Dirac-Wu-Yang Monopoles, Gauge Symmetry, Orientation-Entanglement & Twist, and how these Underlie the Fractional Quantum Hall Effect

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Abstract: The purpose of this paper is to explain the pattern of fill factors observed in the Fractional Quantum Hall Effect (FQHE), which appears to be restricted to odd-integer denominators as well as the sole even-integer denominator of 2. The method is to use the mathematics of gauge theory to develop Dirac monopoles without strings as originally taught by Wu and Yang, while accounting for orientation / entanglement and related "twistor" relationships between spinors and their environment in the physical space of spacetime. We find that the odd-integer denominators are included and the even-integer denominators are excluded if we regard two fermions as equivalent only if both their orientation and their entanglement are the same, i.e., only if they are separated by 4π not 2π. We also find that the even integer denominator of 2 is permitted because unit charges can pair into boson states which do not have the same entanglement considerations as fermions, and that all other even-integer denominators are excluded because only integer charges, and not fractional charges, can be so-paired. We conclude that the observed FQHE fill factor pattern can be fundamentally explained using nothing other than the mathematics of gauge theory in view of how orientation / entanglement / twist applies to fermions but not to bosons, while restricting all but unfractionalized fermions from pairing into bosons.

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Contents

1. Introduction: Wu and Yang and the Dirac Monopole without Strings

The Fractional Quantum Hall Effect (FQHE) observed in two-dimensional systems of electrons at low temperatures in superconducting materials subjected to large magnetic fields is characterized by observed filling factors $\nu = n/m$, where *n* and *m* are each integers, but where *m* is an *odd integer only*, with the exception that *m may also be the even integer 2*. In other words, the apparent pattern, widely reported and studied in the literature, is $\nu = n/m$ with $n = 0, \pm 1, \pm 2, \pm 3...$ and $m = 1, 2, 3, 5, 7, 9, 11...$, see, e.g., [1], [2], [3], [4], [5], [6] generally, and for the even denominator 2, see, e.g., [6], [7] ($\nu = 1/2$), [8] ($\nu = 3/2$), [9] ($\nu = 5/2$) and [10] ($\nu = 7/2$). Two questions arise from this effect: why are the denominators in the filling factor odd but not even (including the quantization of whole unit charges with *m*=1), and why is the even denominator *m*=2 an apparent exception? We show that this pattern of filling factor denominators has a fundamental explanation based on using the mathematics of gauge theory to develop the Dirac Quantization Condition (DQC) for Dirac-Wu-Yang monopoles, in view of how orientation-entanglement applies to fermion spinors but not to bosons, and also in view of a "twisting" associated with orientation-entanglement which appears to have been underreported in the literature.

In 1931 Dirac discovered that the existence of magnetic monopoles implies that the electric charge must be quantized [11]. While charge quantization had been known for several decades based on the experimental work of Thompson [12] and Millikan [13], Dirac was apparently the first to lay out a possible theoretical imperative for this quantization. Using a hypothesized solenoid of singularly-thin width known as the Dirac string to shunt magnetic field lines out to mathematical infinity, Dirac established that a magnetic charge strength μ would be related to the electric charge strength *e* according to $e\mu = 2\pi n$, where *n* is an integer, which became known as the Dirac Quantization Condition (DQC). Subsequently, Wu and Yang used gauge potentials, which are locally- but not globally-exact, to obtain the exact same DQC without strings [14], [15]. Their approach is concisely summarized by Zee on pages 220-221 of [16] and will be briefly reviewed here, because it provides the methodological basis for understanding the pattern of filling factors observed for the FQHE. Throughout we use the natural units of $\hbar = c = 1$.

Using the differential one form $A = A_\mu dx^\mu$ for the electromagnetic gauge field a.k.a. potential and the differential two-form $F = \frac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu = dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu$, a magnetic charge μ may be *defined* as the total net magnetic flux $\mu \equiv \oiint F$ passing through a closed two-dimensional surface S^2 which for convenience and symmetry we may take to be a sphere. Differential exterior calculus in spacetime geometry teaches that the exterior derivative of an exterior derivative is zero, *dd*=0, which means that the three-form equation $dF = ddA = 0$. Thus, via Gauss / Stokes, $\iiint 0 = \iiint dF = \oiint F = \mu$. In classical electrodynamics prior to Dirac, this was taken to mean that the magnetic charge μ=0. But a close consideration of gauge symmetry, which is locally but not globally exact, tells a different story.

When a spin ½ fermion wavefunction (which we may regard to be that of the electron) undergoes a local gauge (really, phase) transformation $\psi(x) \to \psi'(x) = e^{i\Lambda(x)}\psi(x)$, the gauge field one-form transforms as

$$A \to A' = A + e^{-i\Lambda} de^{i\Lambda} / ie \,. \tag{1.1}$$

If we represent F in polar coordinates (r, φ, θ) as $F = (\mu / 4\pi) d\cos\theta d\varphi$, then because $F = dA$ and dd=0, we can deduce that $A = (\mu / 4\pi)\cos\theta d\varphi$. However, $d\varphi$ is not defined on the north and south poles. So we may define a north coordinate patch over which $A_N = (\mu / 4\pi)(\cos\theta - 1) d\varphi$ and a south patch over which $A_S = (\mu / 4\pi)(\cos\theta + 1) d\varphi$. But at places where these patches overlap, these gauge potentials are not the same, and specifically, the difference is $A_S - A_N = (\mu / 2\pi) d\varphi$, or written slightly differently:

$$A_N \to A'_N \equiv A_S = A_N + (\mu / 2\pi) d\varphi \,. \tag{1.2}$$

So comparing this with (1.1), we may regard A_S as a gauge-transformed state A'_N of A_N, for which the gauge transformation is simply:

$$\frac{1}{ie} e^{-i\Lambda} de^{i\Lambda} = \frac{\mu}{2\pi} d\varphi \,. \tag{1.3}$$

This differential equation for Λ and φ in relation to e and μ is satisfied by:

$$\exp(i\Lambda) = \exp\left(ie\mu \frac{\varphi}{2\pi}\right), \tag{1.4}$$

as can be seen simply by plugging $e^{i\Lambda}$ from (1.4) into the left hand side of (1.3) and reducing. This relates the azimuth angle φ which is one of the three spacetime coordinates, to the local gauge (phase) angle Λ, and thereby connects rotations through φ in physical space to rotations through Λ in the gauge space in a manner that we shall now explore in detail.

In polar coordinates, $\varphi = 0$ and $\varphi = 2\pi$ in (1.4) describe exactly the same *orientation* (but not entanglement) on the surface of S^2. So to make sense of (1.4) at like-orientations, we must have:

$$\exp(i\Lambda) = \exp(ie\mu \cdot 0) = 1 = \exp(ie\mu \cdot 1), \tag{1.5}$$

Specifically, this means that $\exp(ie\mu) = 1$. Mathematically, the general solution for an equation of this form is $\exp(i2\pi n) = 1$ for any integer $n = 0, \pm1, \pm2, \pm3...$, which is infinitely degenerate but quantized. As a result, the solution to (1.5) is:

$$\Lambda = e\mu = 2\pi n . \tag{1.6}$$

This, of course, is the Dirac Quantization Condition, which may also specified in relation to the gauge (phase) parameter Λ which is seen to be an quantized integer multiple of 2π. Specifically, (1.6) with simple rearrangement tells us that the electric charge is quantized according to:

$$e = n\frac{2\pi}{\mu} = ne_{\mathrm{u}} = \frac{\Lambda}{\mu}, \tag{1.7}$$

where the n=1 "unit" (u) of electric charge is $e_{\mathrm{u}} \equiv 2\pi/\mu$, defined as 2π times the inverse of the magnetic charge. The customary interpretation of $e = n(2\pi/\mu)$ in (1.7), ever since Dirac first found this relationship, is that if this magnetic charge "exists," then the electric charge is quantized in units of e_{u}. It is important to keep in mind that the converse is not true: the observed quantization of electric charge does *not* imply that the magnetic charges do exist. In fact, as best as is known, this magnetic charge μ has not been observed to date, and the quantization of electric charge is explained not on the basis of these Dirac-Wu-Yang magnetic charges, but on the basis of the charge generators $Q = Y/2 + I^3$ which emerge in Yang-Mills gauge theory following the electroweak symmetry breaking of $SU(2)_W \times U(1)_Y$ down to $U(1)_{em}$.

We may then go back to the original definition $\mu \equiv \oiint F$ and use (1.6) to write:

$$\oiint F = \mu = \frac{2\pi}{e} n = n\mu_{\mathrm{u}} = \frac{\Lambda}{e}, \tag{1.8}$$

where we also define an n=1 unit of magnetic charge $\mu_{\mathrm{u}} \equiv 2\pi/e$, similarly quantized. By appropriate local gauge transformation, and specifically by choosing n=0 which is the same as choosing the phase angle $\Lambda = 0$, the nonzero surface integral can be made to vanish, $\oiint F = 0$. But this does not invalidate (1.7) and (1.8) nor does it prevent us from seeking to draw physical conclusions from these. It simply means that n=0 with no monopoles and no electric charges is one of the permitted states. But again, the meaning of the whole range of charges $e = ne_{\mathrm{u}}$ for $n \neq 0$ has been physically-interpreted since Dirac, as suggesting that the "existence" of a magnetic charge would imply electric charge quantization, with the further understanding that the converse is not true.

In the lowest positive non-zero n=1 state, from (1.6), we have $\Lambda = e\mu = 2\pi$. So if we define a reduced $\not\Lambda \equiv \Lambda/2\pi$, then by (1.6), the reduced gauge parameter $\not\Lambda = n$ is simply the charge quantum number n. So every gauge transformation adding an angle of 2π adds one unit of electric and magnetic charge. Thus, although $\Lambda = 2\pi, 4\pi, 6\pi ...$ are *mathematically*-equivalent angles, they do appear (if these monopoles "exist") to be *physically*-distinguishable because of their connection $\not\Lambda = n$ to charge quantization. This is an important observation which will show itself in a number of ways throughout the forthcoming development.

This is how Wu and Yang obtain Dirac monopoles and the DQC without strings.

2. The Fractional Denominators Indicated by Dirac-Wu-Yang: are they Somehow Related to FQHE?

If we closely study this derivation by Wu and Yang, we see that there are some additional quantum states indicated that have not yet been considered. Referring to (1.5), not only do $\varphi=0$ and $\varphi=2\pi$ describe exactly the same *orientation* (sans entanglement), but so too do $\varphi=4\pi$, $\varphi=6\pi$, $\varphi=8\pi$, etc. So we may now extend (1.5) to:

$$\exp(i\Lambda)=1=\exp(ie\mu\cdot 1)=\exp(ie\mu\cdot 2)=\exp(ie\mu\cdot 3)=\exp(ie\mu\cdot 4)=\exp(ie\mu\cdot 5)=\exp(ie\mu\cdot 6)\ldots \quad (2.1)$$

Each of the above is a separate relationship of the general form $\exp(ie\mu\cdot l)=1$, where $l=1,2,3,4,5,6...$ is an integer not the same as the *n* already in use. At the same time, as noted after (1.5), the general solution for an equation of this form is $\exp(i2\pi n)=1$ with this integer $n=0,\pm1,\pm2,\pm3\ldots$. Comparing $\exp(ie\mu\cdot l)=1$ with $\exp(i2\pi n)=1$ means that more generally, $e\mu\cdot l=2\pi n$, or, restated (also using $\Lambda=2\pi n$ from (1.6)):

$$e=\frac{n}{l}\frac{2\pi}{\mu}=\frac{n}{l}e_{\mathrm{u}}=\nu e_{\mathrm{u}}=\frac{\Lambda}{l}\frac{1}{\mu}, \quad (2.2)$$

where we define a "filling factor"

$$\nu\equiv\frac{n}{l}; \quad n=0,\pm1,\pm2,\pm3...; \quad l=1,2,3,4,5,6\ldots. \quad (2.3)$$

So this tells us that if these monopoles "exist," not only is the electric (and magnetic) charge quantized, but each unit of electric charge e_{u} (or magnetic charge μ_{u}) can be fractionalized into any $\nu=n\cdot(1/l)$ quantized *n* fraction $1/l$ of itself.

Taking (1.7) together with (2.2) and (2.3), this means that if the Dirac-Wu-Yang monopole "exists," then all particles carrying electromagnetic charge must obey (1.7). But they will also obey the more permissive conditions of (2.2) which lead to fractionalized charges. We then see, that (1.7), $e=ne_{\mathrm{u}}$, is a special case of (2.2) and (2.3), $e=(n/l)e_{\mathrm{u}}$ with $l=1,2,3,4,5,6...$, in the particular circumstance where *l*=1.

Now, one may take the view that (2.2) and (2.3) are just a "trivial" extension of (1.7), because mathematically, it is certainly trivial that the angles $\varphi=2\pi$, 4π, 6π, 8π, etc. have the same trigonometric properties as the angle $\varphi=0$, and rotational orientation is indistinguishable as between any set of angles $\varphi=\varphi_0+2\pi n$ with $n=0,\pm1,\pm2,\pm3...$ differing from some base angle φ_0 by an integer multiple of 2π. And this may explain why (2.2) and (2.3) do not appear to have been developed or studied in the literature to nearly the same degree as the DQC. But in the context of (2.2) and (2.3), the fact that this trivial trigonometric concurrence of $\varphi = 0$ with $\varphi = 2\pi$, 4π, 6π, 8π leads to fractional charges is anything but trivial: Just as the DQC motivates us to consider whether the Dirac monopoles exist and if so in what fashion and under what circumstances, the logical extension of the DQC via Wu and Yang motivates us to ask the same questions about fractional charges which we ask about Dirac monopoles, because (2.2) and (2.3) package all of these questions inseparably together: If these monopoles exist, then *gauge theory itself* inexorably implies that electric charge is quantized, *and also that electric charge is fractionalized*. This means that the DQC is really a DQFC, Dirac Quantization *and Fractionalization* Condition. And although this emanates from something that is mathematically trivial, namely the trigonometric indistinguishability of rotational angles $\varphi = 2\pi$, 4π, 6π, 8π etc. from one another, the fact that gauge theory tells us that the DQC is really a DQFC, is highly non-trivial and must be explored.

So, given that the Dirac-Wu-Yang arguments do lead not only to charge quantization, but inexorably to the *quantization of fractionalized charges*, i.e., to filling factors $\nu = n \cdot (1/l)$, and given that fractionalized charges *are experimentally observed* in the FQHE, one is motivated to explore the question whether there is a connection between the two. So we now pose the question which will be the subject of the remainder of this paper: might the DQFC (2.2) and (2.3) actually be related in some way to the FQHE? And if so, how?

For there to be a valid connection between the fractionalized charge states of Dirac-Wu-Yang monopoles and the fractionalized quasiparticle states of FQHE, there are two main problems that must be overcome, one experimental and arithmetical, the other theoretical and physical. First, *experimentally and arithmetically*, while the fractional charge denominators permitted by Dirac-Wu-Yang in (2.3) may assume any integer value $l = 1, 2, 3, 4, 5, 6...$, the denominators *observed* in the FQHE are more restricted: they only take the values $l = 1, 2, 3, 5, 7, 9...$. That is, the observed denominators are always odd integers, with the exception that the even integer $l = 2$ is also observed, see again, [6], [7], [8], [9], [10]. So, the Dirac-Wu-Yang approach – if it applies at all to FQHE – must be able to explain this arithmetically-restricted experimental pattern of observed denominators. The result in (2.3) is simply too inclusive, i.e., it includes states which are not observed alongside states which are.

Second, *theoretically and physically*, even if the denominator pattern $l = 1, 2, 3, 5, 7, 9...$ can be explained, applying the Dirac-Wu-Yang arguments to the fractionally charged quasiparticles in FQHE systems is still physically nontrivial. The Dirac-Wu-Yang theoretical argument is developed within the three-dimensional physical space of spacetime geometry, and is understood to apply to systems of electrons, protons, and neutrons for which no fractionally charged particles and no Dirac-Wu-Yang magnetic monopoles have ever been observed. But at the level of analysis where the quasiparticle language applies, the system is fundamentally two-dimensional, because the superconducting materials used together with the ultra-low temperatures and large magnetic fields applied to stimulate the observed FQHE, combine in some fashion to substantially remove one degree of spatial freedom from the electrons and so restrict the electrons to two space dimensions. And in some way that needs to be understood, these all synergistically coact to produce the $l = 1, 2, 3, 5, 7, 9...$ denominator pattern. Because of this difference between the three-dimensional space of Dirac-Wu-Yang and the two-dimensional restricted space of FQHE, one might take the *a priori* view that there is no connection between Dirac-Wu-Yang and FQHE. So for certain, at the very least, if there is some hidden, not-yet-understood connection between these two fundamentally-different environments of Dirac-Wu-Yang and FQHE, it is important for such a connection to be carefully developed and articulated.

3. Orientation-Entanglement, and the Odd-Integer FQHE Denominators

Spinors, which includes electrons, reverse sign upon a spatial rotation through an angle φ by an odd multiple of 2π. Specifically, as Misner, Thorne and Wheeler (MTW) point out in one of the most widely-regarded discussions of this topic in [17] at section 41.5, the spin matrix of a rotation $R = \cos(\varphi/2) - i(\mathbf{n} \cdot \boldsymbol{\sigma})\sin(\varphi/2)$ (see MTW [41.48]) reverses sign upon a rotation through an odd multiple of 2π, as does the sign of a spinor under $\xi \to \xi' = R\xi$ (MTW [41.50]). This sign reversal does not, however, appear in the transformation law for a vector, $X \to X' = RXR^*$ (MTW [41.49]).

Misner, Thorne and Wheeler provide a visual, macroscopic, intuitive, essentially-topological understanding for this result by considering the orientation and entanglement of an object relative to its surrounding environment, because while orientation is restored under a 2π rotation, it takes a 4π rotation to restore the object's state of entanglement, i.e., to restore the complete "version" of the object. They do, however, at page 1148 of [17], make the statement:

> "Whether there is also a detectable difference in the physics . . . for two inequivalent versions of an object is not known."

This question of whether MTW orientation-entanglement brings about detectable physics in the study of physical systems will occupy a fair share of the analysis to follow in this paper.

Now, the gauge transformation $\psi(x) \to \psi'(x) = e^{i\Lambda(x)}\psi(x)$ with which we originally started at (1.1) acts on a Dirac fermion wavefunction which we may take to be that of an electron. And electrons are Dirac spinors. So as such, the sign of this wavefunction will reverse under any rotation from a given φ to $\varphi + 2\pi$ and will only be restored under two rotations, i.e., when rotated from φ to $\varphi + 4\pi$. Therefore, let us suppose that some weight needs to be ascribed to the *version* of the electron and not only its orientation, and therefore revisit (2.1) where we equated the entire set of rotations differing from one another by only 2π, not 4π.

Specifically, let us now explore the consequences of taking the more-restrictive view that two terms $\exp(ie\mu(\varphi/2\pi))$ in (1.4) may be equated *if and only if they differ from one another by* 4π. Then, let us start with a Dirac fermion in the $\varphi = 2\pi$ state, which as seen in the derivation leading to (1.6) is the $n = 1$ state for which the DQC gives $e\mu = 2\pi$. Thus, for this state, we have $\exp(ie\mu(\varphi/2\pi)) = \exp(i2\pi(2\pi/2\pi)) = 1$. Then, because we are starting out with a fermion oriented to $\varphi = 2\pi$, the equivalent *versions* will be those for which $\varphi = 6\pi$,

$\varphi = 10\pi$, $\varphi = 14\pi$, etc. As a result, when we apply this entanglement restriction to (1.4), then in lieu of (2.1), we now obtain:

$$\exp(i\Lambda) = 1 = \exp\left(ie\mu \frac{\varphi}{2\pi}\right) = \exp(ie\mu \cdot 1) = \exp(ie\mu \cdot 3) = \exp(ie\mu \cdot 5) = \exp(ie\mu \cdot 7).... \tag{3.1}$$

So this consideration of entanglement in addition to orientation naturally discards all of the even-numbered states such as $\exp(ie\mu \cdot 2)$, $\exp(ie\mu \cdot 4)$, etc.

Each of the above is now a separate relationship $\exp(ie\mu(1+2l)) = 1$ where l continues to be an integer with the values $l = 0,1,2,3...$, so that $2l+1 = 1,3,5,7...$ is an *odd integer*. Comparing $\exp(ie\mu(1+2l)) = 1$ with the mathematical relationship $\exp(i2\pi n) = 1$ means that $e\mu(1+2l) = 2\pi n$, or, restated:

$$e = \frac{n}{1+2l}\frac{2\pi}{\mu} = \frac{n}{1+2l} e_{\mathrm{u}} = \nu e_{\mathrm{u}} , \tag{3.2}$$

with a redefined filling factor:

$$\nu = \frac{n}{1+2l}; \quad n = 0, \pm 1, \pm 2, \pm 3...; \quad l = 0,1,2,3.... \tag{3.3}$$

In contrast to (2.3), this filling factor *will always have an odd denominator*. And with the exception of the even denominator 2, this is what is observed in FQHE.

This means that if regard two electron states to be equivalent in (1.4) only if they have the same version i.e., differ from one another by 4π not 2π , then all of the fill factors will only have an odd-integer denominator, and all even-integer denominators will be excluded because they involve inequivalent versions. Other than the even denominator of 2, this fully accords with the fractional charges states observed in FQHE. So other than the question of the even-numbered denominator of 2, this does solve the *experimental and arithmetical problem* of matching the observed fractional charge denominators. But because of the role of entanglement in reaching this result, we are now led to entertain not only whether the FQHE is in some way a physical manifestation of the Dirac Quantization *and Fractionalization* Condition, but whether the *odd-integer* denominators in the FQHE are "also a detectable difference in the physics . . . for two inequivalent versions of an object," which is the question Misner, Thorne and Wheeler have posed. In short: is FQHE an experimental, detectable manifestation not only of Dirac-Wu-Yang monopoles and fractionalization, *but also a detectable physical manifestation of MTW spinor orientation and topological entanglement*? And if so, how?

For ordinary systems of electron and protons and neutrons in which the electrons are unconstrained to two-dimensions by any materials and / or low temperatures and / or large magnetic fields and so are operating in three-dimensional space, neither fractionalization nor

Dirac monopoles nor signs of orientation and entanglement appear to be physically detectable. So as to the *theoretical and physical problem*, the question is whether in some way, and if so how, the physical constraints imposed by certain superconducting materials and low temperatures and high magnetic fields create an environment in which fractionalization and entanglement and even the Dirac-Wu-Yang monopoles themselves are suddenly forced to make a physical appearance.

As to the Dirac-Wu-Yang monopoles, we keep in mind that while the high energies of Grand Unified Theories (GUT) have certain symmetries which are broken at lower energies, so too, *low temperatures* near absolute zero are also thought to cause displays of certain symmetries which become broken at higher temperatures. [18] So the question which comes to the fore is whether an electric / magnetic symmetry of Maxwell's theory in the form of Dirac-Wu-Yang monopoles really does physically manifest from the confluence of low-temperatures near 0K for electrons in certain superconducting materials under large magnetic fields, and then gets broken into the non-observation of monopoles under anything but these limited conditions.

As to entanglement, the question which comes to the fore is whether by tightly constraining the electrons to two rather than three dimensions, and extracting virtually all of their heat energy leaving them only with their Fermi energies, we are forcing the electrons into some highly-constrained topological condition which forces them to reveal their entanglements and to display an electric and magnetic monopole symmetry and a charge fractionalization which they otherwise can keep hidden from observation.

And straddling between the experimental and arithmetical problem and the theoretical and physical problem laid out in section 2, is the following problem as to the boundary between mathematics and physics: Mathematically speaking, an angle of $\varphi = 2\pi$ is indistinguishable from an angle of 4π, 6π, 8π, etc. But physically, for fermions, if we account for entanglement together with orientation, then $\varphi = 0$, 4π, 8π etc. form one set of "equivalent" angles, while $\varphi = 2\pi$, 6π, 10π etc. form a second set of "equivalent" angles, and we do have a basis for *physically* distinguishing these two sets of angles notwithstanding their *mathematical* indistinguishability. In (3.2) and (3.3) we used this to force out the even-numbered fractional denominators and only retain the odd-integer denominators which does accord with the FQHE observations aside from the *m*=2 denominator still to be discussed. But there is still a puzzle for the angles *within each of these two sets of angles*:

For example, an *n*=1 state with $\nu = 1$ corresponds with the azimuth $\varphi = 2\pi$, while an *n*=1 state with $\nu = 1/3$ corresponds with the azimuth $\varphi = 6\pi$. Even with entanglement considered, it therefore seems as though we should still regard $\varphi = 2\pi$ to be equivalent with $\varphi = 6\pi$. And yet, from (3.2) and (3.3), these two angles are connected with *two observably-different physical n=1 states,* namely, $\nu = 1$ for $\varphi = 2\pi$ and $\nu = 1/3$ for $\varphi = 6\pi$. And going further, $\nu = 1/5$ for $\varphi = 10\pi$, etc. So we are also required to consider the following question: How it is that geometric angles such as $\varphi = 2\pi$ and $\varphi = 6\pi$ and $\varphi = 10\pi$ which are equivalent under both orientation *and entanglement, still manage to lead to distinct physical states* such as the respective $\nu = 1$, $\nu = 1/3$ and $\nu = 1/5$? Is there something else that is still being "missed" by the orientation-entanglement analysis which leads to angles such as $\varphi = 2\pi$ and $\varphi = 6\pi$ and $\varphi = 10\pi$ still being distinguishable from one another, notwithstanding their pure trigonometric equivalence and their like-entanglements? Put differently, entanglement makes the $\varphi = 0$, 4π, 8π etc. azimuths different from the $\varphi = 2\pi$, 6π, 10π etc. azimuths, but within each azimuth set, what might make, e.g., $\varphi = 2\pi$ observably physically different from $\varphi = 6\pi$ and those observably physically different from $\varphi = 10\pi$, etc.? As we shall develop starting in section 5, all of these angles are indeed distinguishable, because of a topological "twisting" which also occurs in relation to orientation-entanglement, but which appears to be overlooked is the usual discussions of this subject.

All of the foregoing questions will be considered in detail beginning in section 5. But for the moment, having explained the odd-integer FQHE denominators via (3.2) and (3.3) by only equating angles with matching orientation *and entanglement*, we must now address the next *experimental and arithmetical* question: Why does nature also appear to permit the even denominator 2, in addition to the odd denominators $1+2l$ in (3.3), but not permit any other even denominators?

4. Untangled Electrons Pairing into Bosons, and the *m*=2 FQHE Denominators

Equation (1.7) for the non-fractionalized Dirac Quantization Condition specifies charges $e = n(2\pi/\mu) = ne_{\mathrm{u}}$ which are integer multiples of the unit charge $e_{\mathrm{u}} = 2\pi/\mu$. For this set of integer charges, (1.6) tells us that the gauge parameter $\Lambda = 2\pi n$. This entire set of integer-quantized charges $e = ne_{\mathrm{u}}$ corresponds to the single azimuth $\varphi = 2\pi$ which means that there is a one-to-infinite quantized mapping of φ to Λ. That is, an infinite set of gauge states $\Lambda = 2\pi n$ can all be used to equivalently describe the same azimuth state $\varphi = 2\pi$, yielding quantized multiples of the unit charge.

Now, if we take a single fermion e.g. electron in the $\varphi = 2\pi$ state and do a 4π rotation to a $\varphi = 6\pi$ state which restores the electron to its original orientation and entanglement version, then $\exp(i\Lambda) = 1 = \exp(ie\mu \cdot 3)$ is the portion of (3.1) which describes this new state. Referring again to the general relationship $\exp(i2\pi n) = 1$, the solution is $\Lambda = 2\pi n = 3e\mu$, restated as $e = \Lambda/3\mu = (n/3)(2\pi/\mu) = ne_{\mathrm{u}}/3$. This specifies integer n multiples *of 1/3 of the unit charge* e_{u}. As before, the gauge parameter $\Lambda = 2\pi n$, from which we earlier defined a reduced gauge $\not\Lambda = n$ after (1.8). Similarly, if we generally define a reduced azimuth $\not\varphi \equiv \varphi/2\pi$, we see that for $\varphi = 6\pi$, the fractional denominator $1+2l$ is equal to the reduced azimuth, $1+2l = \not\varphi = 3$. It is readily seen that for the $\varphi = 10\pi$, $\varphi = 14\pi$ etc. states which also maintain the $\varphi = 2\pi$ version of the fermion, that the odd number denominators in (3.2) and (3.3) may be written as $\not\varphi = 1+2l$. Using this information and notation, we rewrite the filling factor of (3.3) as:

$$\nu = \frac{\not\Lambda}{\not\varphi} = \frac{\Lambda}{\varphi}; \quad \not\Lambda = n = 0, \pm 1, \pm 2, \pm 3...; \quad \not\varphi = 1 + 2l = 1, 3, 5, 7, 9...; \quad l = 0, 1, 2, 3.... \tag{4.1}$$

So the *fractionalization* of charge is determined directly by the number of "windings" $\not\varphi = 1 + 2l$ *in the physical space of spacetime*, and the odd-integer denominators occur because two topological windings, not one, are needed to restore a fermion to its original version. Although the $\not\varphi = 1 + 2l = 1, 3, 5, 7, 9...$ states all appear to be mathematically equivalent based on their differing from one another by 4π and so representing the same fermion version, the fact that these windings directly determine the fractionalization denominator is potentially an observable physical effect which belies this apparent equivalence. Similarly, the *quantization* of charge into integer multiples *n* of a fractional charge $1/\not\varphi$ is related to the quantized degeneracy of the gauge parameter $\not\Lambda = n$ which describes an infinite number of equivalent gauge states, and mathematically to the fact that phase angles which differ from one another by 2π are degenerately equivalent. Nonetheless, each "winding" *in the gauge space* adds one unit to the charge quantization number $n = \not\Lambda$, and this too is a physical effect which belies this apparent equivalence of gauge states differing from one another by 2π.

This sharpens the question as to how angles which differ from another by 2π or by 4π and so are trivially mathematically equivalent, can nonetheless yield different observable physical results, because even disregarding fractionalization, the quantum number $n = \not\Lambda$ which represents the local gauge angles $\Lambda = \pm 2\pi n$ will most certainly be an observable if the Dirac-Wu-Yang monopoles "exist." It is also worth noting that the fill factor ν which is certainly observed if it can be related to the FQHE fill factor, is given simply by the ratio $\nu = \Lambda / \varphi$ of the gauge angle to the azimuth angle, even though both these angles are thought to have the same mathematical property of indistinguishability under a 2π or a 4π rotation. Again, all of this this would have to mean that there is something else going on in both the physical and in the gauge space, *above and beyond orientation-entanglement*, which causes different observable effects even when mathematically, two angles with a 2π or 4π difference seem to be indistinguishable. And again, this is a related topological twisting to be develop in the next section.

But first, let us return to the immediate business at hand, which is to understand the only even denominator, 2, which is phenomenologically-observed in the FQHE.

Based on (4.1), it will be easily appreciated that a denominator of 2 corresponds to a winding number $\not\varphi = 2$. The set of quantized states for the unit charge – not a fractional charge – which we now write as $e = \not\Lambda (2\pi / \mu) = \not\Lambda e_u$, corresponds to the winding number $\not\varphi = 1$. So to get from an electron with $\not\varphi = 1$ to some state with $\not\varphi = 2$ we are only making one turn of the azimuth. Thus while we are restoring orientation, we are not restoring entanglement. Nonetheless, $\not\varphi = 2$ is the only even winding number which nature permits, so we have to figure out why we observe a state that is only one turn above $\not\varphi = 1$.

A $\not\varphi = 1$ electron will not be restored to its original version at $\not\varphi = 2$, because fermions need to do two windings to regain their original version. Only bosons can maintain equivalent version with one winding, because for bosons, entanglement is not an issue because they are not spinors. Again, as reviewed at the start of section 3, spinors rotate according to $\xi \rightarrow \xi' = R\xi$, while vector bosons rotate via $X \rightarrow X' = RXR*$. So for an electron to go from $\not\varphi = 1$ to $\not\varphi = 2$, that electron must "disguise" itself as a boson. How might the electron do that? By finding a second electron to "conspire" with the first electron and "pair up" into a single boson system. Then, that pair of electrons can be wound from $\not\varphi = 1$ to $\not\varphi = 2$ without changing its entanglement. And based on (4.1), the filling factor will now be $\nu = \not\Lambda / \not\varphi = n/2$, which yields the denominator of 2.

What does this mean?: it means that while all of the permitted windings of individual electrons yield fractional charges with the 3, 5, 7, 9, etc. odd denominators, the permitted winding for a boson *pair of electrons* yields the one permitted even denominator, namely 2. While the "Cooper pairs" model of electron pairing [19] may well come to mind, for the moment let us not be that specific. Let us simply talk in terms of the requirement that a first electron needs to find some way to pair together with a second electron if a $\not\varphi = 2$ winding state and thus a ½ unit of charge is to be empirically displayed under the right set of conditions – as it is at extremely low temperatures in suitable materials subjected to large magnetic fields – while leaving open the mechanism by which that pairing take place. So, the pairing of electrons into boson states would appear to explain why 2 is *permitted* as an even denominator, and we know that there is some grounding in established theory for such pairing to occur. Now, we have left, for the experimental and arithmetical problem, to explain why 4, 6, 8 and even denominators *other than 2* are *not permitted*.

The next even denominator of course is *m*=4 which we now know corresponds to the winding azimuth $\not\varphi = 4$. And we know that this fractionalization $\nu = n/4$ is *not observed*. So let us start with an electron in the $\not\varphi = 3$ fractional state for which $\nu = n/3$. These are fractional fermion charges, so to get them to an $\not\varphi = 4$ winding which would correspond adding one azimuth turn to $\not\varphi = 3$. This would result in an oppositely-entangled state, which is inequivalent to $\not\varphi = 3$. So, as we did to get from $\not\varphi = 1$ to $\not\varphi = 2$, we would have to "pair up" two of these $\not\varphi = 3$ fractional $\nu = n/3$ fermions into a boson state to get to $\not\varphi = 4$ with a quarter-integer fraction $\nu = n/4$. The fact that we do not observe $\not\varphi = 4$, nor do we observe any other even windings $\not\varphi = 6, 8, 10...$, is nature's way of telling us that *fractional charges cannot be paired up into boson states*. *All boson pairs must be constructed from unfractionalized charge units* $\mu_{\mathrm{u}} = 2\pi / e$. Given that fractional charges are commonly regarded as quasiparticles while unit charges are not, this simply means that only "real" particles, not quasiparticles, can form pairs. This is an *observation* about what nature seems to be telling us by excluding all even denominators except for 2; we still will want to *explain why* nature does so.

Pulling together all of these results, we now supplement (3.1) with $\not\varphi = 2$. Thus, the final result for the overall observed pattern is:

$$\nu = \frac{\not\Lambda}{\not\varphi} = \frac{\Lambda}{\varphi}; \quad \not\Lambda = n = 0, \pm 1, \pm 2, \pm 3 ...; \quad \not\varphi = 2 \quad \text{--or--} \quad \not\varphi = 1 + 2l; \quad l = 0, 1, 2, 3... . \tag{4.2}$$

The $\not{\varphi}=1+2l$ odd-denominator states represent fermions exhibiting fractional charges; the $\not{\varphi}=2$ state represents a boson pair of unit charges that are not fractional; and the absence of $\not{\varphi}=4,6,8...$ states tells us that fractional charges are not capable of forming into boson pairs. In Figure 1 below, which is reproduced from [20] and [21], we have annotated the unit electron charge, the $\nu=\not{\Lambda}/\not{\varphi}=1/3$ fractional charge, and the ground state for a pair of unit electrons forming a boson with $\not{\Lambda}/\not{\varphi}=1/2$. Also added as annotations are apparent $\nu=5/11$, $\nu=6/11$ and $\nu=7/9$ fractional states.

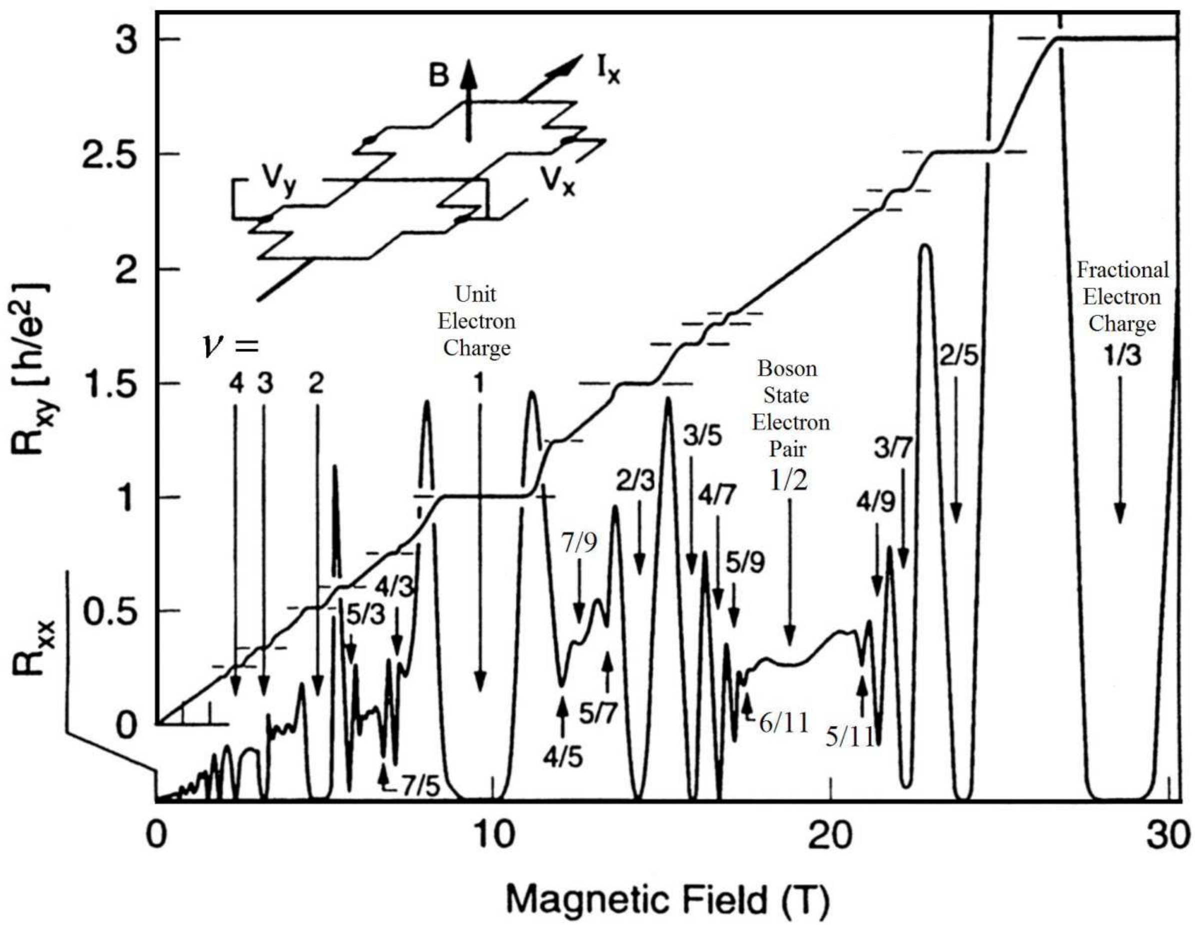

Figure 1: Fractional Quantum Hall Effect, reproduced from [20], [21], with annotation

It is worthwhile comparing the $\not{\Lambda}=n=1$ ground state of the $\not{\varphi}=1$, $\not{\varphi}=2$ and $\not{\varphi}=3$ windings, which are the three states annotated above. For $\nu=\not{\Lambda}/\not{\varphi}=1$ we of course have a unit electron charge. For $\nu=\not{\Lambda}/\not{\varphi}=1/3$ we have 1/3 fractional charge. But for the boson pair with $\nu=\not{\Lambda}/\not{\varphi}=1/2$ with a 1/2 unit of fractional charge, there are *two electrons not one* contributing to the half unit of charge. Therefore, each electron actually contributes a 1/4 unit of charge. Given that electrons naturally repel one another so that any pair formation mechanism must overcome this repulsion, it will be easier for two electrons to assume charges of 1/4 unit apiece and then pair into a boson, than to stay in the unit charge state or in the 1/3 charge state and then pair up. It is the 1/4 charge-per-electron paired state which minimizes the repulsion and therefore provides the most energetically-favored configuration.

Finally, we return to the original definition $\mu \equiv \oiint F$ of the Dirac monopoles. If rewrite $e = \nu e_{\mathrm{u}} = \nu\left(2\pi / \mu\right)$ with the complete filling factor (4.2) in terms of μ, then using the "unit" of magnetic charge $\mu_{\mathrm{u}} = 2\pi / e$, what we learn about the permitted monopole fluxes is that:

$$\oiint F = \nu\mu_{\mathrm{u}} = \left(\not{A} / \not{\varphi}\right)\mu_{\mathrm{u}}; \quad \not{A} = 0, \pm 1, \pm 2, \pm 3...; \quad \not{\varphi} = 2 \quad \text{--or--} \quad \not{\varphi} = 1 + 2l; \quad l = 0,1,2,3.... \tag{4.3}$$

This extended understanding of Dirac monopoles to include fractionalized charges, should put into a somewhat different perspective how one thinks about these monopoles, at least based on Dirac's quantization absent further developments such as t'Hooft / Polyakov monopoles [22], [23] which rely on Yang-Mills gauge theory which is not needed for Dirac monopoles alone. Although the Dirac monopoles when fully developed using Wu and Yang's gauge approach are *fractionalized as well as quantized*, these fractional charges are not observed except under very limited conditions at extremely low temperatures in suitable superconducting materials. Thus, to the degree that the filling factors (4.2) do describe a feature of the natural world but only under these specialized conditions, and because (4.3) is integrally related to (4.2), it would appear that the non-zero magnetic fluxes $\oiint F = \nu\mu_{\mathrm{u}}$ of Dirac monopoles (as distinguished from other types of monopole) would only evidence themselves in nature under equally-restricted conditions.

Therefore, from an experimental and arithmetic standpoint, we conclude that a complete analysis of the gauge symmetries of Dirac Monopoles following the approach pioneered by Wu and Yang [14], [15] results in electric and magnetic charges which are quantized *and fractionalized* in the manner observed in the Fractional Quantum Hall Effect. Because fermions rotated through an azimuth over 2π regain their orientation but not their entanglement, the 4π rotation needed to restore both orientation and entanglement is responsible for the observation of odd-integer denominators and the skipping of most even-integer denominators. The only observed even-integer denominator of 2 appears to be the result of pairing two integer-charged fermions into a boson, and the absence of any larger even denominators appears to indicate that only integer charges, and not fractional charges, can be so-paired. The simplicity of the fill factor $\nu = \not{A} / \not{\varphi}$, and the ability to derive this strictly from gauge theory via Dirac-Wu-Yang approach in view of orientation-entanglement arguments on an arithmetic basis that does match experimental observation, is certainly intriguing. But now we must study these results deeply from a *theoretical and physical* standpoint to see if our suspicion that Dirac-Wu-Yang is connected to the FQHE is real, or illusory.

5. Twist: The Missing Ingredient from Orientation-Entanglement, and how may lead to a Topological Understanding of Quantization

In section 3 we began to review Misner, Thorne and Wheeler's classic treatment of orientation-entanglement (OE) in section 41.5 of [17]. In this section we shall continue this discussion, focusing on one aspect of this subject which requires further development, namely, the topological "twisting" of the threads which are used to track orientation-entanglement. Specifically, we do this because the questions now raised as to how states which differ from one another even by the 4π, two winding rotation needed to restore version, still might leave certain observable signposts in the spaces of these rotations which render these states physically and observably-distinct from one another. Thus, we lay the foundation for physically understanding how, say, seemingly-equivalent rotational states with, e.g. $\varphi = 2\pi$ and $\varphi = 6\pi$ and $\varphi = 10\pi$ might nonetheless be connected to distinctly-observed related quasiparticle states with respective inequivalent $\nu = n$ and $\nu = n/3$ and $\nu = n/5$ quantized fractional charges. And in the process, we also lay the foundation for understanding how the Dirac-Wu-Yang arguments based on $U(1)_{em}$ gauge theory in three space dimensions generally without restriction, may connect to the FQHE observed for highly-restricted electrons forced into two dimensional restraint by superconducting materials and ultra-low temperatures under very large magnetic fields. Thus, by developing these topological twisting features fully, we lay the foundation for connecting the quantized fractional charge results in (4.2) to the FQHE, not only arithmetically and experimentally as we have already done, but theoretically and physically as we now must do.

D. K. Ross in [24] "hypothesize[s] that the OE relations are important to physics [and] represent the deep relationship between any particle or material body and its environment." He proceeds to show (reference renumbered here) "that Dirac [11] magnetic monopoles do not satisfy the OE relationship" and "hypothesize[s] that this is the reason they have never been seen despite extensive searches . . . and despite having a natural and elegant theory underlying them . . . going back to the more natural symmetry of Maxwell's equations with magnetic monopole sources present. " He then states that "[s]ince all known particles satisfy the OE relations and we show that Dirac magnetic monopoles which have not been seen do not satisfy these relationships, it is hopes that this paper will stimulate further work on the OE relations themselves and their topological role in physics." This further work on the OE relationships is precisely the subject of this section, and will lead us to understand the fractional quasiparticles of FQHE as those particles which "do not satisfy these [OE] relationships."

Figure 41.6 of MTW's [17] which is also posted online at [25], shows a spherical "object connected to its surroundings by elastic threads." Indeed, it is these "threads" and various configurations of these "threads" which most directly illustrate the "deep relationship between any particle or material body and its environment" mentioned by [24]. It is also these "threads" themselves which will be the object of the present discussion. As is well-understood, it is always possible following a 720° rotation or integer multiples thereof of an object connected to its environment with "untwisted threads," to remove all entanglement from the connections of an object to its surroundings. But of particular importance, as we shall now develop here, the sequences of disentangling the "threads" from one another are *not unique*. Depending upon the sequence chosen, even after disentanglement, *the "threads" may still each maintain individual twists*, or they may have all twisting removed and have been returned to an untwisted state.

To simplify this development without any loss of information, rather than use the spherical object and the spherical environment and the "threads" employed in Figure 41.6 of [17], let us employ a first "bar" or "stick" which represents the environment and a second "bar" or "stick" which represents the object, and a pair of "ribbons" which represent the connections of this object to its environment. These two ways of representing OE do topologically map into one another as is shown below in Figure 2, which is why we can use the "bars and ribbons" as an alternative way of representing Figure 41.6 of [17].

Specifically, to verify this mapping, one may start with the OE system shown in drawing 1 from Figure 41.6 of [17], and as shown in Figure 2(a) below, to maintain points of reference, may label the northern and southern hemispheres of the object as shown, which hemispheres also have northerly and southerly "thread" connections to the environment. Then, as shown in Figure 2(b) below, one may topologically deform the object by stretching it in into a vertical elongation, while relocating the threads to the right along the northeastern and southeastern regions of the environment. Then, one can take the entire Figure 2(b) and rotate it 90° counterclockwise to arrive at Figure 2(c) below. In this final step, the *environment* is simply represented by a top "bar" or a "stick" at the top of 2(c), the *object* is represented by a bottom "bar" or a "stick" at the bottom of 2(c) which maintains the "north" and "south" labels and now also introduces a directional vector running from north to south, and the north and south thread pair are merged together into a pair of "ribbons" which represent the *entanglement* between the object and its environment. The benefit of employing "ribbons" (or a thick "threads") rather than thin threads is that it is much easier with a two-sided ribbon to illustrate and track any twist which may occur in the course of performing OE operations.

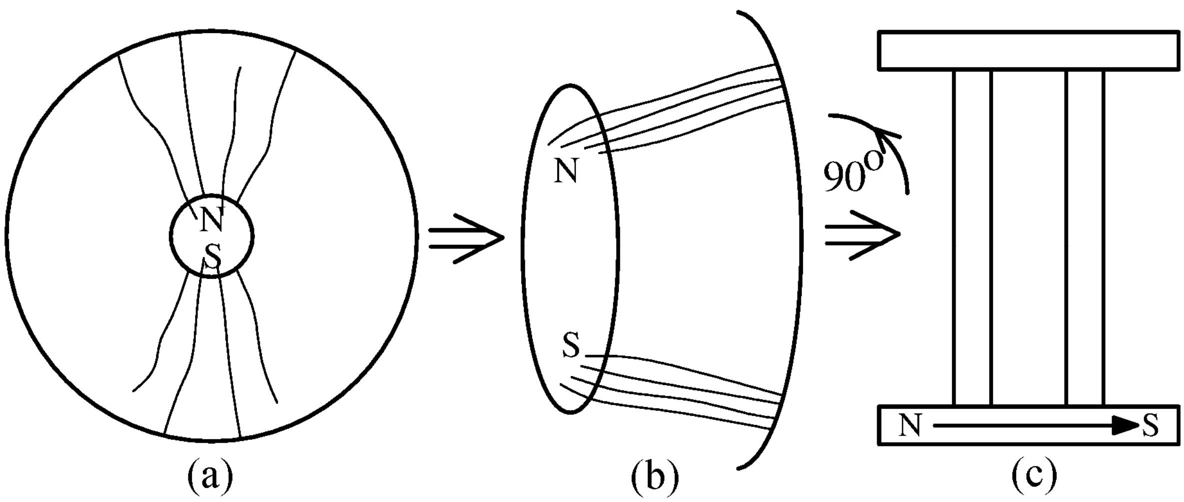

Figure 2: Topological Deformation of Figure 41.6 of [17] (MTW) into a "Bar and Ribbon" Configuration

This "bar and ribbon" configuration is often used in illustrations of the OE relationships, see, for example, an online animation at [26]. It is easy and advisable for the interested reader to construct a physical apparatus from Figure 2(c) by taking two sticks or even pencils, and then gluing or stapling two ribbons or shoelaces or rubber bands to the sticks in the configuration illustrated. It helps to color each side of the ribbons differently for monitoring twists.

So now, starting with the bar and ribbon configuration of Figure 2(c) above, let us immobilize the top "environmental" bar, and rotate the bottom bar – which from now on we shall simply call the bottom "vector" – by a $\varphi \to \varphi + 4\pi = \varphi + 720°$ rotation counterclockwise about the z axis through the angle φ in the x-y plane, as shown in Figure 3(a) below, to arrive at the configuration of Figure 3(b) below. It is worth noting that all three x, y, z dimensions are utilized in this operation, and also worth keeping in mind that for electrons frozen in two dimensions at low temperature in superconducting materials in the FQHE environment, one degree of spatial freedom is removed. It is also noted that this angle φ is an azimuth angle of rotation in three space dimensions, just as was the azimuth angle φ first introduced after (4.1) when we write the electromagnetic field strength as $F = \left(\mu / 4\pi\right) d \cos\theta d\varphi$.

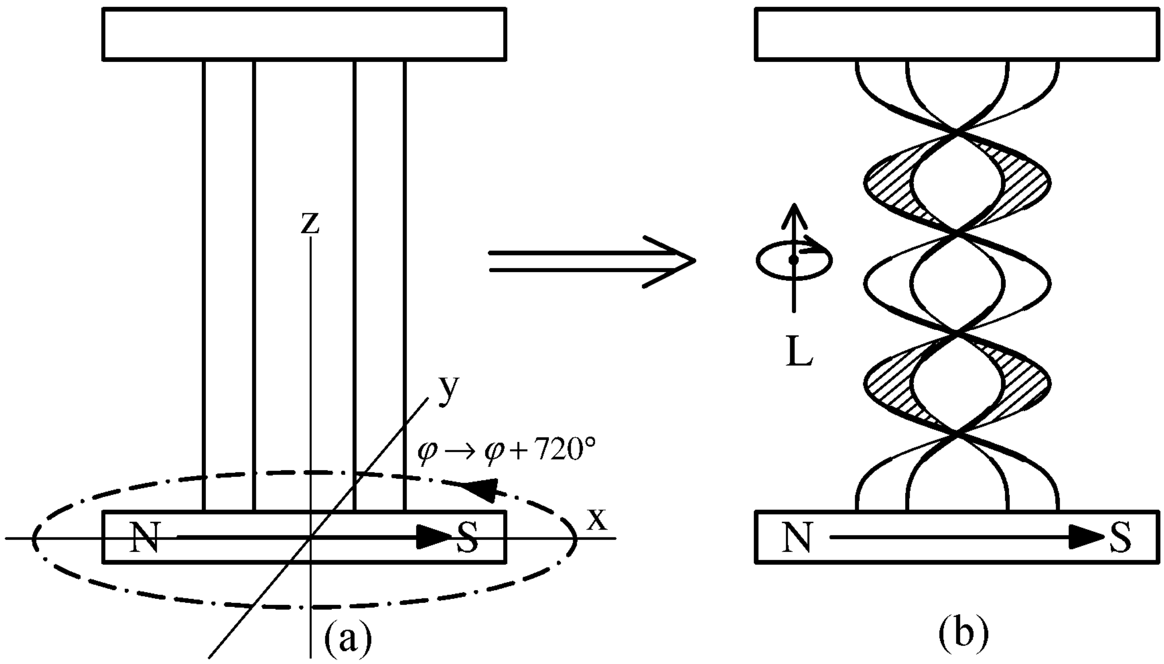

Figure 3: Environmental OE Consequences of Rotating a Vector through 720°

In Figure 3(b), the wider lines illustrated on each ribbon are passing in front of the narrower lines illustrated on each ribbon, and diagonal hash lines are used to illustrate the opposite face of the ribbon relative to the face shown in Figure 3(a). In 3(b), we reach a state in which the ribbons are entangled with one another, with the entanglement forming a left-handed helix as illustrated. And in addition, each of the two individual ribbons also is twisted into a left-handed helix as illustrated. And in addition, both the entanglement helix and the twist helixes are *double* helixes, in the sense that there are two full helix rotations of $-4\pi = -720°$, using a convention in which a right helix has a positive sign and a left helix has a negative sign. Now we seek to disentangle the ribbons, while immobilizing both the environmental "bar" and the N→S vector, by moving the ribbons around the ends of the vector.

Now, if the initial rotation in Figure 3 had been through only $+2\pi = +360°$, Figure 3(b) would contain all *single* helixes, and as is well known there would be no way to disentangle the two ribbons from each other using only manipulations of the ribbons. But from Figure 3(b), because of the double helix entanglement which results from the double winding rotation through $\varphi \rightarrow \varphi + 4\pi = \varphi + 720°$ a.k.a. $\not{\varphi} \rightarrow \not{\varphi} + 2$ using the reduced azimuth $\not{\varphi} \equiv \varphi / 2\pi$ earlier defined, disentanglement is possible using only ribbon manipulations. And specifically, in order to disentangle the two ribbons using only operations of the ribbons with both the environment and the vector remaining immobile, one must perform *two* ribbon operations, and there are three choices for this.

For the first choice, as shown in Figure 4 below, for the first operation, one can take the *north ribbon*, wind it past the north "pole," wind it beneath the entire vector, and then wind it back above the vector past the south "pole." Then, for the second operation, one can take the *south ribbon*, wind it past the north "pole," wind it beneath the entire vector, and then wind it back above the vector past the south "pole." This can be done in either order, that is, one can use the south ribbon in the first operation and the north ribbon in the second operation and end up with the exact same result, which, as shown in Figure 4 below, not only disentangles the two ribbons from each other, but also removes the individual twists in each ribbon. We denote this by placing the number "0" next to each ribbon to indicate that it has no residual twist.

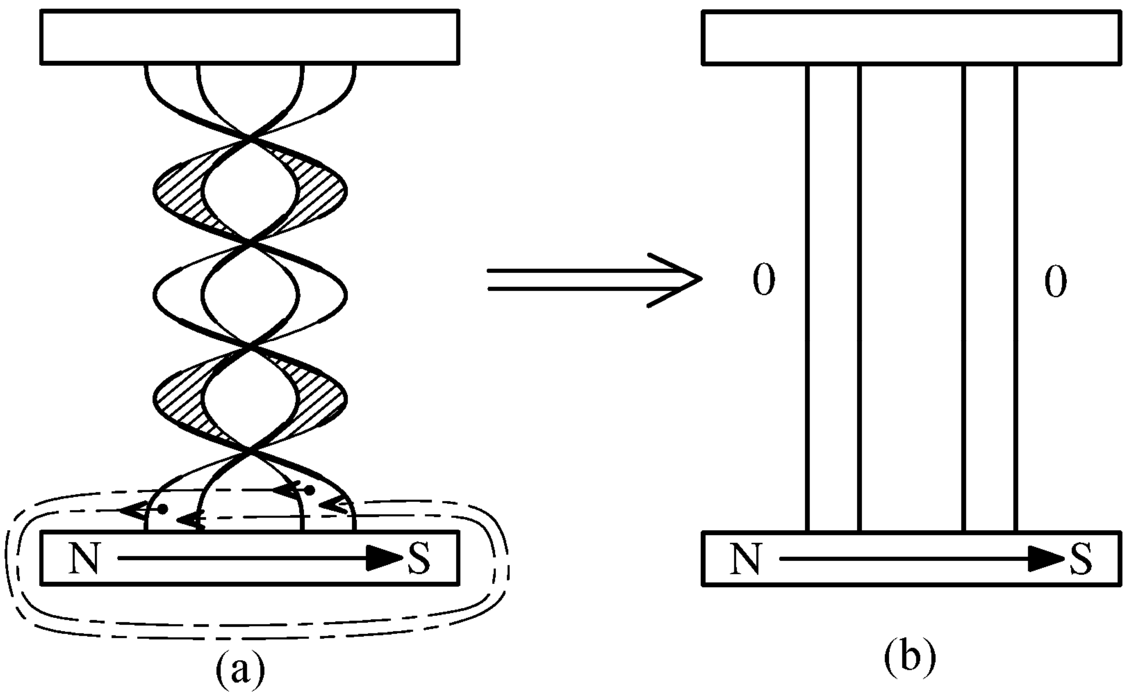

Figure 4: The Disentanglement Operation $0,0 \rightarrow \not{\varphi}+2 \rightarrow \{N,S\} \rightarrow 0,0$

In either case, however, whether the north or south ribbon is operated first, the ribbon windings *must go from north to south*, that is, the ribbons must be first brought around the north pole, then wound beneath the vector, then be brought back up past the south pole. If the ribbons are wound from south to north, they will become even further entangled, and the net effect will be that of having performed a $\varphi \rightarrow \varphi+8\pi$ a.k.a. a $\not{\varphi} \rightarrow \not{\varphi}+4$ quadruple rotation starting from Figure 3(a). The question occurs why there is this apparent asymmetry in which the ribbons must be brought past the north pole first, but that is explained by the fact that the Figure 3 rotation was done counterclockwise and thus was positively signed, $\varphi \rightarrow \varphi+4\pi$. Had the rotation been clockwise hence negative according to the customary conventions for defining angular rotation, i.e., $\varphi \rightarrow \varphi-4\pi$ a.k.a. $\not{\varphi} \rightarrow \not{\varphi}-2$, then disentanglement would have required winding the ribbons first over the south and then over the north pole. So that in fact there is an overall symmetry to these operations.

We shall use the shorthand $0,0 \to \varphi+2 \to N/N, S/N \to 0,0$ to represent this operation in Figure 4 where both the north and south ribbons start with no twists $0,0$, the azimuth is positively rotated through two windings $\varphi+2$, the north and then south ribbons are wound over the north pole $N/N, S/N$, and the disentangled state also restores no twists $0,0$. The fact the order of ribbon operations does not matter means that $0,0 \to \varphi+2 \to S/N, N/N \to 0,0$ as well. Thus, $0,0 \to \varphi-2 \to N/S, S/S \to 0,0$ and $0,0 \to \varphi-2 \to S/S, N/S \to 0,0$ are also operations which restore the initial disentangled state with no twists when the initial rotation is negative $\varphi \to \varphi-2$ rather than positive $\varphi \to \varphi+2$. From here, we shall work only with positive rotations, which means that ribbons must always go first over the north pole to achieve disentanglement. We also keep in mind that the final configuration is invariant under the order in which the north and south ribbons are operated, i.e., under either temporal ordering (N,S) or (S,N) of the permutated ribbon set $\{N,S\}$. Thus, we can simplify the shorthand to write the Figure 4 operation as $0,0 \to \varphi+2 \to \{N,S\} \to 0,0$, simply indicating that either (N,S) or (S,N) over the north pole will restore an untangled, untwisted state following a $\varphi+2$ rotation of a vector.

For the second choice, one can take the north ribbon and wind it *twice* about the north pole, then under the vector, then back over the south pole and the ribbons will disentangle. But here, there will be a residual twist in each ribbon, as now shown below in Figure 5.

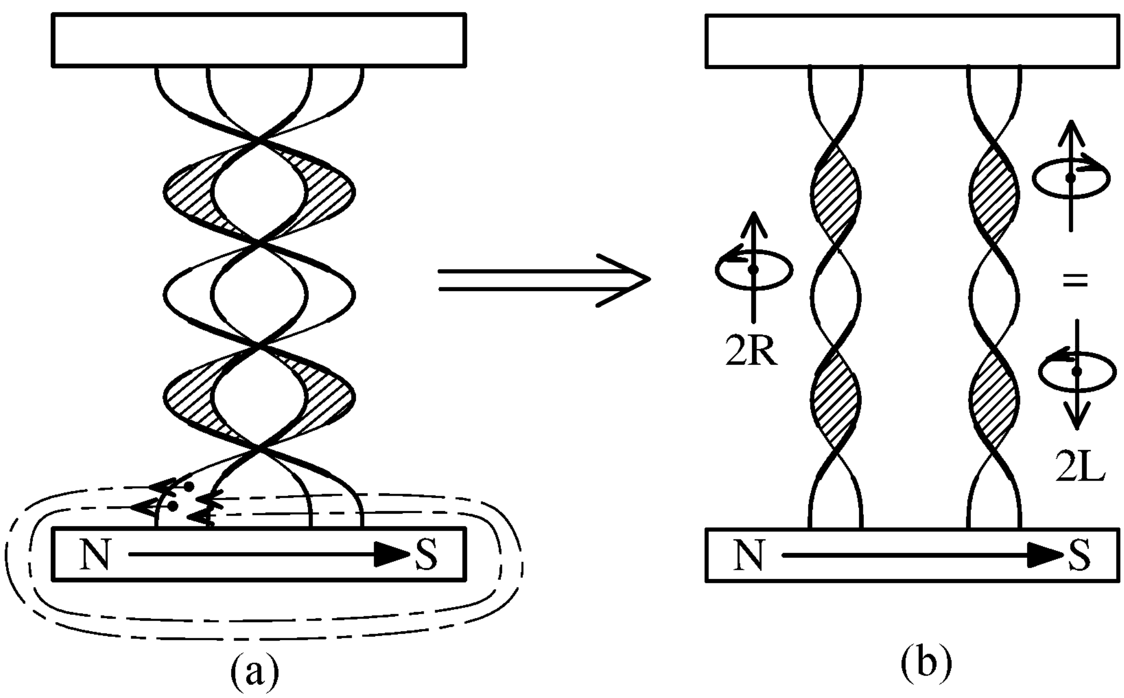

Figure 5: The Disentanglement Operation $0,0 \to \varphi+2 \to \{N,N\} \to 2R$

Now, because we have used the ribbon set $\{N,N\}$ to disentangle the ribbons, the north ribbon maintains a double helix twist with right-handed parity which we denote by 2R, while the south ribbon also has a double helix twist but with left-handed parity which we denote as 2L. If we adopt a "right-hand rule," then we may also refer to these as a double twist "up" for the north ribbon, and a double twist "down" for the south ribbon. The ribbons are fully disentangled, and yet, the end state in Figure 5(b) is observably, physically-distinct from the end state of Figure 4(b), based wholly on the operation that was used to disentangle the ribbons. *So even though a rotation of a vector through $\varphi \to \varphi + 4\pi$ yields the exact same orientation and the exact same entanglement for that vector, the final, physical state can still be different from the starting state, wholly dependent upon how the disentanglement operation has taken place.*

It is because of this, that we can now begin to think about how, for example, the FQHE filling factor $\nu = n/1$ for which $\not\varphi = 1$ a.k.a. $\varphi = 2\pi$ in (4.2), can exhibit different observable physics from the filling factor $\nu = n/3$ for which $\not\varphi = 3$ a.k.a. $\varphi = 6\pi$ in (4.2), even though these angles differ from one another by 4π and so would be physically indistinguishable if we only considered orientation-entanglement (OE) without twist. So as we now see, the complete physics of vector rotations requires us to consider orientation-entanglement-twist all together, which we abbreviate as OET, and once we do consider twist, then angles which differ from one another by 2π or by 4π, despite their trivial trigonometric equivalence, are *all* distinct based on their OET relationships to the surrounding environment. The a vector rotated to the angle 2π is different from a 4π vector is different from 6π is different from 8π is different from 10π, *ad infinitum*, once the OET of the vector is also taken fully into account. This is a place where physics informs and extends mathematics.

Using the notation developed above, we may use $0,0 \to \not\varphi + 2 \to \{N,N\} \to 2R, 2L$ to denote the final state of Figure 5(b) in which the north ribbon ends up with a double right-handed helix and the south ribbon ends up with a double left-handed helix. It will be apparent, however, that the left and right twists are offsetting, which is to say that the *net twist* of the overall system remains zero as it was when it started in Figure 3(a), and in general, this "conservation of twist" result will carry through to all OET disentanglements. So, if we know that the north ribbon has ended up with *2R*, then we automatically know that the south ribbon has ended up with *2L*. Thus, we can used twist conservation under OET disentanglement to simplify the summary of the Figure 5 operations to $0,0 \to \not\varphi + 2 \to \{N,N\} \to 2R$, showing only *2R* as the end state for the north ribbon, and deducing by implication that *2L* is therefore the twist for the south ribbon.

For the third and final choice, one can take the south ribbon and wind it *twice* about the north pole, then under the vector, then back over the south pole and the ribbons will again disentangle. But here, there will be a residual twist in each ribbon oppositely to that shown in Figure 5, as now shown below in Figure 6.

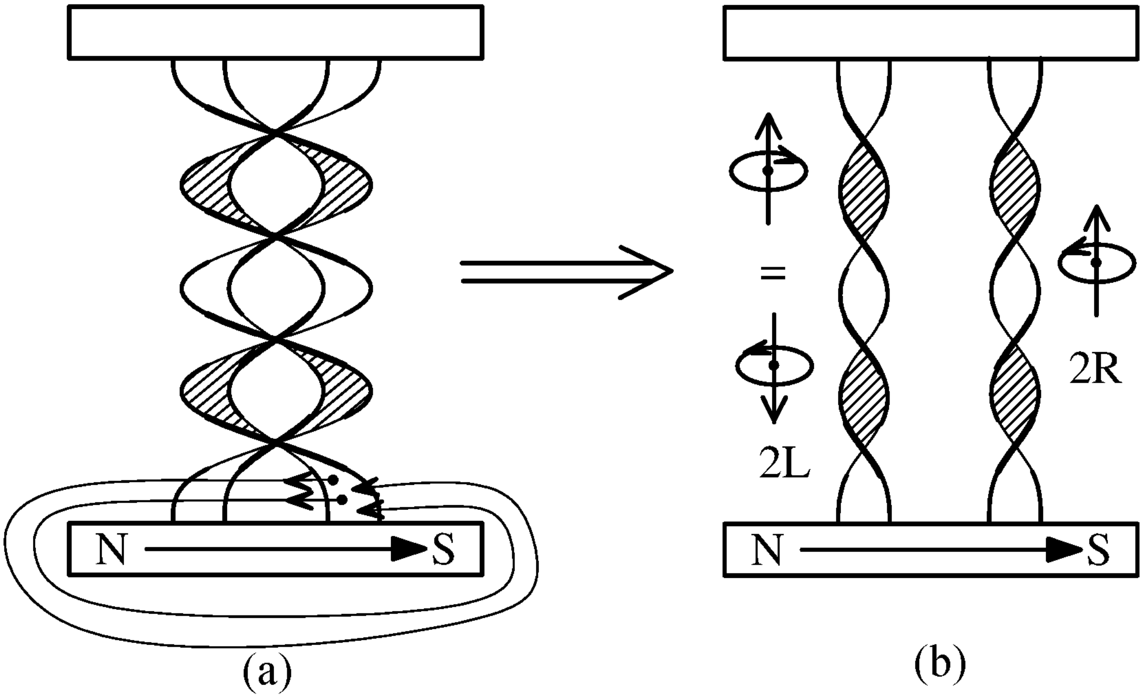

Figure 6: The Disentanglement Operation $0,0 \to \not\varphi + 2 \to \{S,S\} \to 2L$

Here, we have used the ribbon set $\{S,S\}$ to disentangle the ribbons, and the north ribbon maintains a double helix twist but now with left-handed parity which we denote by 2L, while the south ribbon also has a double helix twist but with left-handed parity which we denote as 2R. Twist is still conserved, i.e., the net ribbon twist is zero, so continuing to represent the end result simply by the *2L* state of the north ribbon, $0,0 \to \not\varphi + 2 \to \{S,S\} \to 2L$ now summarizes the Figure 6 operation.

Returning to Figure 4, because we now know that twist is conserved, we further consolidate the summary of this operation by setting $0,0 \to 0$, that is, by using a single zero to represent the twist state of the north ribbon with the south ribbon implicitly also having zero twist because of net twist conservation. Thus we now write this as $0 \to \not\varphi + 2 \to \{N,S\} \to 0$. Figure 5 is then $0 \to \not\varphi + 2 \to \{N,N\} \to 2R$ and Figure 6 is $0 \to \not\varphi + 2 \to \{S,S\} \to 2L$.

How let us now make some final changes to our notation. Because the non-zero twist end results always contain a left- or right-handed double twist, let us count use the number 1 to define a single double-twist, and let us use the "+" sign to denote a right handed and "-" to represent a left-handed twist, and let us refer to the quantum variable which represents the number and handedness of double twists in the resultant north ribbon as m'. Thus, the end state $2R \to m' = +1$, $2L \to m' = -1$, and $0 \to m' = 0$. The number m' is of course quantized, but there is no mystery to this because it represents the number of topological twists and once OE is restored between a vector and its environment, the number of double twists will always be either zero or an integral number. Secondly, because we must always perform a two-winding, 4π

rotation to be able to restore OE, let us also talk about this in terms of making one (1) double turn, rather than two terns. Let us designate the number of double turns as l', so that in all of Figures 4, 5 and 6, we have started out making $l'=+1$ double turns. Therefore, we rewrite $\not\varphi+2$ as $\not\varphi+2l'$ with $l'=+1$, and we refer to all of the results in Figures 4, 5 and 6 as the $l'=+1$ OET states, and we pick out the nickname "principal-prime" for these $l'=+1$ states and abbreviate this by "p'." Pulling all of this together enables us to summarize the three results from Figures 4, 5 and 6 as follows:

$$p' \equiv l' = +1: \quad 0 \to \not\varphi + 2l' \to \begin{cases} \{N,N\} \to m' = +1 \\ \{N,S\} \to m' = 0 \\ \{S,S\} \to m' = -1 \end{cases} . \tag{5.1}$$

Now let's repeat everything we have just done, but instead of a single double-winding $l'=+1$, let's start with Figure 3a, and do two double-windings, $\varphi \to \varphi \pm 8\pi$, i.e., $\not\varphi \to \not\varphi + 4$. This is now an $l'=+2$ state, and it requires four ribbon operations. But instead of using more drawings, let's just use the consolidate notation to represent the results. As discussed earlier, ribbons must always be drawn first over the north and then over the south pole, because the rotation is a positive rotation. Doing otherwise with create further entangling, rather than detangling. As also reviewed, the temporal order with which one draws the ribbons does not matter because as with $l'=+1$ the final twist results are invariant with respect to this order. So operational sets of ribbons that can be used in any temporal permutation are $\{N,N,N,N\}$, $\{N,N,N,S\}$, $\{N,N,S,S\}$, $\{N,S,S,S\}$ and $\{S,S,S,S\}$. Let us nickname this $l'=+2$ as "diffuse-prime," abbreviated d'. What we now have, in place of five more Figures, are the five resulting states:

$$d' \equiv l' = +2: \quad 0 \to \not\varphi + 2l' \to \begin{cases} \{N,N,N,N\} \to m' = +2 \\ \{N,N,N,S\} \to m' = +1 \\ \{N,N,S,S\} \to m' = 0 \\ \{N,S,S,S\} \to m' = -1 \\ \{S,S,S,S\} \to m' = -2 \end{cases} . \tag{5.2}$$

So now we start to see the pattern of OET. In general, l' which represents the number of double windings is an integer which always has the positive value $l'=0,1,2,3,...$ ($l'=0$ is represented by Figure 3(a) and has the single state $m'=0$ with no twists), and the resultant twist of the north ribbon flowing detangling ranges over the integers m' for which $-l' \le m' \le +l'$. So, for example, if we go next to $l'=+3$ with three double windings, and call this "fundamental-prime," abbreviated f'. For f', with $l'=+3$, we have seven states $m'=0,\pm1,\pm2,\pm3$. Strikingly, this is the same pattern of orbital angular momentum and magnetization, as represented by the quantum numbers *l* and *m* which characterize the electrons in an atom. And also strikingly, the azimuth angle φ is the same azimuth in physical three-dimensional space against which angular momentum is specified.

This leads us top two questions: Are these concurrences merely coincidental, or can OET be used to provide a fully-topological understanding of electronic structure (and by extension nuclear structure which is subject to the same quantized exclusion principles)? And, how does this all relate to the FQHE which motivated this discussion because of the need to physically-distinguish rotational states with the same OE, i.e., states differing by a 4π rotation. Clearly, OET provides the basis for asserting that even states which differ by $4\pi n$ rotations from one another are not trivially-identical once all of OE and Twist are considered, and that the differences between these inequivalent states parallel the orbital and magnetic quantum structures of the atom and the nucleus and the quantum numbers which force exclusion.

This leads us to consider the possibility that these atomic quantum numbers may in fact be indicators of states of twist, and it will leads us to propose an experiment by which the fractional quantum Hall states are examined for properties reminiscent of electrons in the s, p, d, f, g, h, i, k… orbital states of electrons in atoms to possibly validate or contradict these apparent parallels.

6. To be added

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