Derivation of the Equation $E = m c^2$ from the Universal Uncertainty Principles

Rodolfo A. Frino – December 2014 - v4
Electronics Engineer
Degree from the National University of Mar del Plata

Abstract

The present paper is concerned with the derivation of the Einstein's formula of equivalence of mass and energy, $E = m c^2$ *, from the universal uncertainty relations. These relations are a generalization of the original uncertainty relations developed by Werner Heisenberg in 1927. Thus, this approach unifies two of the most important laws of physics as provides the proof of a) the quantum mechanical nature of the above formula, and, b) the correctness of the universal uncertainty relations that I found in 2012.*

Keywords: *Heisenberg uncertainty principle, universal uncertainty principle, universal momentum-position uncertainty relations, universal energy-time uncertainty relation, spatial uncertainty principle, temporal uncertainty principle, annihilation, Planck time, Planck length, Planck force, Planck energy, Planck's constant, reduced Planck's constant, Newton's law of universal gravitation.*

1. Introduction

In 1927 Werner Heisenberg proposed two fundamental relations that would revolutionize quantum mechanics. These relations are known as the Heisenberg uncertainty relations or principles. In 2012 I generalized these relations by incorporating the effects of gravity into the momentum-position uncertainty relations and into the energy-time uncertainty relation. The results of this generalization are called the universal uncertainty relations (See the paper: *The Universal Uncertainty Principle* [1]). I won't consider the general universal uncertainty relations (or GUUP) in this paper. The special universal uncertainty relations (or SUUP) are:

1. The universal momentum-position uncertainty relations (or special spatial universal uncertainty principle or **spatial SUUP**):

$$\Delta p_x \Delta x \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_x} \tag{1.1a}$$

$$\Delta p_y \Delta y \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_y} \tag{1.1b}$$

$$\Delta p_z \Delta z \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_z} \tag{1.1c}$$

where L_P is the Planck length

and

2. the universal energy-time uncertainty relation (or special temporal universal uncertainty principle or **temporal SUUP**):

$$\Delta E \Delta t \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t} \tag{1.2}$$

where T_P is the Planck time

Based on these relations, and considering the results of particle-antiparticle annihilation experiments, the Planck force and the Newton law of universal gravitation, I shall derive the equation of equivalence of mass and energy

$$E = m c^2 \tag{1.3}$$

The purpose of using the Planck force and the Newton's law of universal gravitation is to determine the value of the real proportionality constant *k* I shall introduce in the next section (See **Appendix 1** for the nomenclature used in this paper).

2. From the Universal Uncertainty Relations to Einstein's Formula of Equivalence of Mass and Energy

Let us consider the special spatial uncertainty principle given by relation (1.1a)

$$\Delta p_x \Delta x \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_x} \tag{2.1a}$$

and the special temporal uncertainty principle given by relation (1.2)

$$\Delta E \Delta t \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t} \tag{2.1b}$$

I can make preliminary good estimates by writing the above relations as approximations

$$\Delta p_x \Delta x \approx \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_x} \tag{2.2a}$$

$$\Delta E \Delta t \approx \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t} \tag{2.2b}$$

Later I shall transform these preliminary approximations into equations. Before I begin to derive the formula of equivalence of mass and energy, let me outline the general strategy I shall follow. Because I apply these two uncertainty principles to the same particle, at the same time, the uncertainty in the position of the particle, Δx, must be related to the time Δt taken by the particle to travel a distance equal to Δx. Because Δx is a position uncertainty, Δt must be a time uncertainty. In other words I shall assume that the uncertainty in the speed of the particle is given by the ratio between the uncertainties in position and time. Then I shall assume that when the maximum uncertainty in the velocity of the particle is the speed of light, the maximum uncertainty in the energy of the particle will be the energy of the particle itself. We know that this is so because of annihilation experiments. However, before annihilation, a particle can be either at rest or moving with respect to the observer who performs the measurements. These two cases have been analysed in a previous paper [2], and because the concepts are exactly the same in both papers, there is no need to duplicate the analysis here.

The derivation is very simple as requires basic mathematical knowledge. To start the derivation let us divide expression (2.2a) by expression (2.2b). Mathematically we write

$$\frac{\Delta p_x \Delta x}{\Delta E \Delta t} \approx \frac{\sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_x}}{\sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t}} \tag{2.3}$$

To simplify the equations I shall used Δp instead of Δp_x . This gives

$$\frac{\Delta p \, \Delta x}{\Delta E \, \Delta t} \approx \frac{\sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p}}{\sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t}} \tag{2.4}$$

Now I raise eq. (2.4) to the power of 2 to remove the square roots of the second side of the equation. This gives

$$\frac{\Delta p^2 \Delta x^2}{\Delta E^2 \Delta t^2} \approx \frac{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p}{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t} \tag{2.5}$$

Multiplying numerator and denominator by $\frac{4}{\hbar}$ we get

$$\frac{\Delta p^2}{\Delta E^2} \left(\frac{\Delta x}{\Delta t} \right)^2 \approx \frac{\hbar - L_P \Delta p}{\hbar - T_P \Delta t} \tag{2.6}$$

Now I define the uncertainty in the velocity of the particle as

$$\Delta v \equiv \frac{\Delta x}{\Delta t} \tag{2.7}$$

(Δv should not be confused with the average velocity of the particle)
Where, to simplify the equations, I have used Δv instead of Δv_x . Then, according to eq. (2.7) we can rewrite eq. (2.6) as

$$\frac{\Delta p^2}{\Delta E^2}\Delta v^2 \approx \frac{\hbar - L_P \Delta p}{\hbar - T_P \Delta t} \tag{2.8}$$

Taking into account that

$$\Delta p = m \Delta v \tag{2.9}$$

(It is worthy to observe that, as postulated by Heisenberg in his uncertainty relations, the mass of the particle, *m*, does not include any uncertainties). Thus we can write

$$\frac{(m \Delta v)^2}{\Delta E^2}\Delta v^2 \approx \frac{\hbar - L_P \Delta p}{\hbar - T_P \Delta t} \tag{2.10}$$

$$\frac{m^2 \Delta v^4}{\Delta E^2} \approx \frac{\hbar - L_P \Delta p}{\hbar - T_P \Delta t} \tag{2.11}$$

$$m^2\, \Delta v^4 \left(\hbar - T_P \Delta E\right) \approx \left(\hbar - L_P\, m \Delta v\right) \Delta E^2 \tag{2.12}$$

$$\hbar\, m^2\, \Delta v^4 - \left(m^2\, \Delta v^4\, T_P\right) \Delta E - \left(\hbar - m \Delta v\, L_P\right) \Delta E^2 \approx 0 \tag{2.13}$$

$$\hbar\, m^2\, \Delta v^4 - \left(m^2\, \Delta v^4\, T_P\right) \Delta E + \left(m \Delta v\, L_P - \hbar\right) \Delta E^2 \approx 0 \tag{2.14}$$

$$\left(m \Delta v\, L_P - \hbar\right) \Delta E^2 - \left(m^2\, \Delta v^4\, T_P\right) \Delta E + \hbar\, m^2\, \Delta v^4 \approx 0 \tag{2.15}$$

Thus we arrive to a quadratic equation with the following coefficients

$$A \equiv m \Delta v\, L_P - \hbar \tag{2.16a}$$

$$B \equiv - m^2\, \Delta v^4\, T_P \tag{2.16b}$$

$$C \equiv \hbar\, m^2\, \Delta v^4 \tag{2.16c}$$

The solution is

$$\Delta E = \frac{-B \pm \sqrt{B^2 - 4\, A C}}{2\, A} \tag{2.17}$$

Substituting the values of *A*, *B* and *C* into equation (2.17) with the second side of equations (2.16a), (2.16b) and (2.16c) respectively we get

$$\Delta E \approx \frac{m^2\, \Delta v^4\, T_P \pm \sqrt{m^4\, \Delta v^8\, T_P^2 - 4\left(m \Delta v\, L_P - \hbar\right) \hbar\, m^2\, \Delta v^4}}{2\left(m \Delta v\, L_P - \hbar\right)} \tag{2.18}$$

$$\Delta E \approx \frac{m^2\, \Delta v^4\, T_P \pm \sqrt{m^4\, \Delta v^8\, T_P^2 - 4\, \hbar\, m^3\, \Delta v^5\, L_P + 4\, \hbar^2\, m^2\, \Delta v^4}}{2\left(m \Delta v\, L_P - \hbar\right)} \tag{2.19}$$

Taking $m^2\, \Delta v^4$ as a common factor inside the square root we get

$$\Delta E \approx \frac{m^2\,\Delta v^4\,T_P \pm m\,\Delta v^2 \sqrt{m^2\,\Delta v^4\,T_P^2 - 4\,\hbar\,m\,\Delta v\,L_P + 4\,\hbar^2}}{2\left(m\,\Delta v\,L_P - \hbar\right)} \tag{2.20}$$

Now I shall assume that the maximum physically possible uncertainty in the velocity of the particle is the speed of light, *c*. Thus, I take the limit on both sides of expression (2.20) when Δv tends to *c*. According to the appendices of the previous paper entitled "*Derivation of the Equation* $E = m c^2$ *from the Heisenberg Uncertainty Principles*" [2] we can substitute the approximate sign with an equal sign if we make the second side proportional to the first side through a proportionality constant (denoted by *k*). Thus, the approximate expression (2.20) transforms into an equation

$$\lim_{\Delta v \to c} \Delta E = k \lim_{\Delta v \to c} \left[\frac{m^2\,\Delta v^4\,T_P \pm m\,\Delta v^2 \sqrt{m^2\,\Delta v^4\,T_P^2 - 4\,\hbar\,m\,\Delta v\,L_P + 4\,\hbar^2}}{2\left(m\,\Delta v\,L_P - \hbar\right)} \right] \tag{2.21}$$

$$\lim_{\Delta v \to c} \Delta E = k \left[\frac{m^2 c^4\,T_P \pm m\,c^2 \sqrt{m^2 c^4\,T_P^2 - 4\,\hbar\,m\,c\,L_P + 4\,\hbar^2}}{2\left(m\,c\,L_P - \hbar\right)} \right] \tag{2.22}$$

Considering that the Planck length and the Planck time are related through the following relationship

$$L_P = c\,T_P \tag{2.23}$$

we can substitute L_P with $c\,T_P$ in eq. (2.22). This yields

$$\lim_{\Delta v \to c} \Delta E = k \left[\frac{m^2 c^4\,T_P \pm m\,c^2 \sqrt{m^2 c^4\,T_P^2 - 4\,\hbar\,m c^2\,T_P + 4\,\hbar^2}}{2\left(m c^2\,T_P - \hbar\right)} \right] \tag{2.24}$$

Considering that

$$\left(m\,c^2\,T_P - 2\,\hbar\right)^2 = m^2 c^4\,T_P^2 - 4\,\hbar\,m\,c^2\,T_P + 4\,\hbar^2 \tag{2.25}$$

we can rewrite eq. (2.24) in terms of the first side of eq. (2.25). This produces

$$\lim_{\Delta v \to c} \Delta E = k \left[\frac{m^2 c^4\,T_P \pm m\,c^2 \sqrt{\left(m\,c^2\,T_P - 2\,\hbar\right)^2}}{2\left(m\,c^2\,T_P - \hbar\right)} \right] \tag{2.26}$$

Simplifying we get

$$\lim_{\Delta v \to c} \Delta E = k \left[\frac{m^2 c^4\,T_P \pm m\,c^2 \left(m\,c^2\,T_P - 2\,\hbar\right)}{2\left(m c^2\,T_P - \hbar\right)} \right] \tag{2.27}$$

Taking $m\,c^2$ as common factor we can write

$$\lim_{\Delta v \to c} \Delta E = k m c^2 \left[\frac{m c^2 T_P \pm \left(m c^2 T_P - 2\hbar \right)}{2\left(m c^2 T_P - \hbar \right)} \right] \tag{2.28}$$

Now I shall consider two cases, one for each sign of the square root: a) Positive root and b) Negative root.

a) Positive root

I shall show that the positive root produces the formula of equivalence of mass and energy. Thus, considering the positive sign of the square root of eq. (2.28) we get

$$\lim_{\Delta v \to c} \Delta E = k m c^2 \left[\frac{m c^2 T_P + m c^2 T_P - 2\hbar}{2\left(m c^2 T_P - \hbar \right)} \right] \tag{2.29}$$

or

$$\lim_{\Delta v \to c} \Delta E = k m c^2 \left[\frac{2 m c^2 T_P - 2\hbar}{2\left(m c^2 T_P - \hbar \right)} \right] \tag{2.30}$$

Because the value inside the square bracket is 1, we can write

$$\lim_{\Delta v \to c} \Delta E = k m c^2 \tag{2.31}$$

This result agrees with thousand of annihilation experiments: if the kinetic energy of a given particle before annihilation increases, the energy of the corresponding photon after annihilation increases proportionally. Thus the use of the proportionality constant *k* is justified.

According to **Appendix 3** of paper [2] the allowed values of the proportionality constant *k* are 1 and -1. I shall adopt *k*=1 and neglect *k*=-1. This produces the following equation

$$\lim_{\Delta v \to c} \Delta E = m c^2 \tag{2.32}$$

Considering the annihilation experiments (See reference [2])

$$\lim_{\Delta v \to c} \Delta E = E \tag{2.33}$$

where *E* is the total relativistic energy of the particle. Finally, from equations (2.32) and (2.33) we get

$$E = m c^2 \tag{2.34}$$

Thus we have derived the formula of equivalence of mass and energy from the universal uncertainty relations.

b) Negative root

I shall show that the negative root is capable of producing an undefined result. Therefore this root has no physical meaning. Thus, considering the negative sign of the square root of eq. (2.28) we get

$$\lim_{\Delta v \to c} \Delta E = k m c^2 \left[\frac{m c^2 T_P - m c^2 T_P + 2\hbar}{2\left(m c^2 T_P - \hbar\right)} \right] \tag{2.35}$$

$$\lim_{\Delta v \to c} \Delta E = k m c^2 \frac{\hbar}{m c^2 T_P - \hbar} \tag{2.36}$$

Dividing numerator and denominator by $\hbar$ we get

$$\lim_{\Delta v \to c} \Delta E = k m c^2 \frac{1}{m c^2 \frac{T_P}{\hbar} - 1} \tag{2.37}$$

Now we observe that the ratio $\frac{\hbar}{T_P}$ is the Planck energy, E_P

$$\frac{\hbar}{T_P} = E_P \tag{2.38}$$

Thus eq. (2.37) can be written in terms of the Planck energy

$$\lim_{\Delta v \to c} \Delta E = k m c^2 \left(\frac{1}{\frac{m c^2}{E_P} - 1} \right) \tag{2.39}$$

Taking k=1 and the limit on the first side as E, as we did before, we get

$$E = m c^2 \left(\frac{1}{\frac{m c^2}{E_P} - 1} \right) \tag{2.40}$$

Now I shall analyse the particular case where the total energy of the particle equals the Planck energy. Thus we have

$$E = E_P \left(\frac{1}{\frac{E_P}{E_P} - 1} \right) \tag{2.41}$$

Because the denominator of this equation is zero (which produces an undefined result) we are forced to discard the negative square root. Therefore the positive square root of eq. (2.28) is the only one with physical meaning.

3. Conclusions

This paper shows that the Einstein's equation of equivalence of mass and energy (eq. 3.1) is a special case of the universal uncertainty relations when the uncertainty in the velocity of the particle equals the speed of light. The conventional derivation of the formula of equivalence of mass and energy is based on the relativistic mass law

$$m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

, the work-energy theorem and on Newton's second law of motion. This derivation can be found in most books dealing with special relativity and on the web [3]. It is worthy to remark that I derived Einstein's equation without using the above law. So I have also proved that this law, along with the other above mentioned laws, are sufficient but not necessary to derive the famous formula of equivalence of mass and energy.

One point that could be criticized is that this research is based not only on first principles but also on annihilation experiments. However this is also the case of other theories. Einstein's special theory of relativity, for example, is based on the results of experiments (Michelson and Morley) which proved that the speed of light in vacuum is independent of the motion of the light source (a postulate known as: the invariance of c). Einstein adopted this experimental result as one of his postulates to formulate his remarkable and revolutionary theory. In summary, the conclusions of this theory are:

1. the special theory of relativity turned out to be based on quantum mechanical principles (the universal uncertainty relations), which means that two of the most important laws of physics have now been unified.

2. the universal uncertainty relations are correct.

3. the universal uncertainty relations underline a deeper truth than that of the Einstein's equation of equivalence of mass and energy.

In summary what I have shown is that the Heisenberg uncertainty relations have implications that are even more profound and general than previously thought.

Finally it is worthy to remark that the introduction of the universal uncertainty principle coupled with the Schwarzschild radius has allowed me to develop a more complete model of black holes than that obtained through general relativity alone. The predictions of this new quantum gravitational theory [4] includes a general formula for the temperature of the black hole which encompasses the corresponding Hawking temperature equation as a special case. Thus the UUP should be one of the cornerstones of any theory of quantum gravity.

Appendix 1
Nomenclature

c = speed of light in vacuum.
$\hbar$ = reduced Planck's constant ($h/2\pi$).
m = relativistic mass of the particle.
m_0 = rest mass of the particle.
v = group velocity of the particle.
E = total relativistic energy of a particle.
Δx = uncertainty in the position of the particle along the x axis due to its wave nature (wave packet representing the particle).
Δy = uncertainty in the position of the particle along the y axis due to its wave nature (wave packet representing the particle).
Δz = uncertainty in the position of the particle along the z axis due to its wave nature (wave packet representing the particle).
Δp_x = uncertainty in the momentum of the particle along the x axis due to its wave nature (wave packet representing the particle).
Δp_y = uncertainty in the momentum of the particle along the y axis due to its wave nature (wave packet representing the particle).
Δp_z = uncertainty in the momentum of the particle along the z axis due to its wave nature (wave packet representing the particle).
ΔE = uncertainty in the energy of the particle
Δt = time uncertainty (time taken by the particle to travel a distance equal to the uncertainty in the position).
Δv_x = uncertainty in the speed of the particle along the x axis.
Δv_y = uncertainty in the speed of the particle along the y axis.
Δv_z = uncertainty in the speed of the particle along the z axis.
Δv = uncertainty in the speed of the particle along the any axis (I used Δv instead of Δv_x to simplify the equations).
L_P = Planck length. Uncertainty in the position of the particle due to the quantum fluctuations of space-time. This uncertainty does not include the uncertainties in the positions (Δx , Δy , Δz) due to the wave nature of the wave packet representing the particle.
T_P = Planck time. Time uncertainty of the particle due to the quantum fluctuations of space-time. This uncertainty does not include the time uncertainty, Δt , described above.
E_P = Planck energy.
k = proportionality constant (real number).

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