there is some hidden, not-yet-understood connection between these two fundamentally-different environments of Dirac-Wu-Yang and FQHE, it is important for such a connection to be carefully developed and articulated.

3. Orientation-Entanglement, and the Odd-Integer FQHE Denominators

Spinors, which includes electrons, reverse sign upon a spatial rotation through an angle φ by an odd multiple of 2π. Specifically, as Misner, Thorne and Wheeler (MTW) point out in one of the most widely-regarded discussions of this topic in [17] at section 41.5, the spin matrix of a rotation $R = \cos(\varphi/2) - i(\mathbf{n}\cdot\boldsymbol{\sigma})\sin(\varphi/2)$ (see MTW [41.48]) reverses sign upon a rotation through an odd multiple of 2π, as does the sign of a spinor under $\xi \to \xi' = R\xi$ (MTW [41.50]). This sign reversal does not, however, appear in the transformation law for a vector, $X \to X' = RXR^*$ (MTW [41.49]).

Misner, Thorne and Wheeler provide a visual, macroscopic, intuitive, essentially-topological understanding for this result by considering the orientation and entanglement of an object relative to its surrounding environment, because while orientation is restored under a 2π rotation, it takes a 4π rotation to restore the object's state of entanglement, i.e., to restore the complete "version" of the object. They do, however, at page 1148 of [17], make the statement:

> "Whether there is also a detectable difference in the physics . . . for two inequivalent versions of an object is not known."

This question of whether MTW orientation-entanglement brings about detectable physics in the study of physical systems will occupy a fair share of the analysis to follow in this paper.

Now, the gauge transformation $\psi(x) \to \psi'(x) = e^{i\Lambda(x)}\psi(x)$ with which we originally started at (1.1) acts on a Dirac fermion wavefunction which we may take to be that of an electron. And electrons are Dirac spinors. So as such, the sign of this wavefunction will reverse under any rotation from a given φ to $\varphi + 2\pi$ and will only be restored under two rotations, i.e., when rotated from φ to $\varphi + 4\pi$. Therefore, let us suppose that some weight needs to be ascribed to the *version* of the electron and not only its orientation, and therefore revisit (2.1) where we equated the entire set of rotations differing from one another by only 2π, not 4π.

Specifically, let us now explore the consequences of taking the more-restrictive view that two terms $\exp\left(ie\mu\left(\varphi/2\pi\right)\right)$ in (1.4) may be equated *if and only if they differ from one another by* 4π. Then, let us start with a Dirac fermion in the $\varphi=2\pi$ state, which as seen in the derivation leading to (1.6) is the $n=1$ state for which the DQC gives $e\mu=2\pi$. Thus, for this state, we have $\exp\left(ie\mu\left(\varphi/2\pi\right)\right)=\exp\left(i2\pi\left(2\pi/2\pi\right)\right)=1$. Then, because we are starting out with a fermion oriented to $\varphi=2\pi$, the equivalent *versions* will be those for which $\varphi=6\pi$, $\varphi=10\pi$, $\varphi=14\pi$, etc. As a result, when we apply this entanglement restriction to (1.4), then in lieu of (2.1), we now obtain:

$$\exp\left(i\Lambda\right)=1=\exp\left(ie\mu\frac{\varphi}{2\pi}\right)=\exp\left(ie\mu\cdot 1\right)=\exp\left(ie\mu\cdot 3\right)=\exp\left(ie\mu\cdot 5\right)=\exp\left(ie\mu\cdot 7\right)\ldots. \tag{3.1}$$

So this consideration of entanglement in addition to orientation naturally discards all of the even-numbered states such as $\exp\left(ie\mu\cdot 2\right)$, $\exp\left(ie\mu\cdot 4\right)$, etc.

Each of the above is now a separate relationship $\exp\left(ie\mu\left(1+2l\right)\right)=1$ where *l* continues to be an integer with the values $l=0,1,2,3...$, so that $2l+1=1,3,5,7...$ is an *odd integer*. Comparing $\exp\left(ie\mu\left(1+2l\right)\right)=1$ with the mathematical relationship $\exp\left(i2\pi n\right)=1$ means that $e\mu\left(1+2l\right)=2\pi n$, or, restated:

$$e=\frac{n}{1+2l}\frac{2\pi}{\mu}=\frac{n}{1+2l}e_{\text{u}}=\nu e_{\text{u}}, \tag{3.2}$$

with a redefined filling factor:

$$\nu=\frac{n}{1+2l};\quad n=0,\pm 1,\pm 2,\pm 3...;\quad l=0,1,2,3.... \tag{3.3}$$

In contrast to (2.3), this filling factor *will always have an odd denominator*. And with the exception of the even denominator 2, this is what is observed in FQHE.

This means that if regard two electron states to be equivalent in (1.4) only if they have the same version i.e., differ from one another by 4π not 2π, then all of the fill factors will only have an odd-integer denominator, and all even-integer denominators will be excluded because they involve inequivalent versions. Other than the even denominator of 2, this fully accords with the fractional charges states observed in FQHE. So other than the question of the even-numbered denominator of 2, this does solve the *experimental and arithmetical problem* of matching the observed fractional charge denominators. But because of the role of entanglement in reaching this result, we are now led to entertain not only whether the FQHE is in some way a physical manifestation of the Dirac Quantization *and Fractionalization* Condition, but whether

the *odd-integer* denominators in the FQHE are "also a detectable difference in the physics . . . for two inequivalent versions of an object," which is the question Misner, Thorne and Wheeler have posed. In short: is FQHE an experimental, detectable manifestation not only of Dirac-Wu-Yang monopoles and fractionalization, *but also a detectable physical manifestation of MTW spinor orientation and topological entanglement*? And if so, how?

For ordinary systems of electron and protons and neutrons in which the electrons are unconstrained to two-dimensions by any materials and / or low temperatures and / or large magnetic fields and so are operating in three-dimensional space, neither fractionalization nor Dirac monopoles nor signs of orientation and entanglement appear to be physically detectable. So as to the *theoretical and physical problem*, the question is whether in some way, and if so how, the physical constraints imposed by certain superconducting materials and low temperatures and high magnetic fields create an environment in which fractionalization and entanglement and even the Dirac-Wu-Yang monopoles themselves are suddenly forced to make a physical appearance.

As to the Dirac-Wu-Yang monopoles, we keep in mind that while the high energies of Grand Unified Theories (GUT) have certain symmetries which are broken at lower energies, so too, *low temperatures* near absolute zero are also thought to cause displays of certain symmetries which become broken at higher temperatures, see Volovok's [18]. So the question which comes to the fore is whether an electric / magnetic symmetry of Maxwell's theory in the form of Dirac-Wu-Yang monopoles really does physically manifest from the confluence of low-temperatures near 0K for electrons in certain superconducting materials under large magnetic fields, and then gets broken into the non-observation of monopoles under anything but these limited conditions.

As to entanglement, the question which comes to the fore is whether by tightly constraining the electrons to two rather than three dimensions, and extracting virtually all of their heat energy leaving them only with their Fermi energies, we are forcing the electrons into some highly-constrained topological condition which forces them to reveal their entanglements and to display an electric and magnetic monopole symmetry and a charge fractionalization which they otherwise can keep hidden from observation.

And straddling between the experimental and arithmetical problem and the theoretical and physical problem laid out in section 2, is the following problem as to the boundary between mathematics and physics: Mathematically speaking, an angle of $\varphi = 2\pi$ is indistinguishable from an angle of 4π, 6π, 8π, etc. But physically, for fermions, if we account for entanglement together with orientation, then $\varphi = 0$, 4π, 8π etc. form one set of "equivalent" angles, while $\varphi = 2\pi$, 6π, 10π etc. form a second set of "equivalent" angles, and we do have a basis for *physically* distinguishing these two sets of angles notwithstanding their *mathematical* indistinguishability. In (3.2) and (3.3) we used this to force out the even-numbered fractional denominators and only retain the odd-integer denominators which does accord with the FQHE observations aside from the m=2 denominator still to be discussed. But there is still a puzzle for the angles *within each of these two sets of angles*:

For example, an n=1 state with $\nu = 1$ corresponds with the azimuth $\varphi = 2\pi$, while an n=1 state with $\nu = 1/3$ corresponds with the azimuth $\varphi = 6\pi$. Even with entanglement considered, it

therefore seems as though we should still regard $\varphi = 2\pi$ to be equivalent with $\varphi = 6\pi$. And yet, from (3.2) and (3.3), these two angles are connected with *two observably-different physical n=1 states*, namely, $\nu = 1$ for $\varphi = 2\pi$ and $\nu = 1/3$ for $\varphi = 6\pi$. And going further, $\nu = 1/5$ for $\varphi = 10\pi$, etc. So we are also required to consider the following question: How it is that geometric angles such as $\varphi = 2\pi$ and $\varphi = 6\pi$ and $\varphi = 10\pi$ which are equivalent under both orientation *and entanglement*, *still manage to lead to distinct physical states* such as the respective $\nu = 1$, $\nu = 1/3$ and $\nu = 1/5$? Is there something else that is still being "missed" by the orientation-entanglement analysis which leads to angles such as $\varphi = 2\pi$ and $\varphi = 6\pi$ and $\varphi = 10\pi$ still being distinguishable from one another, notwithstanding their pure trigonometric equivalence and their like-entanglements? Put differently, entanglement makes the $\varphi = 0$, 4π, 8π etc. azimuths different from the $\varphi = 2\pi$, 6π, 10π etc. azimuths, but within each azimuth set, what might make, e.g., $\varphi = 2\pi$ observably physically different from $\varphi = 6\pi$ and those observably physically different from $\varphi = 10\pi$, etc.? As we shall develop starting in section 5, all of these angles are indeed distinguishable, because of a topological "twisting" which also occurs in relation to orientation-entanglement, but which appears to be overlooked is the usual discussions of this subject.

All of the foregoing questions will be considered in detail beginning in section 5. But for the moment, having explained the odd-integer FQHE denominators via (3.2) and (3.3) by only equating angles with matching orientation *and entanglement*, we must now address the next *experimental and arithmetical* question: Why does nature also appear to permit the even denominator 2, in addition to the odd denominators $1+2l$ in (3.3), but not permit any other even denominators?

4. Untangled Electrons Pairing into Bosons, and the *m*=2 FQHE Denominators

Equation (1.7) for the non-fractionalized Dirac Quantization Condition specifies charges $e = n\left(2\pi/\mu\right) = ne_{\text{u}}$ which are integer multiples of the unit charge $e_{\text{u}} = 2\pi/\mu$. For this set of integer charges, (1.6) tells us that the gauge parameter $\Lambda = 2\pi n$. This entire set of integer-quantized charges $e = ne_{\text{u}}$ corresponds to the single azimuth $\varphi = 2\pi$ which means that there is a one-to-infinite quantized mapping of φ to Λ. That is, an infinite set of gauge states $\Lambda = 2\pi n$ can all be used to equivalently describe the same azimuth state $\varphi = 2\pi$, yielding quantized multiples of the unit charge.

Now, if we take a single fermion e.g. electron in the $\varphi = 2\pi$ state and do a 4π rotation to a $\varphi = 6\pi$ state which restores the electron to its original orientation and entanglement version, then $\exp\left(i\Lambda\right) = 1 = \exp\left(ie\mu \cdot 3\right)$ is the portion of (3.1) which describes this new state. Referring again to the general relationship $\exp\left(i2\pi n\right) = 1$, the solution is $\Lambda = 2\pi n = 3e\mu$, restated as $e = \Lambda/3\mu = \left(n/3\right)\left(2\pi/\mu\right) = ne_{\text{u}}/3$. This specifies integer *n* multiples *of 1/3 of the unit charge* e_{u}. As before, the gauge parameter $\Lambda = 2\pi n$, from which we earlier defined a reduced gauge

$\not{\Lambda} = n$ after (1.8). Similarly, if we generally define a reduced azimuth $\not{\varphi} \equiv \varphi / 2\pi$, we see that for $\varphi = 6\pi$, the fractional denominator $1+2l$ is equal to the reduced azimuth, $1+2l = \not{\varphi} = 3$. It is readily seen that for the $\varphi = 10\pi$, $\varphi = 14\pi$ etc. states which also maintain the $\varphi = 2\pi$ version of the fermion, that the odd number denominators in (3.2) and (3.3) may be written as $\not{\varphi} = 1+2l$. Using this information and notation, we rewrite the filling factor of (3.3) as:

$$\nu = \frac{\not{\Lambda}}{\not{\varphi}} = \frac{\Lambda}{\varphi}; \quad \not{\Lambda} = n = 0, \pm 1, \pm 2, \pm 3...; \quad \not{\varphi} = 1+2l = 1,3,5,7,9...; \quad l = 0,1,2,3.... \tag{4.1}$$

So the *fractionalization* of charge is determined directly by the number of "windings" $\not{\varphi} = 1+2l$ *in the physical space of spacetime,* and the odd-integer denominators occur because two topological windings, not one, are needed to restore a fermion to its original version. Although the $\not{\varphi} = 1+2l = 1,3,5,7,9...$ states all appear to be mathematically equivalent based on their differing from one another by 4π and so representing the same fermion version, the fact that these windings directly determine the fractionalization denominator is potentially an observable physical effect which belies this apparent equivalence. Similarly, the *quantization* of charge into integer multiples *n* of a fractional charge $1/\not{\varphi}$ is related to the quantized degeneracy of the gauge parameter $\not{\Lambda} = n$ which describes an infinite number of equivalent gauge states, and mathematically to the fact that phase angles which differ from one another by 2π are degenerately equivalent. Nonetheless, each "winding" *in the gauge space* adds one unit to the charge quantization number $n = \not{\Lambda}$, and this too is a physical effect which belies this apparent equivalence of gauge states differing from one another by 2π.

This sharpens the question as to how angles which differ from another by 2π or by 4π and so are trivially mathematically equivalent, can nonetheless yield different observable physical results, because even disregarding fractionalization, the quantum number $n = \not{\Lambda}$ which represents the local gauge angles $\Lambda = \pm 2\pi n$ will most certainly be an observable if the Dirac-Wu-Yang monopoles "exist." It is also worth noting that the fill factor *v* which is certainly observed if it can be related to the FQHE fill factor, is given simply by the ratio $\nu = \Lambda / \varphi$ of the gauge angle to the azimuth angle, even though both these angles are thought to have the same mathematical property of indistinguishability under a 2π or a 4π rotation. Again, all of this this would have to mean that there is something else going on in both the physical and in the gauge space, *above and beyond orientation-entanglement,* which causes different observable effects even when mathematically, two angles with a 2π or 4π difference seem to be indistinguishable. And again, this is a related topological twisting to be develop in the next section.

But first, let us return to the immediate business at hand, which is to understand the only even denominator, 2, which is phenomenologically-observed in the FQHE.

Based on (4.1), it will be easily appreciated that a denominator of 2 corresponds to a winding number $\not{\varphi} = 2$. The set of quantized states for the unit charge – not a fractional charge – which we now write as $e = \not{\Lambda}(2\pi / \mu) = \not{\Lambda} e_u$, corresponds to the winding number $\not{\varphi} = 1$. So to get from an electron with $\not{\varphi} = 1$ to some state with $\not{\varphi} = 2$ we are only making one turn of the

azimuth. Thus while we are restoring orientation, we are not restoring entanglement. Nonetheless, $\varphi = 2$ is the only even winding number which nature permits, so we have to figure out why we observe a state that is only one turn above $\varphi = 1$.

A $\varphi = 1$ electron will not be restored to its original version at $\varphi = 2$, because fermions need to do two windings to regain their original version. Only bosons can maintain equivalent version with one winding, because for bosons, entanglement is not an issue because they are not spinors. Again, as reviewed at the start of section 3, spinors rotate according to $\xi \to \xi' = R\xi$, while vector bosons rotate via $X \to X' = RXR^*$. So for an electron to go from $\varphi = 1$ to $\varphi = 2$, that electron must "disguise" itself as a boson. How might the electron do that? By finding a second electron to "conspire" with the first electron and "pair up" into a single boson system. Then, that pair of electrons can be wound from $\varphi = 1$ to $\varphi = 2$ without changing its entanglement. And based on (4.1), the filling factor will now be $\nu = \cancel{A} / \varphi = n/2$, which yields the denominator of 2.

What does this mean?: it means that while all of the permitted windings of individual electrons yield fractional charges with the 3, 5, 7, 9, etc. odd denominators, the permitted winding for a boson *pair of electrons* yields the one permitted even denominator, namely 2. While the "Cooper pairs" model of electron pairing [19] may well come to mind, for the moment let us not be that specific. Let us simply talk in terms of the requirement that a first electron needs to find some way to pair together with a second electron if a $\varphi = 2$ winding state and thus a ½ unit of charge is to be empirically displayed under the right set of conditions – as it is at extremely low temperatures in suitable materials subjected to large magnetic fields – while leaving open the mechanism by which that pairing take place. So, the pairing of electrons into boson states would appear to explain why 2 is *permitted* as an even denominator, and we know that there is some grounding in established theory for such pairing to occur. Now, we have left, for the experimental and arithmetical problem, to explain why 4, 6, 8 and even denominators *other than 2* are *not permitted*.

The next even denominator of course is *m*=4 which we now know corresponds to the winding azimuth $\varphi = 4$. And we know that this fractionalization $\nu = n/4$ is *not observed*. So let us start with an electron in the $\varphi = 3$ fractional state for which $\nu = n/3$. These are fractional fermion charges, so to get them to an $\varphi = 4$ winding which would correspond adding one azimuth turn to $\varphi = 3$. This would result in an oppositely-entangled state, which is inequivalent to $\varphi = 3$. So, as we did to get from $\varphi = 1$ to $\varphi = 2$, we would have to "pair up" two of these $\varphi = 3$ fractional $\nu = n/3$ fermions into a boson state to get to $\varphi = 4$ with a quarter-integer fraction $\nu = n/4$. The fact that we do not observe $\varphi = 4$, nor do we observe any other even windings $\varphi = 6,8,10...$, is nature's way of telling us that *fractional charges cannot be paired up into boson states*. *All boson pairs must be constructed from unfractionalized charge units* $\mu_u = 2\pi / e$. Given that fractional charges are commonly regarded as quasiparticles while unit charges are not, this simply means that only "real" particles, not quasiparticles, can form pairs. This is an *observation* about what nature seems to be telling us by excluding all even denominators except for 2; we still will want to *explain why* nature does so.

Pulling together all of these results, we now supplement (3.1) with $\varphi = 2$. Thus, the final result for the overall observed pattern is:

$$\nu = \frac{\mathbb{A}}{\varphi} = \frac{\Lambda}{\varphi}; \quad \mathbb{A} = n = 0, \pm 1, \pm 2, \pm 3...; \quad \varphi = 2 \quad \text{--or--} \quad \varphi = 1 + 2l; \quad l = 0, 1, 2, 3.... \tag{4.2}$$

The $\varphi = 1 + 2l$ odd-denominator states represent fermions exhibiting fractional charges; the $\varphi = 2$ state represents a boson pair of unit charges that are not fractional; and the absence of $\varphi = 4, 6, 8...$ states tells us that fractional charges are not capable of forming into boson pairs. In Figure 1 below, which is reproduced from [20] and [21], we have annotated the unit electron charge, the $\nu = \mathbb{A} / \varphi = 1/3$ fractional charge, and the ground state for a pair of unit electrons forming a boson with $\mathbb{A} / \varphi = 1/2$. Also added as annotations are apparent $\nu = 5/11$, $\nu = 6/11$ and $\nu = 7/9$ fractional states.

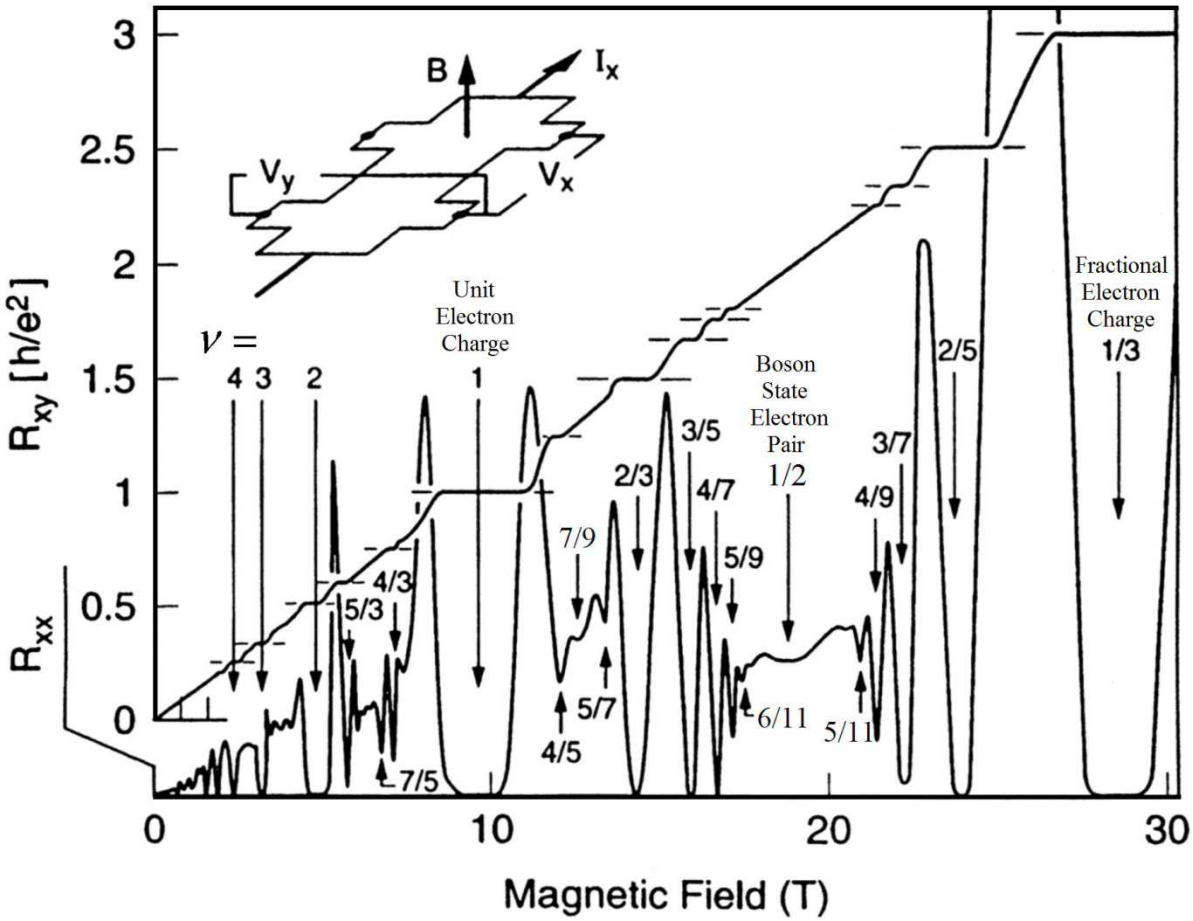

Figure 1: Fractional Quantum Hall Effect, reproduced from [20], [21], with annotation

It is worthwhile comparing the $\mathbb{A} = n = 1$ ground state of the $\varphi = 1$, $\varphi = 2$ and $\varphi = 3$ windings, which are the three states annotated above. For $\nu = \mathbb{A} / \varphi = 1$ we of course have a

unit electron charge. For $\nu = \cancel{A} / \cancel{\varphi} = 1/3$ we have 1/3 fractional charge. But for the boson pair with $\nu = \cancel{A} / \cancel{\varphi} = 1/2$ with a 1/2 unit of fractional charge, there are *two electrons not one* contributing to the half unit of charge. Therefore, each electron actually contributes a 1/4 unit of charge. Given that electrons naturally repel one another so that any pair formation mechanism must overcome this repulsion, it will be easier for two electrons to assume charges of 1/4 unit apiece and then pair into a boson, than to stay in the unit charge state or in the 1/3 charge state and then pair up. It is the 1/4 charge-per-electron paired state which minimizes the repulsion and therefore provides the most energetically-favored configuration.

Finally, we return to the original definition $\mu \equiv \oiint F$ of the Dirac monopoles. If rewrite $e = \nu e_{\text{u}} = \nu \left(2\pi / \mu \right)$ with the complete filling factor (4.2) in terms of μ, then using the "unit" of magnetic charge $\mu_{\text{u}} = 2\pi / e$, what we learn about the permitted monopole fluxes is that:

$$\oiint F = \nu \mu_{\text{u}} = \left(\cancel{A} / \cancel{\varphi} \right) \mu_{\text{u}}; \quad \cancel{A} = 0, \pm 1, \pm 2, \pm 3...; \quad \cancel{\varphi} = 2 \quad \text{--or--} \quad \cancel{\varphi} = 1 + 2l; \quad l = 0, 1, 2, 3.... \qquad (4.3)$$

This extended understanding of Dirac monopoles to include fractionalized charges, should put into a somewhat different perspective how one thinks about these monopoles, at least based on Dirac's quantization absent further developments such as t'Hooft / Polyakov monopoles [22], [23] which rely on Yang-Mills gauge theory which is not needed for Dirac monopoles alone. Although the Dirac monopoles when fully developed using Wu and Yang's gauge approach are *fractionalized as well as quantized*, these fractional charges are not observed except under very limited conditions at extremely low temperatures in suitable superconducting materials. Thus, to the degree that the filling factors (4.2) do describe a feature of the natural world but only under these specialized conditions, and because (4.3) is integrally related to (4.2), it would appear that the non-zero magnetic fluxes $\oiint F = \nu \mu_{\text{u}}$ of Dirac monopoles (as distinguished from other types of monopole) would only evidence themselves in nature under equally-restricted conditions.

Therefore, from an experimental and arithmetic standpoint, we conclude that a complete analysis of the gauge symmetries of Dirac Monopoles following the approach pioneered by Wu and Yang [14], [15] results in electric and magnetic charges which are quantized *and fractionalized* in the manner observed in the Fractional Quantum Hall Effect. Because fermions rotated through an azimuth over 2π regain their orientation but not their entanglement, the 4π rotation needed to restore both orientation and entanglement is responsible for the observation of odd-integer denominators and the skipping of most even-integer denominators. The only observed even-integer denominator of 2 appears to be the result of pairing two integer-charged fermions into a boson, and the absence of any larger even denominators appears to indicate that only integer charges, and not fractional charges, can be so-paired. The simplicity of the fill factor $\nu = \cancel{A} / \cancel{\varphi}$, and the ability to derive this strictly from gauge theory via Dirac-Wu-Yang approach in view of orientation-entanglement arguments on an arithmetic basis that does match experimental observation, is certainly intriguing. But now we must study these results deeply from a *theoretical and physical* standpoint to see if our suspicion that Dirac-Wu-Yang is connected to the FQHE is real, or illusory.

5. Twist: The Missing Ingredient from Orientation-Entanglement, and how this may lead to a Topological Understanding of Quantization

In section 3 we began to review Misner, Thorne and Wheeler's classic treatment of orientation-entanglement (OE) in section 41.5 of [17]. In this section we shall continue this discussion, focusing on one aspect of this subject which requires further development, namely, the topological "twisting" of the threads which are used to track orientation-entanglement. Specifically, we do this because the questions now raised as to how states which differ from one another even by the 4π, two winding rotation needed to restore version, still might leave certain observable signposts in the spaces of these rotations which render these states physically and observably-distinct from one another. Thus, we lay the foundation for physically understanding how, say, seemingly-equivalent rotational states with, e.g. $\varphi = 2\pi$ and $\varphi = 6\pi$ and $\varphi = 10\pi$ might nonetheless be connected to distinctly-observed related quasiparticle states with respective inequivalent $\nu = n$ and $\nu = n/3$ and $\nu = n/5$ quantized fractional charges. And in the process, we also lay the foundation for understanding how the Dirac-Wu-Yang arguments based on $U(1)_{em}$ gauge theory in three space dimensions generally without restriction, may connect to the FQHE observed for highly-restricted electrons forced into two dimensional restraint by superconducting materials and ultra-low temperatures under very large magnetic fields. Thus, by developing these topological twisting features fully, we lay the foundation for connecting the quantized fractional charge results in (4.2) to the FQHE, not only arithmetically and experimentally as we have already done, but theoretically and physically as we now must do.

D. K. Ross in [24] "hypothesize[s] that the OE relations are important to physics [and] represent the deep relationship between any particle or material body and its environment." He proceeds to show (reference renumbered here) "that Dirac [11] magnetic monopoles do not satisfy the OE relationship" and "hypothesize[s] that this is the reason they have never been seen despite extensive searches . . . and despite having a natural and elegant theory underlying them . . . going back to the more natural symmetry of Maxwell's equations with magnetic monopole sources present. " He then states that "[s]ince all known particles satisfy the OE relations and we show that Dirac magnetic monopoles which have not been seen do not satisfy these relationships, it is hopes that this paper will stimulate further work on the OE relations themselves and their topological role in physics." This further work on the OE relationships is precisely the subject of this section, and will lead us to understand the fractional quasiparticles of FQHE as those particles which "do not satisfy these [OE] relationships."

Figure 41.6 of MTW's [17] which is also posted online at [25], shows a spherical "object connected to its surroundings by elastic threads." Indeed, it is these "threads" and various configurations of these "threads" which most directly illustrate the "deep relationship between any particle or material body and its environment" mentioned by [24]. It is also these "threads" themselves which will be the object of the present discussion. As is well-understood, it is always possible following a 720° rotation or integer multiples thereof of an object connected to its environment with "untwisted threads," to remove all entanglement from the connections of an object to its surroundings. But of particular importance, as we shall now develop here, the sequences of disentangling the "threads" from one another are *not unique*. Depending upon the

sequence chosen, even after disentanglement, *the "threads" may still each maintain individual twists*, or they may have all twisting removed and have been returned to an untwisted state.

To simplify this development without any loss of information, rather than use the spherical object and the spherical environment and the "threads" employed in Figure 41.6 of [17], let us employ a first "bar" or "stick" which represents the environment and a second "bar" or "stick" which represents the object, and a pair of "ribbons" which represent the connections of this object to its environment. These two ways of representing OE do topologically map into one another as is shown below in Figure 2, which is why we can use the "bars and ribbons" as an alternative way of representing Figure 41.6 of [17].

Specifically, to verify this mapping, one may start with the OE system shown in drawing 1 from Figure 41.6 of [17], and as shown in Figure 2(a) below, to maintain points of reference, may label the northern and southern hemispheres of the object as shown, which hemispheres also have northerly and southerly "thread" connections to the environment. Then, as shown in Figure 2(b) below, one may topologically deform the object by stretching it in into a vertical elongation, while relocating the threads to the right along the northeastern and southeastern regions of the environment. Then, one can take the entire Figure 2(b) and rotate it 90° counterclockwise to arrive at Figure 2(c) below. In this final step, the *environment* is simply represented by a top "bar" or a "stick" at the top of 2(c), the *object* is represented by a bottom "bar" or a "stick" at the bottom of 2(c) which maintains the "north" and "south" labels and now also introduces a directional vector running from north to south, and the north and south thread pair are merged together into a pair of "ribbons" which represent the *entanglement* between the object and its environment. It will be appreciated that the ribbons capture the same topological information as the threads (just think of the two lines bounding the ribbon widths as being two threads and then add a few more threads for good measure). The benefit of employing "ribbons" (or thick "threads" with discernable width) rather than thin threads is that it is much easier with a two-sided ribbon to illustrate and track any twist which may occur in the course of performing OE operations, which will be very central to the ensuing discussion.

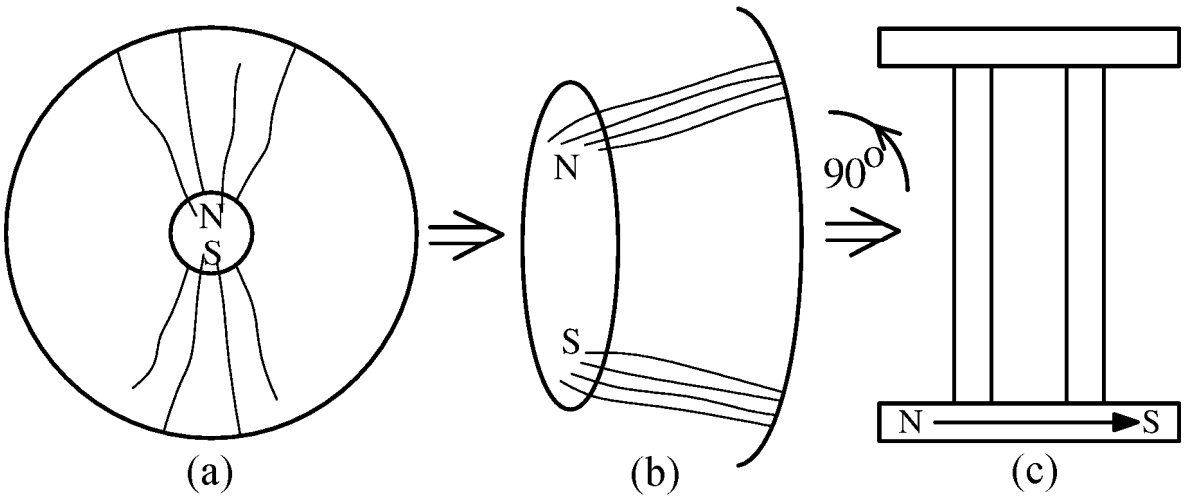

Figure 2: Topological Deformation of Figure 41.6 of [17] (MTW) into a "Bar and Ribbon" Configuration

This "bar and ribbon" configuration is often used in illustrations of the OE relationships, see, for example, an online animation at [26]. To help the interested reader to follow the forthcoming development, tt is easy and advisable, to construct a physical apparatus resembling Figure 2(c) by taking two sticks or dowels or even pencils, and then gluing or stapling two ribbons or shoelaces or rubber bands to the sticks in the configuration illustrated. It also helps to color each side of the ribbons differently for monitoring twists. The apparatus constructed by the author for this purpose, may be viewed at https://jayryablon.files.wordpress.com/2014/12/figure.jpg in the upper-left photograph.

So now, starting with the bar and ribbon configuration of Figure 2(c) above, let us immobilize the top "environmental" bar, and rotate the bottom bar – which from now on we shall simply call the bottom "vector" – by a $\varphi \to \varphi + 4\pi = \varphi + 720°$ rotation counterclockwise about the z axis through the angle φ in the x-y plane, as shown in Figure 3(a) below, to arrive at the configuration of Figure 3(b) below. It is worth noting that all three x, y, z dimensions are utilized in this operation, and also worth keeping in mind that for electrons frozen in two dimensions at low temperature in superconducting materials in the FQHE environment, one degree of spatial freedom is removed. It is also noted that this angle φ is an azimuth angle of rotation in three space dimensions, just as was the azimuth angle φ first introduced after (4.1) when we write the electromagnetic field strength as $F = (\mu / 4\pi) d\cos\theta d\varphi$.

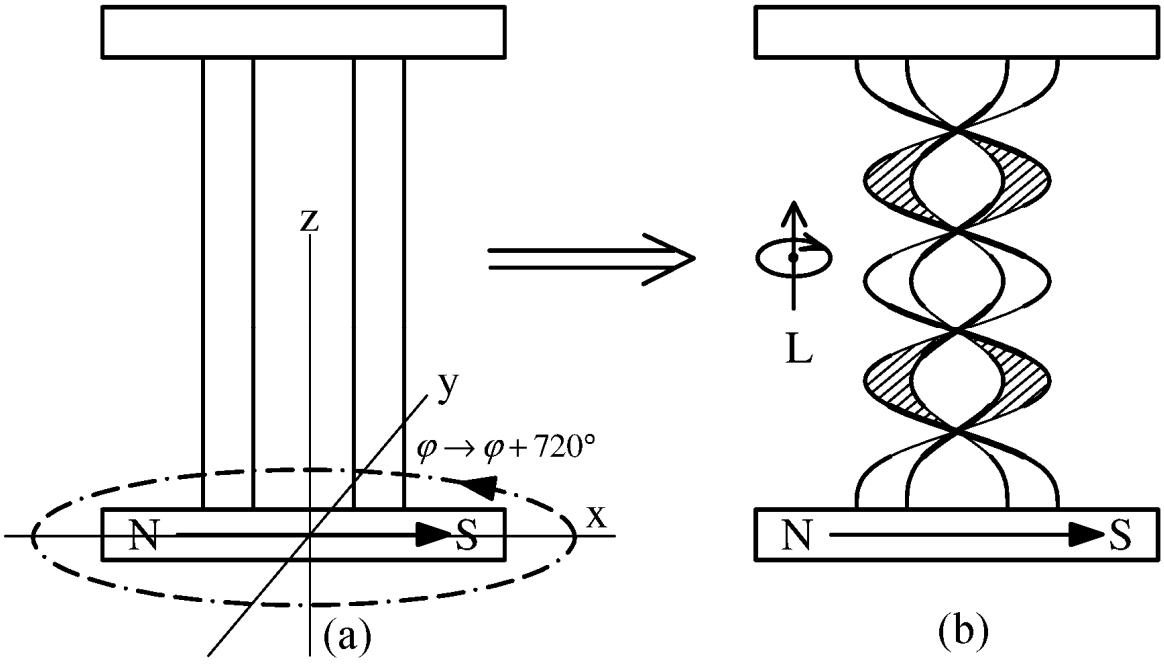

Figure 3: Environmental OE Consequences of Rotating a Vector through 720°

In Figure 3(b), the wider lines illustrated on each ribbon are passing in front of the narrower lines illustrated on each ribbon, and diagonal hash lines are used to illustrate the opposite face of the ribbon relative to the face shown in Figure 3(a). In 3(b), we reach a state in which the ribbons are entangled with one another, with the entanglement forming a left-handed helix as illustrated. And in addition, each of the two individual ribbons also is twisted into a left-handed helix as

illustrated. And in addition, both the entanglement helix and the twist helixes are *double* helixes, in the sense that there are two full helix rotations of $-4\pi = -720°$, using a convention in which a right helix has a positive sign and a left helix has a negative sign. Now we seek to disentangle the ribbons, while immobilizing both the environmental "bar" and the N→S vector, by moving the ribbons around the ends of the vector.

Now, if the initial rotation in Figure 3 had been through only $+2\pi = +360°$, Figure 3(b) would contain all *single* helixes, and as is well known there would be no way to disentangle the two ribbons from each other using only manipulations of the ribbons. But from Figure 3(b), because of the double helix entanglement which results from the double winding rotation through $\varphi \to \varphi + 4\pi = \varphi + 720°$ a.k.a. $\not{\varphi} \to \not{\varphi} + 2$ using the reduced azimuth $\not{\varphi} \equiv \varphi / 2\pi$ earlier defined, disentanglement is possible using only ribbon manipulations. And specifically, in order to disentangle the two ribbons using only operations of the ribbons with both the environment and the vector remaining immobile, one must perform *two* ribbon operations, and there are three choices for this.

For the first choice, as shown in Figure 4 below, for the first operation, one can take the *north ribbon*, wind it past the north "pole," wind it beneath the entire vector, and then wind it back above the vector past the south "pole." Then, for the second operation, one can take the *south ribbon*, wind it past the north "pole," wind it beneath the entire vector, and then wind it back above the vector past the south "pole." This can be done in either order, that is, one can use the south ribbon in the first operation and the north ribbon in the second operation and end up with the exact same result, which, as shown in Figure 4 below, not only disentangles the two ribbons from each other, but also removes the individual twists in each ribbon. We denote this by placing the number "0" next to each ribbon to indicate that it has no residual twist.

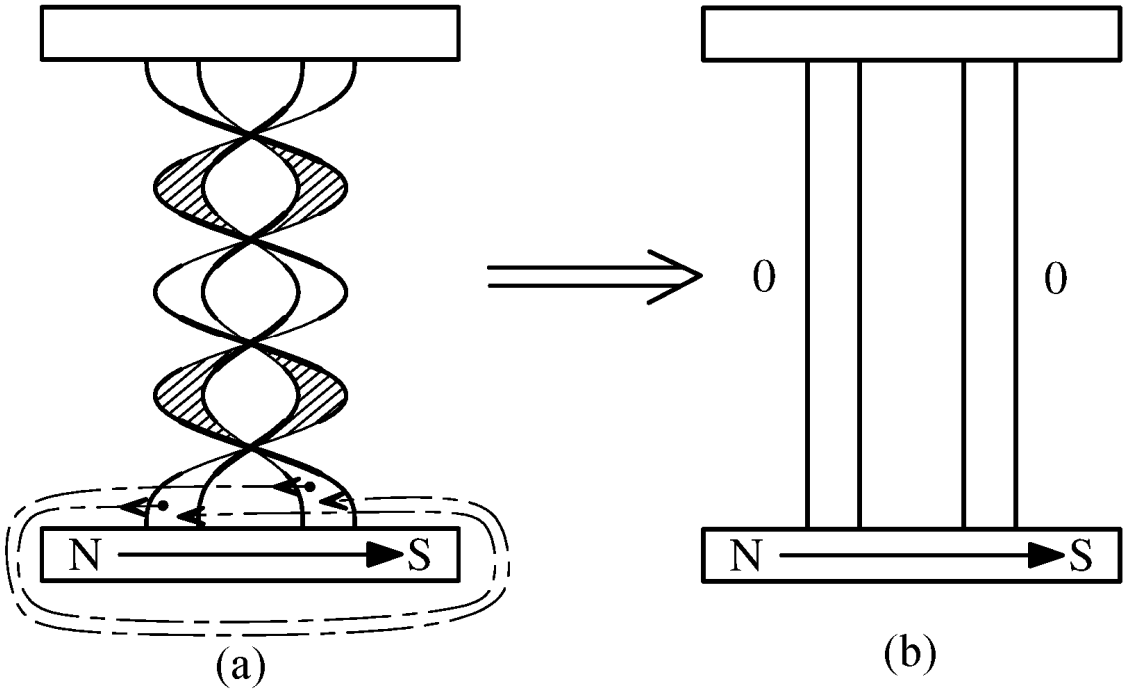

Figure 4: The Disentanglement Operation $0,0 \to \not{\varphi} + 2 \to \{N, S\} \to 0,0$

In either case, however, whether the north or south ribbon is operated first, the ribbon windings *must go from north to south*, that is, the ribbons must be first brought around the north pole, then wound beneath the vector, then be brought back up past the south pole. If the ribbons are wound from south to north, they will become even further entangled, and the net effect will be that of having performed a $\varphi \to \varphi + 8\pi$ a.k.a. a $\not{\varphi} \to \not{\varphi} + 4$ quadruple rotation starting from Figure 3(a). The question occurs why there is this apparent asymmetry in which the ribbons must be brought past the north pole first, but that is explained by the fact that the Figure 3 rotation was done counterclockwise and thus was positively signed, $\varphi \to \varphi + 4\pi$. Had the rotation been clockwise hence negative according to the customary conventions for defining angular rotation, i.e., $\varphi \to \varphi - 4\pi$ a.k.a. $\not{\varphi} \to \not{\varphi} - 2$, then disentanglement would have required winding the ribbons first over the south and then over the north pole. So that in fact there is an overall symmetry to these operations.

We shall use the shorthand $0,0 \to \not{\varphi} + 2 \to N/N, S/N \to 0,0$ to represent this operation in Figure 4 where both the north and south ribbons start with no twists $0,0$, the azimuth is positively rotated through two windings $\not{\varphi} + 2$, the north and then south ribbons are wound over the north pole $N/N, S/N$, and the disentangled state also restores no twists $0,0$. The fact the order of ribbon operations does not matter means that $0,0 \to \not{\varphi} + 2 \to S/N, N/N \to 0,0$ as well. Thus, $0,0 \to \not{\varphi} - 2 \to N/S, S/S \to 0,0$ and $0,0 \to \not{\varphi} - 2 \to S/S, N/S \to 0,0$ are also operations which restore the initial disentangled state with no twists when the initial rotation is negative $\not{\varphi} \to \not{\varphi} - 2$ rather than positive $\not{\varphi} \to \not{\varphi} + 2$. From here, we shall work only with positive rotations, which means that ribbons must always go first over the north pole to achieve disentanglement. We also keep in mind that the final configuration is invariant under the order in which the north and south ribbons are operated, i.e., under either temporal ordering (N,S) or (S,N) of the permutated ribbon set $\{N,S\}$. Thus, we can simplify the shorthand to write the Figure 4 operation as $0,0 \to \not{\varphi} + 2 \to \{N,S\} \to 0,0$, simply indicating that either (N,S) or (S,N) over the north pole will restore an untangled, untwisted state following a $\not{\varphi} + 2$ rotation of a vector.

For the second choice, one can take the north ribbon and wind it *twice* about the north pole, then under the vector, then back over the south pole and the ribbons will disentangle. But here, there will be a residual twist in each ribbon, as now shown below in Figure 5.

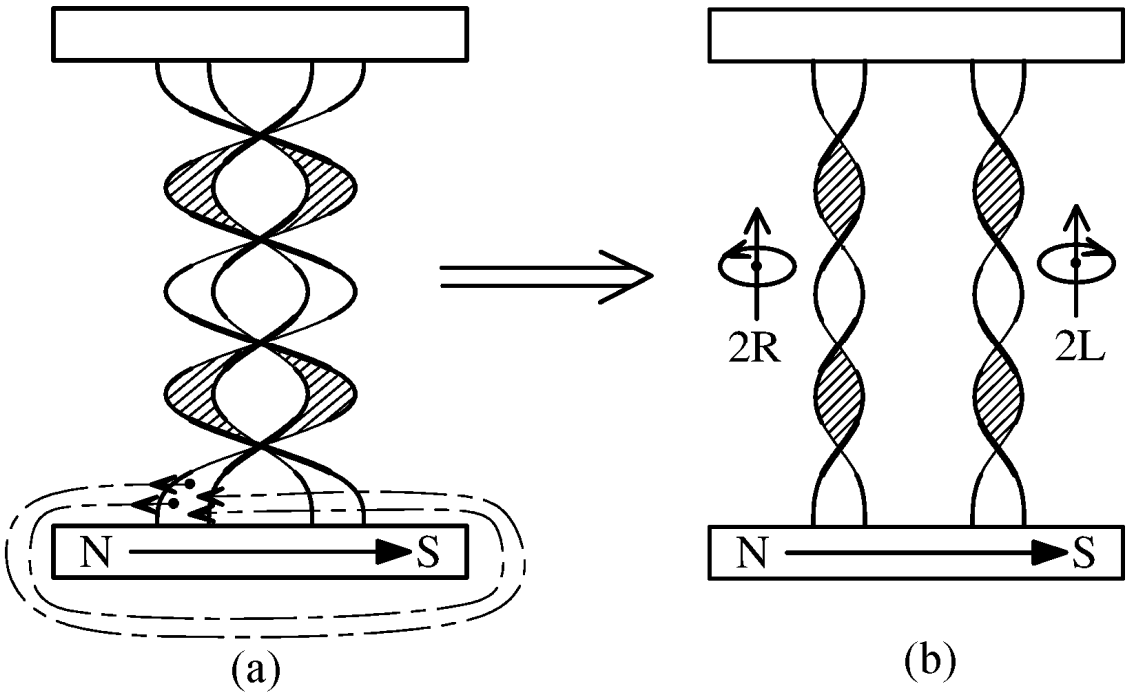

Figure 5: The Disentanglement Operation $0,0 \rightarrow \varphi + 2 \rightarrow \{N, N\} \rightarrow 2R$

Now, because we have used the ribbon set $\{N, N\}$ to disentangle the ribbons, the north ribbon maintains a double helix twist with right-handed parity as defined along the +z axis (see Figure 3(a)) which we denote by 2R, while the south ribbon also has a double helix twist but with left-handed parity which we denote as 2L. If we adopt a "right-hand rule," then we may also refer to these as a double twist "up" for the north ribbon, and a double twist "down" for the south ribbon. The ribbons are fully disentangled, and yet, the end state in Figure 5(b) is observably, physically-distinct from the end state of Figure 4(b), based wholly on the operation that was used to disentangle the ribbons. *So even though a rotation of a vector through* $\varphi \rightarrow \varphi + 4\pi$ *yields the exact same orientation and the exact same entanglement for that vector, the final, physical state can still be different from the starting state, wholly dependent upon how the disentanglement operation has taken place.*

It is because of this, that we can now begin to think about how, for example, the FQHE filling factor $\nu = n/1$ for which $\varphi = 1$ a.k.a. $\varphi = 2\pi$ in (4.2), can exhibit different observable physics from the filling factor $\nu = n/3$ for which $\varphi = 3$ a.k.a. $\varphi = 6\pi$ in (4.2), even though these angles differ from one another by 4π and so would be physically indistinguishable if we only considered orientation-entanglement (OE) without twist. So as we now see, the complete physics of vector rotations requires us to consider orientation-entanglement-twist all together, which we abbreviate as OET, and once we do consider twist, then angles which differ from one another by 2π or by 4π, despite their trivial trigonometric equivalence, are *all* distinct based on their OET relationships to the surrounding environment. The a vector rotated to the angle 2π is different from a 4π vector is different from 6π is different from 8π is different from 10π, *ad*

infinitum, once the OET of the vector is also taken fully into account. This is a place where physics informs and extends mathematics.

Using the notation developed above, we may use $0,0 \rightarrow \not{\varphi} + 2 \rightarrow \{N, N\} \rightarrow 2R, 2L$ to denote the final state of Figure 5(b) in which the north ribbon ends up with a double right-handed helix and the south ribbon ends up with a double left-handed helix. It will be apparent, however, that the left and right twists are offsetting, which is to say that the *net twist* of the overall system remains zero as it was when it started in Figure 3(a), and in general, this "conservation of twist" result will carry through to all OET disentanglements. So, if we know that the north ribbon has ended up with *2R*, then we automatically know that the south ribbon has ended up with *2L*. Thus, we can used twist conservation under OET disentanglement to simplify the summary of the Figure 5 operations to $0,0 \rightarrow \not{\varphi} + 2 \rightarrow \{N, N\} \rightarrow 2R$, showing only *2R* as the end state for the north ribbon, and deducing by implication that *2L* is therefore the twist for the south ribbon. In this way, we adopt a *convention* whereby the helicity of the "north" ribbon is used to characterize the helicity of the overall OET system following disentanglement.

For the third and final choice, one can take the south ribbon and wind it *twice* about the north pole, then under the vector, then back over the south pole and the ribbons will again disentangle. But here, there will be a residual twist in each ribbon oppositely to that shown in Figure 5, as now shown below in Figure 6.

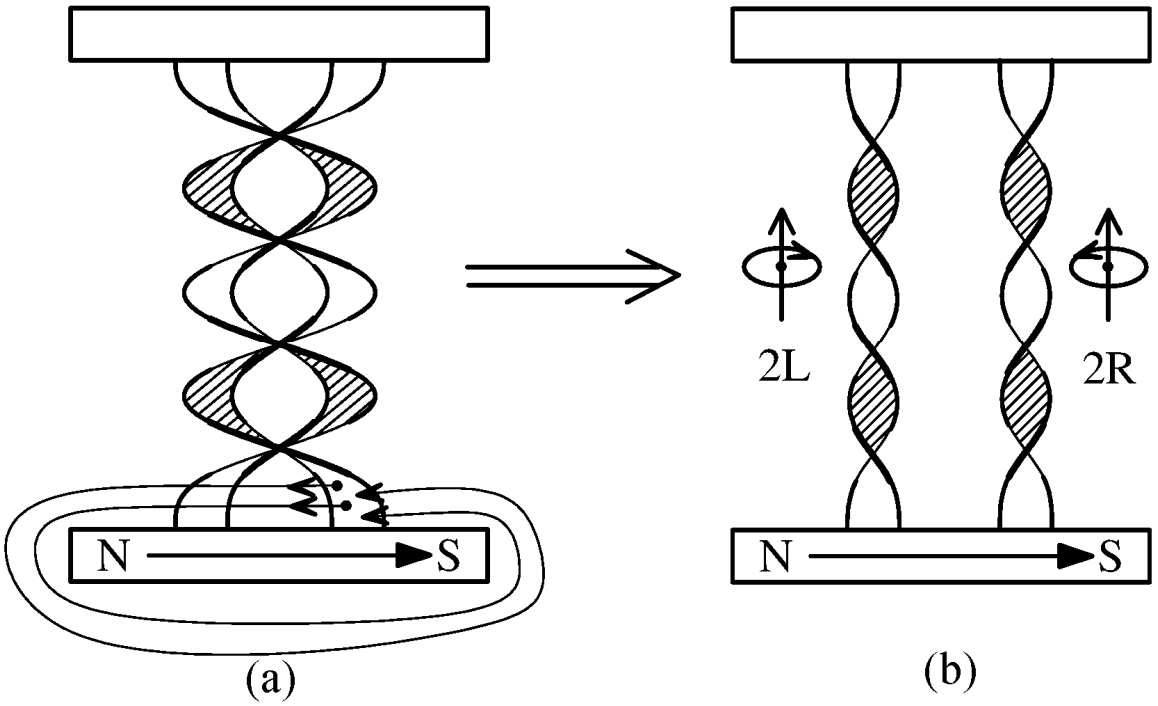

Figure 6: The Disentanglement Operation $0,0 \rightarrow \not{\varphi} + 2 \rightarrow \{S, S\} \rightarrow 2L$

Here, we have used the ribbon set $\{S, S\}$ to disentangle the ribbons, and the north ribbon maintains a double helix twist but now with left-handed parity which we denote by 2L, while the

south ribbon also has a double helix twist but with left-handed parity which we denote as 2R. Twist is still conserved, i.e., the net ribbon twist is zero, so continuing to represent the end result simply by the *2L* state of the north ribbon, $0,0 \rightarrow \varphi + 2 \rightarrow \{S,S\} \rightarrow 2L$ now summarizes the Figure 6 operation.

Returning to Figure 4, because we now know that twist is conserved, we further consolidate the summary of this operation by setting $0,0 \rightarrow 0$, that is, by using a single zero to represent the twist state of the north ribbon with the south ribbon implicitly also having zero twist because of net twist conservation. Thus we now write this as $0 \rightarrow \varphi + 2 \rightarrow \{N,S\} \rightarrow 0$. Figure 5 is then $0 \rightarrow \varphi + 2 \rightarrow \{N,N\} \rightarrow 2R$ and Figure 6 is $0 \rightarrow \varphi + 2 \rightarrow \{S,S\} \rightarrow 2L$.

How let us now make some final changes to our notation. Because the non-zero twist end results always contain a left- or right-handed double twist, let us count use the number 1 to define a single double-twist, and let us use the "+" sign to denote a right handed and "-" to represent a left-handed twist, and let us refer to the quantum variable which represents the number and handedness of double twists in the resultant north ribbon as m'. Thus, the end state $2R \rightarrow m' = +1$, $2L \rightarrow m' = -1$, and $0 \rightarrow m' = 0$. The number m' is of course quantized, but there is no mystery to this because it represents the number of topological twists and once OE is restored between a vector and its environment, the number of double twists will always be either zero or an integral number. Secondly, because we must always perform a two-winding, 4π rotation to be able to restore OE, let us also talk about this in terms of making one (1) double turn, rather than two terns. Let us designate the number of double turns as l', so that in all of Figures 4, 5 and 6, we have started out making $l' = +1$ double turns. Therefore, we rewrite $\varphi + 2$ as $\varphi + 2l'$ with $l' = +1$, and we refer to all of the results in Figures 4, 5 and 6 as the $l' = +1$ OET states, and we pick out the nickname "principal-prime" for these $l' = +1$ states and abbreviate this by "p'." Pulling all of this together enables us to summarize the three results from Figures 4, 5 and 6 as follows: TEACH m=0 FOR SS AND m=0 STARTING POINT

$$p' \equiv l' = +1\text{:}\quad 0 \rightarrow \varphi + 2l' \rightarrow \begin{cases} \{N,N\} \rightarrow m' = +1 & \left(\left|l' = +1, m' = +1\right\rangle\right) \\ \{N,S\} \rightarrow m' = 0 & \left(\left|l' = +1, m' = 0\right\rangle\right) \\ \{S,S\} \rightarrow m' = -1 & \left(\left|l' = +1, m' = -1\right\rangle\right) \end{cases} . \qquad (5.1)$$

Now let's repeat everything we have just done, but instead of a single double-winding $l' = +1$, let's start with Figure 3a, and do two double-windings, $\varphi \rightarrow \varphi \pm 8\pi$, i.e., $\varphi \rightarrow \varphi + 4$. This is now an $l' = +2$ state, and it requires four ribbon operations. But instead of using more drawings, let's just use the consolidate notation to represent the results. As discussed earlier, ribbons must always be drawn first over the north and then over the south pole, because the rotation is a positive rotation. Doing otherwise with create further entangling, rather than detangling. As also reviewed, the temporal order with which one draws the ribbons does not matter because as with $l' = +1$ the final twist results are invariant with respect to this order. So operational sets of ribbons that can be used in any temporal permutation are $\{N,N,N,N\}$,

$\{N,N,N,S\}$, $\{N,N,S,S\}$, $\{N,S,S,S\}$ and $\{S,S,S,S\}$. Let us nickname this $l'=+2$ as "diffuse-prime," abbreviated d'. What we now have, in place of five more Figures, are the five resulting states:

$$d'\equiv l'=+2:\quad 0\rightarrow\varphi+2l'\rightarrow\begin{cases}\{N,N,N,N\}\rightarrow m'=+2 & \left(\left|l'=+2,m'=+2\right\rangle\right)\\ \{N,N,N,S\}\rightarrow m'=+1 & \left(\left|l'=+2,m'=+1\right\rangle\right)\\ \{N,N,S,S\}\rightarrow m'=0 & \left(\left|l'=+2,m'=0\right\rangle\right)\\ \{N,S,S,S\}\rightarrow m'=-1 & \left(\left|l'=+2,m'=-1\right\rangle\right)\\ \{S,S,S,S\}\rightarrow m'=-2 & \left(\left|l'=+2,m'=-2\right\rangle\right)\end{cases}\quad . \tag{5.2}$$

So now we start to see the pattern of OET. In general, l' which represents the number of double windings is an integer which always has the positive value $l'=0,1,2,3,...$ ($l'=0$ is represented by Figure 3(a) and has the single state $m'=0$ with no twists), and the resultant twist of the north ribbon flowing detangling ranges over the integers m' for which $-l'\le m'\le+l'$. So, for example, if we go next to $l'=+3$ with three double windings, and call this "fundamental-prime," abbreviated f'. For f', with $l'=+3$, we have seven states $m'=0,\pm1,\pm2,\pm3$. Strikingly, this is the same pattern of orbital angular momentum and magnetization, as represented by the quantum numbers *l* and *m* which characterize the electrons in an atom. And also strikingly, the azimuth angle φ is the same azimuth in physical three-dimensional space against which angular momentum is specified.

This leads us to two questions: Are these concurrences merely coincidental, or can OET be used to provide a fully-topological understanding of electronic structure (and by extension nuclear structure which is subject to the same quantized exclusion principles for proton and neutrons)? And, how does this all relate to the FQHE which motivated this discussion because of the need to physically-distinguish rotational states with the same OE, i.e., states differing by a 4π rotation. Clearly, OET provides the basis for asserting that even states which differ by $4\pi n$ rotations from one another are not trivially-identical once all of OE and Twist are considered, and that the differences between these inequivalent states parallel the orbital and magnetic quantum structures of the atom and the nucleus and the quantum numbers which force exclusion. We also note in passing that the web animation at [26] which follows the same winding procedure we have used here albeit displayed from a bottom perspective view, is one example of how OE discussions often overlook Twist: This animation performs the disentanglement operation $\{S,S\}$, so while it does indeed untangle the threads, it still leaves the routinely-overlooked twist which in this case is $\left|l'=+1,m'=-1\right\rangle$ of Figure 6.

This leads us to consider the possibility that these atomic quantum numbers may in fact be indicators of states of twist, and it will leads us to propose an experiment by which the fractional quantum Hall states are examined for properties reminiscent of electrons in the s, p, d, f, g, h, i, k… orbital states of electrons in atoms to possibly validate or contradict these apparent parallels.

6. Twist Part 2: Topological Modelling of Fermions and Conservation of Orbital Plus Spin Angular Momentum

The reason we started to study topology in the last section was to see if physical azimuths φ which differ from one another by 2π and so are geometrically indistinct, or especially by 4π which are indistinct under geometric orientation and topological entanglement, might nevertheless be topologically distinct based on topological twist. If we can show topological distinctness, this would provide a basis for there being observable, physical differences based on different azimuth or gauge angles, even when those angles differ by integer multiples of 2π or 4π and so are indistinct under, respectively, orientation alone or orientation-entanglement alone. If established, this in turn would provide a basis for trying to connect the quantized and fractionalized fill factors in (4.2) to FQHE assuming we can also physically and theoretically explain the carryover of a result derived in three-dimensional gauge theory to a fundamentally two-dimensional, ultra-low temperature phenomenon.

We saw in the last section that when twist is considered, the topological pattern could be summarized by quantum numbers $l' = 0,1,2,3,...$ and $-l' \le m' \le +l'$, which follow the precise pattern that unprimed versions of these quantum numbers follow in atomic structure. This does confirm that even azimuths separated by 4π are physically distinct when Twist is made part of the Orientation-Entanglement analysis. So referring to the proposed FQHY fill factor relationship $\nu = \Lambda / \varphi$ of (4.2), this means that the use of angles separated by 2π in the case of the gauge angle Λ and by 4π in the case of the azimuth angle φ *are* physically distinct when their topological Orientation-Entanglement-Twist are considered, and that *these angles therefore can be used in the relationship* $\nu = \Lambda / \varphi$ *to specify a physical observable.* Or, to be precise: the fact that angles differing by 2π are trivially indistinct based on geometric trigonometry alone cannot be used to disqualify $\nu = \Lambda / \varphi$ as a possible observed physical relationship based on the combined consideration of *geometry and topology*, in particular, once topological twisting has been taken into consideration. Even once this hurdle is cleared, however, the question of how a three-dimensional Dirac-Wu-Yang (DWY) U(1) gauge-theory analysis resulting in U(1) QWY magnetic monopoles has any relevance to the two-dimensional system of electrons in FQHE still does remain for consideration.

It is entirely possible to turn right now to this question of how the three-dimensional DWY analysis for the U(1) gauge theory of electrons in three dimensions can possibly be related, *theoretically and physically*, to FQHE systems in which electrons are fundamentally constrained to two dimensions. (For shorthand, we shall refer to this as the "two-dimensional constraint problem.") But, not only have we now established that angles separated by 2π or 4π are topologically distinct, *but we have established that they are topologically distinct in a fashion that can be characterized by* $l' = 0,1,2,3,...$ where l' is the number of OET double rotations, and by $-l' \le m' \le +l'$ where m' is the number of double twists that remain following disentanglement using one of $2l'+1$ disentanglement operations. Because the removal of the "primes" from these relationships yields the precise relationships for the orbital quantum numbers of electrons in atoms, and because *it is these very same electrons which have three dimensional freedom in general at higher temperatures but become constrained to two dimensions in superconductors at low temperatures*, it seems that we will want to know as much as possible about the topological

behavior of these electrons before to attempt to tackle the two-dimensional constraint problem. Why?

Consider even these elementary questions: The electrons in the atomic shells of a superconducting material at high temperature (meaning, not too close to 0K) certainly possess the quantum numbers *n, l, m, s* which establish the basis for fermionic exclusion in the filling of orbital shells. And these quantum numbers yield certain shell structures and wavefunction distributions (see e.g. [27]) which are very familiar and well established. So: what will happen to these quantum numbers as we cool down this material toward 0K and phase transitions occur which enable FQHE? Will these quantum numbers just go away? Or will they still be there? That is, are electrons in a superconducting material close to 0K still carriers of the *n, l, m, s* quantum numbers which subject them to exclusion in atomic orbitals? And, if we suppose that, yes, these quantum numbers do continue to exist for the electrons in a superconductor near 0K, then another question arises: What effect will these *n, l, m, s* quantum numbers have on what is observed when the superconductor is subjected to large magnetic field near 0K? Instead of manifesting in the three-dimensional shell structures reviewed in e.g. [27] which will most certainly be constrained in some fashion to two dimensions near 0K, might *n, l, m, s* instead manifest in some other way? And, if they do manifest in some other way, might they in fact manifest via the fill factor $\nu = \Lambda / \varphi$; which is driven by gauge Λ and azimuth φ angles that are topologically-distinct under 2π or 4π rotations; which angles based on what we found in section 5 may well bear a one-to-one relationship with *n, l, m, s*; and which fill factor does correctly represent the observed FQHE based on what we discussed in section 4?

By laying out the questions in this way, we have deliberately "shown our hand," which is to say, this is exactly the line of reasoning that we shall seek to develop and support in the balance of this paper. Namely, we shall seek to show that FQHE is to low temperature electronic behavior in atoms, what the shell structure is to higher temperature electronic behavior in atoms. But, if it is the *n, l, m, s* quantum numbers which are the common denominator between low and high temperatures, invariantly characterizing electrons no matter what the temperature, and with the temperature dependence being expressed by how these unvarying quantum numbers observably exhibit themselves, then these *n, l, m, s* become the fundamental point of reference for guiding our understanding from high to low temperatures and back again.

What is fundamentally important about what we uncovered in section 5 – beyond the showing that angles differing by 2π or 4π are topologically distinct – is the finding that the topological pattern $l' = 0, 1, 2, 3, ...$ and $-l' \le m' \le +l'$ matches the electronic pattern for orbital angular momentum, which raises the prospect that these *n, l, m, s* quantum numbers themselves, in turn, can be completely grounded in a topological understanding. And if that can be established, then it becomes the topology itself – OET when performed in three dimensions versus OET when performed under the restraint of removing the degree of freedom of one space dimension and so being restrained to two dimensions – which becomes our key guidepost for tacking the two-dimensional constraint problem.

Moreover, if in fact the *n, l, m, s* quantum numbers of electronic structure can be grounded entirely in the topology of OET in three dimensions whether or not one of the space dimensions is removed as a degree of freedom, then even when we consider ordinary

temperatures well above 0K where all three space dimensions are available, it would become possible to understand atomic structure itself squarely on the basis of geometry and topology. This in turn would be extraordinarily beneficial to the ultimate fulfillment of Wheeler's geometrodynamic program [28], [29] in the spirit of Einstein and Weyl, of establishing that the entirety of the observed natural world is a manifestation of spacetime geometry and spacetime topology.

For all of these reasons, we shall now continue forward with the development from section 5, to further develop this possible connection between this OET topology and electronic atomic structure.

We found in section 5 that when rotating a vector (meaning, the arrow in the N→S bar) through $l' = 0,1,2,3,...$ double windings (through 4π), that the number of double twists (4π twists) that remain following disentanglement using one of $2l'+1$ available disentanglement operations is $-l' \le m' \le +l'$. And this made us attentive to a *possible* connection with the analogous $l = 0,1,2,3,...$ and $-l \le m \le +l$ in one of $2l+1$ states from atomic structure. However, even if this possible is established to be a *true* connection, the finding that angles differing by 4π are topologically distinct still does not entirely solve the problem of physical observability because the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ to which we seek a connection is not by itself a physical observable. Observables must commute with the Hamiltonian, and when commuted with the Dirac Hamiltonian $\left[H, \mathbf{L}\right] = -i\left(\boldsymbol{\alpha} \times \mathbf{P}\right)$. To fashion an *observable* angular momentum analogy, we need to also include the spin operator for which $\text{diag}\left(\boldsymbol{\Sigma}\right) = \left(\boldsymbol{\sigma}, \boldsymbol{\sigma}\right)$, and for which $\left[H, \mathbf{S}\right] = +2i\left(\boldsymbol{\alpha} \times \mathbf{P}\right)$. Then, forming the total angular momentum $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\Sigma}$, we obtain $\left[H, \mathbf{J}\right] = 0$, and find that it is $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\Sigma}$ which is the conserved observable. Thus, if our goal is to develop a topological understanding of *observable* physics – as it must be – then we must advance the results of section 5 to provide a topological understanding of the intrinsic spin ½ of fermion, and its interplay with orbital angular momentum. Thus will be the main objective of this section.

We saw in the last section through the notation developed in advance of (5.5) namely $2R \to m' = +1$, $2L \to m' = -1$, and $0 \to m' = 0$, that a double twist of 4π resulting from detangling following one double rotation through 4π which we represented by $l' = +1$, corresponds with one unit of m' which appears analogous to the magnetization quantum number *m*. And the z-component of total angular momentum $j_z = m + s_z$ combines *m* with $s_z = \pm\frac{1}{2}$ unit of angular momentum which is intrinsic spin, as a downstream consequence of $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\Sigma}$ and $\left[H, \mathbf{J}\right] = 0$. So carrying through this analogy, if we wish to use these topological twists to represent spin ½, a single twist through 2π would correspond to a half unit of m', and analogize to a spin of $s_z = \pm\frac{1}{2}$. So we shall now proceed to represent a fermion, using the "bar and ribbon" topological device of Figure 2(c), by introducing a single 2π twist into each ribbon, but with opposite helicity as between these two ribbons so as to conserve twist. Because we are using a convention utilizing twist conservation whereby the north ribbon specifies the OET helicity, we will need two such bar and ribbon systems: one in which the north ribbon has right-handed

helicity to represent spin up, and the other in which it has left-handed helicity to represent spin down. We denote the right-helicity ribbon with one half of a right-handed double twist using the quantum number $s_z' = +\frac{1}{2}$ and the left-helicity ribbon likewise as $s_z' = -\frac{1}{2}$, and we continue the convention where the "north" ribbon represents the overall topological system. We also define $j_z' = m' + s_z'$. Because we have not yet performed any $+4\pi$ rotations over the +z azis and then disentangled anything, we these are configurations in the state $\left| l' = 0, m' = 0 \right\rangle$. This is shown in Figure 7 below.

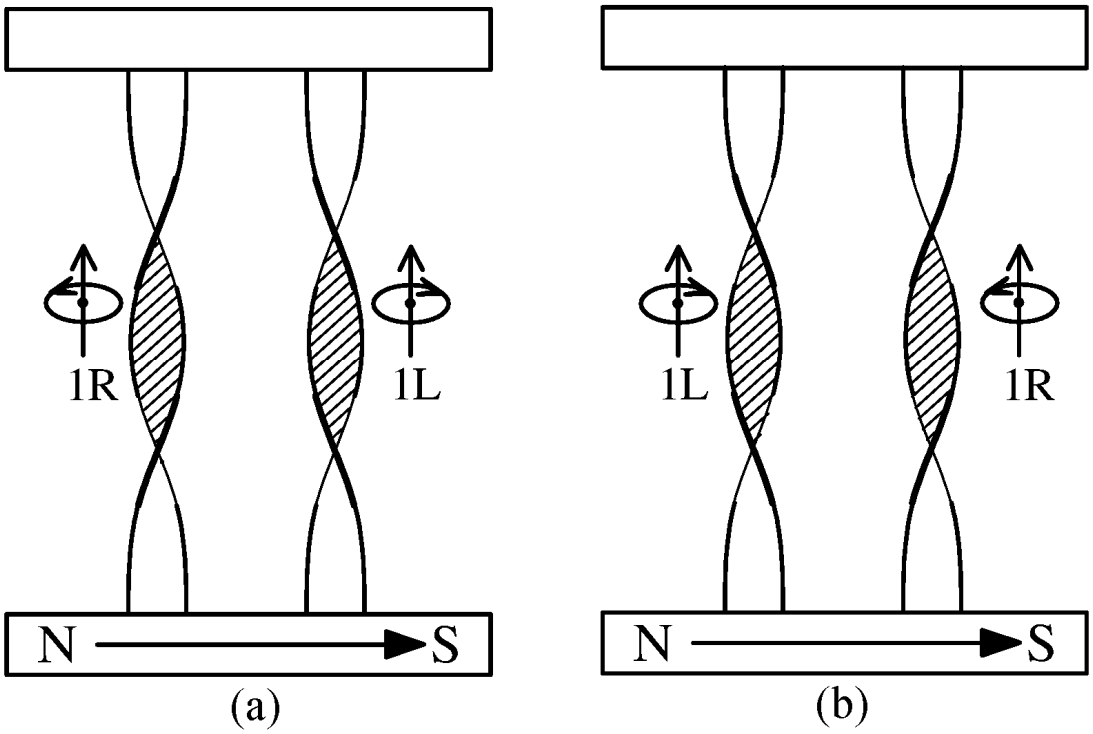

Figure 7: Topological Representation of (a) $\left| l' = 0, m' = 0, s_z' = +\frac{1}{2}, j_z' = +\frac{1}{2} \right\rangle$ and (b) $\left| l' = 0, m' = 0, s_z' = -\frac{1}{2}, j_z' = -\frac{1}{2} \right\rangle$

Now, given that we intend to use the configuration of Figure 7 above to represent spin up and spin down fermions $s_z = +\frac{1}{2}$ and $s_z = -\frac{1}{2}$, a practical question arises which may point to a deep physical result: Let us suppose that the reader has built the apparatus in Figure 3(a) and now wishes produce the apparatuses of Figures 7(a) and 7(b). Is the reader required to unstaple or unglue (detach) the ribbons from Figure 3(a), given them each a twist, and then reattach them back? Or, can the reader merely perform some set of operations of rotation and disentanglement to get from Figure 3(a) to Figures 7? This is a practical question, but it is also a theoretical question, namely: is there some way in which Figure 3(a) can be topologically deformed into Figures 7? That is to say, *are these topologically equivalent, or, are they topologically different?* The reason this is important is that if are eventually intending to have the number of double twists be a topological representation of total angular momentum about the z-axis, then Figure 3 will eventually represent spin 0 and Figures 7 will eventually represent spin ½, and so the topological deformation of Figure 3 into Figures 7 would suggest that one can use topological

deformations to go from spin 0 scalars to a spin ½ fermions and by induction to higher spins. So the deep question this raises is *whether topological deformations in the three-dimensional physical space can be used to connect particles of different spins, or whether such different-spin particles are topologically distinct.*

As it turns out, which the reader who has built Figure 3(a) will find, there are no OET operations about either the z or y axes that will deform Figure 3 into Figures 7. The z-rotations have already been explored in the last section, and it turns out that rotations about the y axis followed by detangling produce results equivalent to the z rotations. However, it is possible to deform Figure 3(a) into Figures 7 by rotating the bottom N→S bar about the x axis, and then detangling. Specifically, using +x to define the axis of rotation, of one does a *right-handed* rotation of the N→S bar about the x axis through 2π (one rotation not two) letting the ribbons wind once about the bar, and then untangles the north ribbon over the north pole and the south ribbon over the south pole, the resulting configuration is $s_z' = +\frac{1}{2}$ in Figure 7(a). With a *left handed* rotation of the N→S bar about the x axis through 2π followed by the same untangling, the result is Figure 7(b). So our representation of spin 0 can be topologically deformed into our representation of spin ½, but only if we are allowed to rotate the N→S bar about the x axis. If we are prevented from this rotation, and are only permitted rotations about y and z, then the different spins cannot be deformed into one another.

Although tangential to the main development here, this raises the physical question whether there are natural situations in three-dimensional space where rotations can occur about two of the three axes but not about the third. And in thinking about this question, one is drawn to the dynamical property of physical space whereby if one has an object with differing length > width > depth, say a length of 12 inches, a width of 6 inches and a depth of 1 inch, and then throws that object into the air while imparting rotation, a stable rotation can be achieved about the length and depth axes, *but not about the width axis*. So there is a physical precedent for "excluding" rotations about one of the three space axes. While we shall not pursue this collateral question here, it is perhaps worth musing whether this dynamical property of three-dimensional space can help is try to understand spin supersymmetry, or its apparent observed absence.

So returning to the main development, let us now rotate the azimuth φ of the N→S bar of each of Figures 7(a) and (b) through one double winding of $+4\pi$, then disentangle using the $\{N,N\}$, $\{N,S\}$ and $\{S,S\}$ operations developed and discussed in section 5. We can draw some more Figures for this, but that is now unnecessary because we have developed some shorthand for representing the results. Starting with Figure 7(a) which is in the state $\left|l'=0, m'=0, s_z'=+\frac{1}{2}, j_z'=+\frac{1}{2}\right\rangle$ with one-half of a right-handed double twist in the north ribbon, we rotate $\varphi \to \varphi + 4\pi$, which brings about the $\left|l'=1\right\rangle$ state representing one double rotation. (Again, every whole integer represents the number of *double* twists or rotations.) We learned at (5.1) and (5.2) that an untangling which equally balances *N* and *S* restores the original number of and handedness of double twists, i.e., leaves m' unchanged from its original state. If one carries out these above operations it will become clear that j_z' tells us the number of double twists in each final state. Thereafter, for each of the three combinations of disentangling, we arrive at:

$$\left|l'=0,m'=0,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle \rightarrow +4\pi \rightarrow \begin{cases}\{N,N\}\rightarrow\left|l'=1,m'=+1,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{3}{2}\right\rangle \\ \{N,S\}\rightarrow\left|l'=1,m'=0,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle \\ \{S,S\}\rightarrow\left|l'=1,m'=-1,s'_z=+\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle\end{cases} . \qquad (6.1)$$

Likewise, if we start with Figure 7(a) which is in the state $\left|l'=0,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle$ and again rotate via $\varphi\rightarrow\varphi+4\pi$ to $\left|l'=1\right\rangle$, each of the disentangling options produces:

$$\left|l'=0,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle \rightarrow +4\pi \rightarrow \begin{cases}\{N,N\}\rightarrow\left|l'=1,m'=+1,s'_z=-\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle \\ \{N,S\}\rightarrow\left|l'=1,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle \\ \{S,S\}\rightarrow\left|l'=1,m'=-1,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{3}{2}\right\rangle\end{cases} . \qquad (6.2)$$

We see here, and can confirm generally, that when the maximally-stretched all-N or all-S untangling operations are applied, every two-winding rotation of $+4\pi$ will have added or subtracted exactly one double twist from the original state. This means that we should now also assign the azimuth angles $\varphi=2\pi$ to each of Figures 7, i.e., that $l'=0$ with $s'_z=\pm\tfrac{1}{2}$ should be assigned the azimuth angle $\varphi=2\pi$ indicating that one twist is already built into this ground state.

But the most important result is that were we to simply remove the "primes" from all the quantum numbers in (6.1) and (6.2), these would precisely reproduce all six of the angular momentum-related quantum numbers for the p shell electronic structure of an atom, which, for example, appear in the Periodic Table of the Elements sequence B, C, N, O, F and Ne. This appears to validate that atomic structure may in fact be explainable simply on the basis of topological OET in three space dimensions. Does this pattern continue for larger rotations?

Let's now start with each of Figures 7 which we now know have $\varphi=2\pi$ and rotate four times to $\varphi=10\pi$ which brings them into the $l'=2$ state of two double azimuth rotations. To save on drawings, let us simply represent the results in the same way. Now we have:

$$\left|l'=0,m'=0,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle \rightarrow +8\pi \rightarrow \begin{cases}\{N,N,N,N\}\rightarrow\left|l'=2,m'=+2,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{5}{2}\right\rangle \\ \{N,N,N,S\}\rightarrow\left|l'=2,m'=+1,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{3}{2}\right\rangle \\ \{N,N,S,S\}\rightarrow\left|l'=2,m'=0,s'_z=+\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle \\ \{N,S,S,S\}\rightarrow\left|l'=2,m'=-1,s'_z=+\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle \\ \{S,S,S,S\}\rightarrow\left|l'=2,m'=-2,s'_z=+\tfrac{1}{2},j'_z=-\tfrac{3}{2}\right\rangle\end{cases} .(6.3)$$

Likewise, if we start with Figure 7(a) which is in the state $\left|l'=0,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle$ and again rotate via $\varphi\rightarrow\varphi+4\pi$ to $\left|l'=1\right\rangle$, each of the disentangling options produces:

$$\left|l'=0,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle \rightarrow +8\pi \rightarrow \begin{cases} \{N,N,N,N\}\rightarrow \left|l'=2,m'=+2,s'_z=-\tfrac{1}{2},j'_z=+\tfrac{3}{2}\right\rangle \\ \{N,N,N,S\}\rightarrow \left|l'=2,m'=+1,s'_z=-\tfrac{1}{2},j'_z=+\tfrac{1}{2}\right\rangle \\ \{N,N,S,S\}\rightarrow \left|l'=2,m'=0,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{1}{2}\right\rangle \\ \{N,S,S,S\}\rightarrow \left|l'=2,m'=-1,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{3}{2}\right\rangle \\ \{S,S,S,S\}\rightarrow \left|l'=2,m'=-2,s'_z=-\tfrac{1}{2},j'_z=-\tfrac{5}{2}\right\rangle \end{cases} \quad .(6.4)$$

This now yields ten sets of the $\left|l'=2\right\rangle$ quantum numbers which are precisely analogous to those of the d shell electrons which, for example, run from Sc to Zn. For $\varphi=14\pi$, i.e., by adding $+12\pi$ a.k.a. doing an $l'=3$ rotation, there are fourteen states that precisely reproduce the f shell angular momentum electronic structure seen in the Lanthanoids and Actinoids. So this pattern – *derived wholly from the topology of OET in three space dimensions* – will continue to perfectly match the observed atomic structure for angular momentum, all the way through the entire periodic table.

These results deepen our attention to a possible connection between OET topology and electronic atomic structure, especially now that we have a j'_z quantum number representing the total number of double twists which are *observable* after all untangling has been performed, which analogizes to the *observable* $\mathbf{J}=\mathbf{L}+\frac{1}{2}\mathbf{\Sigma}$ for which $\left[H,\mathbf{J}\right]=0$ which has the associated half-integer states $j_z=m+s_z=\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{5}{2},\pm\frac{7}{2}...$ for the eigenvalue j_z in $J_z\left|\xi\right\rangle=j_z\left|\xi\right\rangle$ when J_z operates on the spinor state $\left|\xi\right\rangle$. So the observable numbers of doubles twists in the topological pattern now matches up precisely with the empirically-observable angular momentum. And in general the permitted states and numbers of these states also match perfectly with all of the angular momentum quantum numbers *l, m, s, j*. Because these results which seem so reminiscent of atomic structure are directly deduced from studying twists in the orientation-entanglement regime articulated by MTW in [17], a return to these origins of OET study is now highly warranted.

7. Fermions as Twisted Strings, and the Four Laws of Topological Least Action

When we talk about orientation-entanglement (and should also be talking about twist), such as is done in Misner, Thorne and Wheeler's premier exposition [17], the discussion is always approached schematically in terms of an "object" (e.g., the "ball" in Figure 41.6 of MTW's [17] or the "bar" into which that is deformed in Figure 2) which is entangled via "threads" (e.g., the north and south threads of Figure 41.6 or the "ribbons" of Figure 2) with its external "environment" (e.g., the outer sphere in MTW and the top bar into which that is deformed in Figure 2). The "threads" or "ribbons" or related devices are then used to topologically track the "object" relative to its "environment."

But throughout all of this is an unstated assumption which we now state explicitly, that the "object" is representative of a spinor such as an electron and that the "environment" is something to which the threads can be "tied," such as nearby nuclei. But if OE&T is ever to

progress from an analogy to a direct model for real observed physics, we must progress from this schematic, and ask come very direct physical questions: Physically speaking, what is the "object," what is "the environment," and most importantly, *what are the threads?* Understanding the physical nature of the "threads" or "ribbons" is critical for advancing OE&T from an analogy to a physical model of electrons and electronic (and thereafter nuclear) structure.

On might be reflexively inclined to think of the threads as "field lines" emanating from the "object / electron" in the fashion of Faraday, but that is a classical picture and a quantum field is not a set of lines but a set of quantum parameters at every point in space. But we do not even have to answer this question in this way, if we note something very basic about the results in the last section: Starting with Figures 7, after we have performed l' double rotations, and after we have used some permutations of N and S disentanglement operations to remove all entanglement, we will *visually observe* at a twist-conserved (zero total twist) configuration in which the left ribbon (used as a convention to label the result) has $j_z' = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{7}{2}...$ double twists (+ = right handed; - = left-handed) which is a number exactly equal to the observable $j_z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{7}{2}...$ which tells us the number of $\hbar$ units of angular momentum for an electron in the absolutely analogous electronic state in an atom. Simply stated: After all is rotated and then disentangled, *it is the threads themselves, through their number of double twists, which carry the angular momentum quantum numbers ascribed to the electron (or protons or neutron) itself.* And from this we make a very important inference about physical reality:

Given that the electron (or proton or neutron) is what carries the observed j_z in the real world, and given as demonstrated that the threads / ribbons are what carry the analog j_z' in an isomorphic topological mapping, then if this topological modelling is to be used to describe real observed fermions, the *it is the threads or ribbons themselves which must be the fermions!* Mich like the "Wizard of Oz" distracting us from the "man behind the curtain," the "object" and the "environment" in the usual OE discussion, like a clever natural magician, are actually distracting us from what is really going on, especially if the assume without real examination of our assumptions that the OE "object" should be an observable fermion. *The threads or ribbons are what carry the quantum orbital properties associated with electrons; it is therefore these threads or ribbons or whatever one names them which should be regarded as the electrons.*

Indeed, in retrospect, when we topologically deformed Figure 2(a) from MTW into Figure 2(c) which has been employed here and later advanced to Figure 7 once intrinsic spin was considered, we stopped one deformation short of making the ribbon into electrons. Specifically, the distinctions (lines) between the ribbons and the "object" and environment," topologically, are totally irrelevant. The lines between ribbons and bars can be removed, the bars can be shortened and rerouted to merge them continuously and seamless with the ribbons, the earlier N and S designations can be relocated and we can add "west" W and "east" E designations, and at the end of this further topological deformation, the representation of the $m' = 0, s_z' = \pm\frac{1}{2}$ electrons in Figure 7 can be topologically deformed into Figures 8 below:

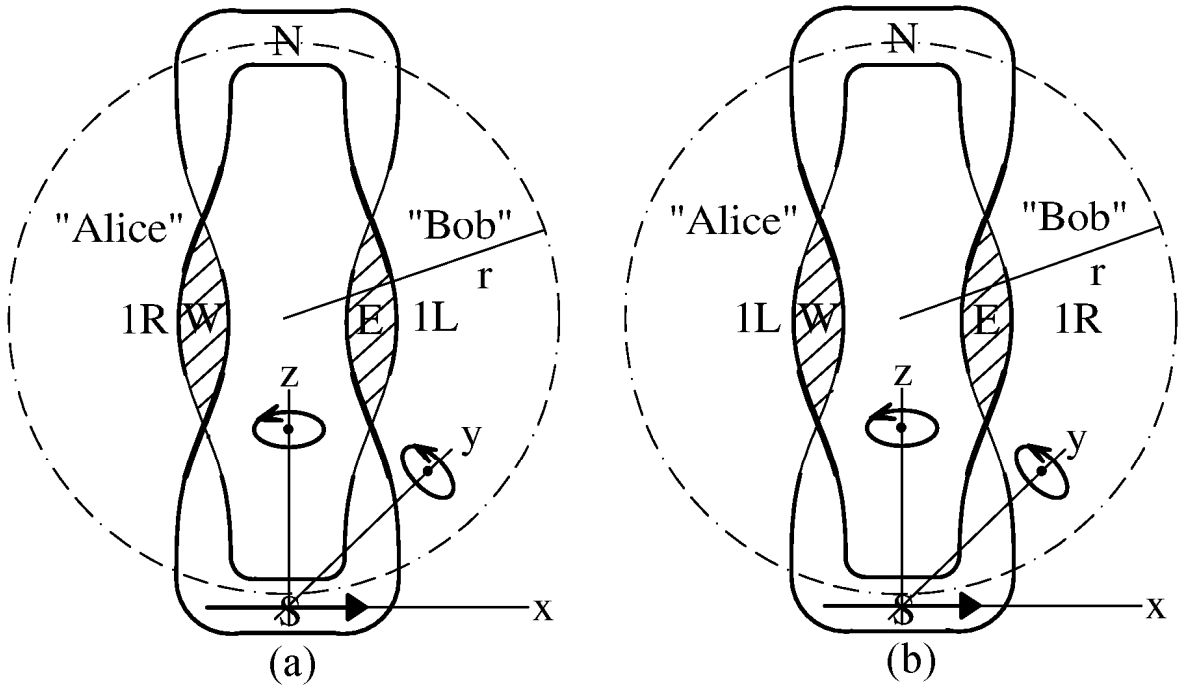

Figure 8: Topological Model of (a) Fermion observed with Spin Up $\left|m'=0,s'_z=+\frac{1}{2}\right\rangle$ and (b) a Fermion observed with Spin Down $\left|m'=0,s'_z=-\frac{1}{2}\right\rangle$

It should be apparent that Figures 8 are completely topologically equivalent to Figures 7. But in Figures 8, there is no longer any conceptual distinction between the "object" on the one hand, and the "environment" and the "threads" on the other. Figure 8(a) is the topological model of a spin up electron and Figure 8(b) is the topological model of a spin down electron. All of the additional states analogous to the observed orbital angular momenta, such as those in equations (6.1) to (6.4), involve merely rotating the vector at the renamed south pole about the z-axis, and then untangling the W and E threads in various permutations about the southwest SW corner until all that remains are twists without entanglements, precisely as was done before. It is easily seen that these figures can be further stretched and rounded without changing their inherent topologies, which is indicated schematically by the circles overlaid with each of these figures.

But now, we are not rotating of an "object" relative to an "environment" followed by disentanglement of distinct "threads." Now, the *electron itself* is seen as a "ribbon" or "loop" or "ring" or, yes, a "string" (with some finite thickness), which is inherently-twisted to contain two oppositely-handed twists, and which, like a pretzel of sorts, can have some of its parts rotated relative to its other parts about the y or z axes, and then disentangled about the southwest corner. (As mentioned in the last section, there is never a rotation about the x axis, or more accurately, about any local tangent to the string, because this would change the string's intrinsic twist which analogizes to changing its intrinsic spin.) These angular-momentum-analogs developed in sections 5 and 6 no longer describe OET of some "object" relative to an "environment," but the *pretzel-like OET of a string-like electron in relation to itself.* They are now, loosely speaking, a topological characterization of a pretzel and the deformations use to create the pretzel. After all

of this is disentangled, we end up precisely as before, with states that map in the manner of quantum exclusion to the angular momenta of an s, p, d, f, etc. shell electron.

This fully accounts, by analogy, for three of the four quantum numbers l, m and s_z. As to the principal quantum number n not yet addressed, we are mindful that in the original Bohr model of the atom, this quantum number arose in relation to the orbital angular momentum via $\mathbf{L} = n\hbar$. Of course, this relationship has long since been superseded, but to this day n still represents the overall energy of each orbital, and as orbital energy increases, so too does distance from the nucleus, and thus, the mean radius of the shell. It is for that reason that we have also shown circles with radii r in Figures 8, with the view not being quantified at this time, that this final n quantum number is accounted for by the radius of the electron string, so that n=1 is represented by the Figure 8 twisted strings having some minimum radius, and n=2,3,4 etc. are represented by string radii which are larger in some way still-to-be quantified.

For nomenclature, we shall adopt the word "string" to refer to these models of a fermion in Figure 8, and it is apparent that these are strings with twists. This is meant to validate the string theory approach widely reviewed in the literature, with the caveat that what we mean to validate, specifically, is the idea that an electron or similar particle might be represented by a closed physically-real loop, in physical three-dimensional space, which has a finite radius. Because the twisting of and twists within this closed string are of vital importance in topologically connecting these strings to the observed l, m and s_z, and to the degree that a "twist" in a "string" cannot be discerned unless that string has some finite cross-sectional width, we shall regard these strings as in reality having an inherent, also to-be-determined, finitely dimensioned cross-section in the physical three-dimensional space.

We shall refer to the indicated four sectors of these rings as north, south, east and west, and although we have changed the earlier N and S designations, we maintain a reference vector displayed in the south sector so we can talk easily about rotating the south part of the string about the north portion of the string whereby every 4π rotation adds one double-rotation unit of l'. Because it is helpful to enable discussion of the Einstein, Podolsky & Rosen (EPR) [30] paradox with reference to these twisted rings, we may also refer to the western and eastern sectors, respectively, as thoee of "Alice" and "Bob." And as a reminder of the origin of Figures 8 from Figures 7, we shall refer to the north and south sections of these rings, formerly the "bars," as the "anchors."

As before, rotations which add l' involve holding the north anchor fixed while rotating the vector tattooed on the south anchor right-handedly about the z-axis. This is simply a choice of convention; the results do not change if we maintain an immobile south and a movable north anchor. As before, just with new names, untangling will always proceed about the west side of the south anchor, because using the southeast corner would not untangle but adds further tangling, as has been discussed earlier. The east and west string sections (a.k.a. Alice and Bob sections) can be untangled in any permutations we wish. For example, see (6.3) and (6.4), we can use any of $\{W,W,W,W\}$, $\{W,W,W,E\}$, $\{W,W,E,E\}$, $\{W,E,E,E\}$ OR $\{E,E,E,E\}$ to untangle the $l'=2$ entanglement, and this together with s_z' is what determines which of the $|l',m',s_z',j_z'\rangle$ we end up with after all untangling is complete.

Topologically, a single rotation of -2π and $+2\pi$ about the labelled x axis in figures 8(a) and (b) respectively would remove all twists and turn these into true "rings." So these twisted strings are *not* Mobius strips. But, because such a rotation about the x axis as discussed in the previous section would turn this from a spin ½ representatives to a spin 0 representative, we shall forbid rotations about x and only permit rotations about y and z. Of course, coordinate systems may be defined locally and nobody ought be barred from freely choosing using coordinates other than what and where these are shown in Figures 8. So what we really need to forbid to preserve intrinsic spin, is rotating any localized part of the string about the axis that runs locally parallel (tangentially) to the body of the string, because that would change the twists that we are using to represent the spin. While rotations about y and z (in the orientations locally depicted) are permitted, as earlier discussed, y rotations and z rotations end up with the same results. So as a convention we shall always use z rotations when we first add l' to an electron before we untangle it.

Finally, so we have in mind some definitive possible mapping between the quantum numbers *n*, *l*, *m* and s_z from atomic physics and the topological l', m', s_z' developed so far here, let us formalize a principal topological number n', at least qualitatively, which we now introduce and designate by a positive integer $n' = 1, 2, 3, 4....$ An increase in this n' is to correlate to a presently-unquantified increase in the mean radius of the string loops in Figure 8. Then, if we are using these strings to model the quantum states of electron in atoms, we simply use strings of greater radius, e.g., $n' = 2, 3, 4...$ once we fill all the exclusionary states for a given respective radius $n' = 1, 2, 3....$ Then, because a larger radius hence greater circumference provides enough extra string length to allow more entangling twists, we can put this into a new relation $l' < n'$ analogizing to the atomic relation $l < n$. Simply put: a longer string can be given more tangles and twists than a shorter string.

Now let's use Figures 8 to develop some physics connections.

Dating back to Fermat, Maupertuis, Euler, Lagrange and Hamilton, the principle of least action written in the present day in spacetime as $\delta S = \delta \int d^4x \mathcal{L} = 0$, has been the foundation for describing the dynamical behaviors of physical systems. Albert Einstein gave *geometric* voice to this principle via the geodesic equation $d^2x^\mu / dt^2 + \Gamma^\mu_{\alpha\beta}\left(dx^\alpha / dt\right)\left(dx^\beta / dt\right) = 0$ of General Relativity, in which the Newton's first law that a material body will move in a straight line absent a second-law force acting thereon was generalized to the law that such bodies will move in straight *geodesic* lines when the spacetime containing that movement possesses local Riemannian, non-Euclidean curvature. The prospect raised in Figures 8 above that physical electrons and other fermions might be represented by the topological tanging and detangling and twisting of twisted strings suggests that we ought to supplement this line of development with some form of *topological* least action principle. After all, it is one thing for a person to build an apparatus like that of Figures 7 and then manually apply forces to rotate the bars and untangle the threads. It is quite another thing for an electron bearing some physical relation to Figures 8 to first find itself twisted and tangled up in some way, and then to somehow untangle itself into

configurations represented by states such as those of (6.1) to (6.4) which map to its quantized states as part of an atomic shell structure, all untouched by human hands.

It will be familiar to anybody who has even toured a science museum that planetary orbits are often geodesically modelled by introducing a moving ball into a concave receptacle such as the upper half of http://37.media.tumblr.com/tumblr_l0j2tgKctv1qb0ukuo1_r1_400.jpg and then watching the ball "orbit" about the central neck until frictional losses draw the ball into the neck to disappear from view. Of course, gravitation itself is drawing the ball down into the neck and that is not how the geodesic equation really works, but the model still imparts a true visualization as to how curved geometry influences natural straight-line motion without overtly applying any acceleration-inducing force.

A similar simple visual model can be developed for *topological* least action that will be understood by any parent or child who has ever attended a swing set at a playground: Going back to Figures 7, imagine that the lower bar is the swing seat upon which the child sits, the upper bar is the top fulcrum of the swing set, and the two ribbons are the chains which connect the seat to the top of the swing set, and that these chains happen to each have one twist in them which we are forbidden from removing. Let us think of this in least action or lowest energy terms as the *topological* "ground state" to which the swing seat will always dynamically to return, if it is not otherwise prevented from doing so.

Now let's do what any playful child has ever done, and instead of using the swing as intended, let us twist the swing seat a few times about a z-axis pointing up, so that now the two ribbons (chains in the swing set) have become twisted about and entangled with one another. What will now happen, dynamically? If the child lets go of the swing (removes any force from the swing) and steps back, then the swing will rotate about the z axis oppositely to how the child first wound it, it will go past its original configuration to some extent due to its rotational inertia, it will thereafter slow to a stop. Then it will reverse direction, and do this through a few pendulous cycles until the damping effect of friction has drawn off all energy and the swing seat has returned to its ground state. And if the child sits on the swing while this is all happening, the child will get dizzy and perhaps provide the parent with an opportunity to modify the child's behavior. This is a topological analog of the straight line motion of Newton's first law. The swing, after release, has no restraint imposed along any of the three space dimensions, and so it can rotate freely to untangle itself, until it is restored to its ground state. As with the geodesic museum model, the analogy is not perfect because gravity is drawing the swing seat back to its ground state and we do not expect gravity to be untangling electrons from themselves. But this is still a valid model for *topological* least action dynamics in the in the same manner as the ball in the concave receptacle is a valid model for *geometric* least action geodesic dynamics, and allows us to state several definitions and then postulate a *first law* of topological least action.

The *definitions* are as follows:

> *Rotation: an orientation change in vectorial direction of a section of a string loop relative to other regions of the string loop.*
> *Entanglement: a state in which two or more sections of a string loop have become twisted around one another.*

Twist: a state in which single section of a string loop has become intrinsically twisted unto itself, without reference to any other sections of the string loop.
Untangling: an operation to remove an entanglement without rotation, by moving a first part of a string loop over and around a second part of the string loop.

With these definitions, the *first topological law* of least action is as follows:

1. A string loop which is in a state of entanglement will naturally evolve to become untangled by rotation, absent a force acting on the string loop which prevents its untangling by rotation.

This describes the situation where the child lets go of the swing seat and everything naturally reverts back to the ground sate through a damped pendulous rotation. We note – because this will connect to the three-dimensional environment of the Dirac-Wu-Yang use of gauge theory to derive DWY monopoles and fractionalized, and quantized charges – that this first topological law is a statement of dynamics operating with access all three dimensions. There are no dimensional restraints on this swing.

Now let's suppose that the child at the swing set is particularly clever and mischievous (which often go hand in hand), notices that each chain already has one twist a.k.a. half a double twist ($\left| j' = \pm\tfrac{1}{2} \right\rangle$), and decides he or she would like to leave even more twists in the chains before the next child comes along to use the swing. So being exceptionally clever and realizing that nothing can be untangled following a single rotation, the child introduces two rotations about the z axis ($+4\pi$) which is one double entanglement ($l' = 1$), but then does not release the seat. The child continues to exert a force which prevents seat from rotating. Now the child has imposed a dimensional constraint. If in Figures 8 the child is using force to prevent the vector → at the south anchors from rotating about the z axis, then the overall swing and its chains are confined to the two dimensions of the x-z plane. The y axis is off limits, and least for rotations of the seat.

But suppose the child now decides to untangle its handiwork by passing the left chain twice about the left side of the swing, which is to say, by applying the what is $\{N, N\}$ operation in Figure 7, leading to the final state $\left| l' = 1, m' = +1, s'_z = +\tfrac{1}{2}, j'_z = +\tfrac{3}{2} \right\rangle$ on the top line of (6.1). With that maneuver, the next child to arrive at the swing will encounter three halves of a double twist ($\left| j' = \pm\tfrac{3}{2} \right\rangle$), i.e., three twists in each chain. In Figure 8(a) this entails twice passing the W section of the string loop over and around the southeast corner and then around the south anchor. This is the template for a *second topological law* of least action, which is as follows:

2. If there is a force which prevents a string loop in a state of entanglement from untangling by rotation, that string will naturally evolve to become untangled by untangling, absent a force acting on the string loop which prevents its untangling by untangling.

Now the reader will notice that the child has exploited a small but important loophole to untangle the swing without using a rotation: While the y-axis has been placed off limits to rotation and so the string loop really is restrained to two dimensions, the very act of "moving a first part of a

string over and around a second part of the string" does require that there be at least some nominal incursion into the y axis, albeit very small and very limited, in order to bring the W or E sections of the string around the vector → on the south anchor. Specifically, if carefully done, the untangling operation can be performed beneath the → in the x-z plane without *ever* bringing the ribbon / string section off of the x-z plane, *except* a) when it is brought past the southwest corner and b) when it is against brought past the southeast corner. At each of these corners, the string will need to assume a coordinate position $y=-\varepsilon$, where ε is some small but finite number at least as large as the width of the string itself at is narrowest cross section. It is worth the reader carrying this exercise to witness this. So these two corners represent "chokehold" positions for untangling: if everything is confined to the two dimensions of the x-z plane, and there is a choke against even going to $y=-\varepsilon$ at the SW and SE corners, then there is no way to untangle the string, either by rotation or by untangling.

If the reader has carefully followed these two scenarios, then it will become apparent that there is also a natural dynamical preference for twist over entanglement. In the scenario just described, a system which stated with $\left|l'=0,\, j'=\pm\tfrac{1}{2}\right\rangle$ was rotated through $+4\pi$ ($l'=1$) into a state $\left|l'=1,\, j'=\pm\tfrac{1}{2}\right\rangle$ where the two chains were mutually *entangled* through two turns. Let us represent this "double entanglement" (one double twist of chains or string sections *around one another*) by $\left|e'=1\right\rangle$. Rotation was then removed by the child as an option for untangling. So now there were two options left: First, the system can remain in the state $\left|e'=1,l'=1\right\rangle$ with one double entanglement following a double rotation. Second, the system can be untangled by one of the $\{N,N\}$, $\{N,S\}$ or $\{S,S\}$ operations represented in (6.1) and (6.2). If we use $\{N,N\}$, then $\left|e'=1,l'=1\right\rangle$ evolves to $\left|e'=0,l'=1,\, j'=\pm\tfrac{3}{2}\right\rangle$, and the unit of double-twist entanglement is removed, $e'=1\Rightarrow e'=0$ at the expense of adding one net unit of double twist to each chain, $j'=\pm\tfrac{1}{2}\Rightarrow j'=\pm\tfrac{3}{2}$. If we take note an electrons will evolve, for example, from their free state of $j=\pm\tfrac{1}{2}$ to an in-shell state of $j=\pm\tfrac{3}{2}$ if required to join an atomic shell, then this leads us by analogy to discern a *third topological law* of least action:

> *3. A string loop in a state of entanglement will seek to become untangled, always first by rotation. If there is a dimensional restraint barring rotation, then the string loop will seek to become untangled by untangling, even if this creates more twists.*

Now, let us imagine that we take a single chain and hang a swing seat by that one chain. Whatever twist may reside in that chain, if any, once the pendulous rotation of the swing ceases, we will have found the ground state for the twist. This leads us to a *fourth topological law* of least action:

4. A section of string taken out of its ground state of twist will naturally untwist back toward its ground state of twist, absent a force which prevents its untwisting.

Both topological laws 3 and 4 work together, because while an individual string section will seek its ground state of twist *if it can*, untangling takes precedence over untwisting. So maintaining

an untangled state, in and of itself, may bring about the force which prevents untwisting into the ground sate. We also noted in the earlier discussion that if one were to rotate Figures 8 about the x axis by one turn at the local area where this axis is shown, then the spin ½ could be removed. This is not permitted, and so in Figures 8, it is one-half of a double twist which is the ground state of twist.

Now let's consider the final situation, already previewed, where the child has rotated the swing a few times, does not let go (remove his or her force) to permit a counter-rotation that removes the tangles and twists the child has so-expertly introduced, and does even not allow any of the chains to be passed about the seat so as to at least untangle the chains even if that means adding more twists to each chain. Now the child has choked off the SW and SE corners of the swing. And suppose the child has a bad tantrum and goes so far as to spray water all over the chains so that they rust in place, or to place cement or glue into the chains so that they are all gummed into place, or to spray liquid nitrogen all over the chains so that they are frozen into place (the "frozen" analogy is intentional because this is what will lead us to FQHE). Now, the swing is simply stuck with extra tangles and extra twists that cannot be removed, and when the next child arrives, what was expected to be a swing set will have a somewhat different appearance and not be usable in the way that was expected. And if the swing is really an electron, this choking off of untangling occurs near 0K, and it simultaneously restores a U(1) electric-magnetic symmetry, gives rise to Dirac-Wu-Yang monopoles, and causes the electric and magnetic charges to appear in quantized multiples of fractionalized charges.

At the moment, this is still an analogy under development, with physical connections also under development. But as earlier noted in section 3, just as GUTs have certain symmetries which are broken at lower energies, *low temperatures* near absolute zero are also thought to cause displays of certain symmetries which become broken at higher temperatures, per Volovok. [18] So if the migration from electrons which are free to move without restraint in three dimensions, to electrons which are somewhat topologically constrained to be able to occupy shells in atoms, to electrons which are even more tightly constrained to be wholly confined to two dimension because the temperatures are now right near absolute 0K, can be shown to parallel the migration from the free electron string loops of Figures 8, to configurations where all rotation is prevented for untangling (first law is unavailable), to a configuration where a choke is placed so that the topology is tightly restricted to two dimensions and no untangling whatsoever is permitted, then we will have found a path to physically connect the three-dimensional analysis of DWY, topologically, to the two-dimensional environment of FQHE.

The foregoing four topological least action principles are the first step to developing these physical connections. The next step is previewed by another simple question. Spin and angular momentum all comes in units or half units of $\hbar$. Of course, one can always work in natural units with $\hbar = c = 1$. But when we step back out of natural units, $\hbar$ is a unit of action which is also a unit of action. We have been analogizing units of angular momentum against units of double-twist. Units of double twist are truly dimensionless numbers. So how do twists, which are topological numbers, physically become angular momentums, which represent circular flows of energy? That is the subject of the next section.

8. From Topological Twist to Quantized Angular Momentum (to be added)

To be added.

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