Derivation of the Equation $E = m c^2$ from the Universal Uncertainty Principles

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Abstract

The present paper is concerned with the derivation of the Einstein's formula of equivalence of mass and energy, $E = m c^2$ *, from the universal uncertainty relations. These relations are a generalization of the Heisenberg uncertainty relations. Thus, this approach unifies two of the most important laws of physics as provides the proof of the quantum mechanical interpretation of the above formula, and, at the same time, provides the proof of the correctness of the universal uncertainty relations that I found in 2012.*

Keywords: *Heisenberg uncertainty principle, universal uncertainty principle, universal momentum-position uncertainty relations, universal energy-time uncertainty relation, spatial uncertainty principle, temporal uncertainty principle, annihilation, Planck time, Planck length, Planck force, Planck energy, Planck's constant, reduced Planck's constant, Newton's law of universal gravitation.*

1. Introduction

In 1927 Werner Heisenberg proposed two fundamental relations that would revolutionize quantum mechanics. They are known as the Heisenberg uncertainty relations or principles. In 2012 I generalized these relations by incorporating the effects of gravity into the momentum-position uncertainty relations and into the energy-time uncertainty relation. The result of this generalization were called the universal uncertainty relations:

1. The universal momentum-position uncertainty relations (or spatial UUP [1]):

$$\Delta p_x \Delta x \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_x} \tag{1.1a}$$

$$\Delta p_y \Delta y \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_y} \tag{1.1b}$$

$$\Delta p_z \Delta z \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_z} \tag{1.1c}$$

and

2. the universal energy-time uncertainty relation (or temporal UUP):

$$\Delta E \Delta t \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t} \tag{1.2}$$

Based on these relations, and considering the results of particle-antiparticle annihilation experiments, the Planck force and the Newton law of universal gravitation, I shall derive the equation of equivalence of mass and energy

$$E = m c^2 \tag{1.3}$$

The purpose of using the Planck force and the Newton's law of universal gravitation is to determine the value of the real constant *k* I shall introduce in the next section. (See **Appendix 1** for the nomenclature used in this paper.)

2. From the Universal Uncertainty Relations to Einstein's Formula of Equivalence of Mass and Energy

Let us consider the spatial uncertainty principle given by relation (1.1a)

$$\Delta p_x \Delta x \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_x} \tag{2.1a}$$

and the temporal uncertainty principle given by relation (1.2)

$$\Delta E \Delta t \geq \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t} \tag{2.1b}$$

We can make preliminary good estimates by writing the above relations as approximations

$$\Delta p_x \Delta x \approx \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_x} \tag{2.2a}$$

$$\Delta E \Delta t \approx \sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t} \tag{2.2b}$$

Later we shall transform these preliminary approximations into exact equations. Before I begin to derive the formula of equivalence of mass and energy, let me outline the general strategy I shall follow. Because we apply these two uncertainty principles to the same particle, the uncertainty in the position of the particle, Δx, must be related to the time Δt taken by the particle to travel a distance equal to Δx. Because Δx is a position uncertainty, Δt must be a time uncertainty. In other words we shall assume that the uncertainty in the speed of the particle is given by the ratio between the uncertainties in position and time. Then we shall assume that when the maximum uncertainty in the velocity of the particle is the speed of light, the maximum uncertainty in the energy of the particle will be the energy of the particle itself. We know that this is so because of annihilation experiments. However, before annihilation, a particle can be either at rest or moving with respect to the observer. These two cases have been analyzed in a previous paper [2], and because the concepts are exactly the same in both papers, there is no need to duplicate the results here.

The derivation is very simple as requires basic mathematical knowledge. To start the derivation let us divide the expression (2.2a) by the expression (2.2b). Mathematically we write

$$\frac{\Delta p_x \Delta x}{\Delta E \Delta t} \approx \frac{\sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_x}}{\sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t}} \tag{2.3}$$

To simplify the equations I have used Δv instead of Δv_x, and Δp instead of Δp_x.

$$\frac{\Delta p \, \Delta x}{\Delta E \, \Delta t} \approx \frac{\sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p}}{\sqrt{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t}} \tag{2.4}$$

Now I raise eq. (2.4) to the power of 2 to remove the square roots of the second side of the equation. This gives

$$\frac{\Delta p^2 \, \Delta x^2}{\Delta E^2 \Delta t^2} \approx \frac{\frac{\hbar^2}{4} - \frac{\hbar}{4} L_P \Delta p_x}{\frac{\hbar^2}{4} - \frac{\hbar}{4} T_P \Delta t} \tag{2.5}$$

Multiplying numerator and denominator by $\frac{4}{\hbar}$ we get

$$\frac{\Delta p^2}{\Delta E^2} \left(\frac{\Delta x}{\Delta t} \right)^2 \approx \frac{\hbar - L_P \Delta p_x}{\hbar - T_P \Delta t} \tag{2.6}$$

Considering that

$$\Delta v \equiv \frac{\Delta x}{\Delta t} \tag{2.7}$$

(Δv should not to be confused with the average velocity of the particle)
Where, to simplify the equations, I have used Δv instead of Δv_x

$$\frac{\Delta p^2}{\Delta E^2} \Delta v^2 \approx \frac{\hbar - L_P \Delta p_x}{\hbar - T_P \Delta t} \tag{2.8}$$

Taking into account that

$$\Delta p = m \Delta v \tag{2.9}$$

(It is worthy to observe that, as postulated by W. Heisenberg in his uncertainty principle, the mass of the particle, *m*, does not include any uncertainty). Thus we can write

$$\frac{(m\,\Delta v)^2}{\Delta E^2}\,\Delta v^2 \approx \frac{\hbar - L_P \Delta p_x}{\hbar - T_P \Delta t} \tag{2.10}$$

$$\frac{m^2 \Delta v^4}{\Delta E^2} \approx \frac{\hbar - L_P \Delta p_x}{\hbar - T_P \Delta t} \tag{2.11}$$

$$m^2\,\Delta v^4 \left(\hbar - T_P \Delta E\right) \approx \left(\hbar - L_P\, m\,\Delta v\right) \Delta E^2 \tag{2.12}$$

$$\hbar\, m^2\,\Delta v^4 - \left(m^2\,\Delta v^4\, T_P\right) \Delta E - \left(\hbar - m\,\Delta v\, L_P\right) \Delta E^2 \approx 0 \tag{2.13}$$

$$\hbar\, m^2\,\Delta v^4 - \left(m^2\,\Delta v^4\, T_P\right) \Delta E + \left(m\,\Delta v\, L_P - \hbar\right) \Delta E^2 \approx 0 \tag{2.14}$$

$$\left(m\,\Delta v\, L_P - \hbar\right) \Delta E^2 - \left(m^2\,\Delta v^4\, T_P\right) \Delta E + \hbar\, m^2\,\Delta v^4 \approx 0 \tag{2.15}$$

Thus we arrive to a quadratic equation with the following coefficients

$$A \equiv m\,\Delta v\, L_P - \hbar \tag{2.16a}$$

$$B \equiv -\,m^2\,\Delta v^4\, T_P \tag{2.16b}$$

$$C \equiv \hbar\, m^2\,\Delta v^4 \tag{2.16c}$$

The solution is

$$\Delta E = \frac{-B \pm \sqrt{B^2 - 4\,A\,C}}{2\,A} \tag{2.17}$$

Substituting the values of *A*, *B* and *C* into equation (2.17) with the second side of equations (2.16a), (2.16b) and (2.16c) respectively we get

$$\Delta E \approx \frac{m^2\,\Delta v^4\, T_P \pm \sqrt{m^4\,\Delta v^8\, T_P^2 - 4\left(m\,\Delta v\, L_P - \hbar\right)\hbar\, m^2\,\Delta v^4}}{2\left(m\,\Delta v\, L_P - \hbar\right)} \tag{2.18}$$

$$\Delta E \approx \frac{m^2\,\Delta v^4\, T_P \pm \sqrt{m^4\,\Delta v^8\, T_P^2 - 4\,\hbar\, m^3\,\Delta v^5\, L_P + 4\,\hbar^2\, m^2\,\Delta v^4}}{2\left(m\,\Delta v\, L_P - \hbar\right)} \tag{2.19}$$

Taking $m^2\,\Delta v^4$ as a common factor inside the square root we get

$$\Delta E \approx \frac{m^2\,\Delta v^4\, T_P \pm m\,\Delta v^2 \sqrt{m^2\,\Delta v^4\, T_P^2 - 4\,\hbar\, m\,\Delta v\, L_P + 4\,\hbar^2}}{2\left(m\,\Delta v\, L_P - \hbar\right)} \tag{2.20}$$

Now we shall assume that the maximum physically possible uncertainty in the velocity of the particle is the speed of light, *c*. Thus, we take the limit on both sides of expression

(2.20) when Δv tends to *c*. According to the appendices of the previous paper entitled "*Derivation of the Equation* $E = m c^2$ *from the Heisenberg Uncertainty Principles*" [2] we can substitute the approximate sign with an equal sign if we make the second side proportional to the first side through a proportionality constant (denoted by *k*). Thus, the approximate expression (2.20) transforms into an equation

$$\lim_{\Delta v_x \to c} \Delta E = k \lim_{\Delta v_x \to c} \left[\frac{m^2 \, \Delta v^4 \, T_P \pm m \Delta v^2 \sqrt{m^2 \, \Delta v^4 \, T_P^2 - 4 \hbar m \Delta v L_P + 4 \hbar^2}}{2 \left(m \Delta v L_P - \hbar \right)} \right] \tag{2.21}$$

$$\lim_{\Delta v_x \to c} \Delta E = k \left[\frac{m^2 c^4 T_P \pm m c^2 \sqrt{m^2 c^4 T_P^2 - 4 \hbar m c L_P + 4 \hbar^2}}{2 \left(m c L_P - \hbar \right)} \right] \tag{2.22}$$

Considering that the Planck length and the Planck time are related through the following relationship

$$L_P = c T_P \tag{2.23}$$

we can substitute L_P with $c T_P$ in eq. (2.22). This yields

$$\lim_{\Delta v_x \to c} \Delta E = k \left[\frac{m^2 c^4 T_P \pm m c^2 \sqrt{m^2 c^4 T_P^2 - 4 \hbar m c^2 T_P + 4 \hbar^2}}{2 \left(m c^2 T_P - \hbar \right)} \right] \tag{2.24}$$

Considering that

$$\left(m c^2 T_P - 2 \hbar \right)^2 = m^2 c^4 T_P^2 - 4 \hbar m c^2 T_P + 4 \hbar^2 \tag{2.25}$$

we can rewrite eq. (2.24) in terms of the first side of eq. (2.25). This produces

$$\lim_{\Delta v_x \to c} \Delta E = k \left[\frac{m^2 c^4 T_P \pm m c^2 \sqrt{\left(m c^2 T_P - 2 \hbar \right)^2}}{2 \left(m c^2 T_P - \hbar \right)} \right] \tag{2.26}$$

Simplifying we get

$$\lim_{\Delta v_x \to c} \Delta E = k \left[\frac{m^2 c^4 T_P \pm m c^2 \left(m c^2 T_P - 2 \hbar \right)}{2 \left(m c^2 T_P - \hbar \right)} \right] \tag{2.27}$$

Taking $m c^2$ as common factor we can write

$$\lim_{\Delta v_x \to c} \Delta E = k \, m c^2 \left[\frac{m c^2 T_P \pm \left(m c^2 T_P - 2 \hbar \right)}{2 \left(m c^2 T_P - \hbar \right)} \right] \tag{2.28}$$

Now we shall consider two cases, one for each sign of the square root: a) Positive root and b) Negative root.

a) Positive root

I shall show that the positive root produces the formula of equivalence of mass and energy. Let us begin considering the positive sign of the square root of eq. (2.28)

$$\lim_{\Delta v_x \to c} \Delta E = k\, m c^2 \left[\frac{m c^2 T_P + m c^2 T_P - 2\hbar}{2\left(m c^2 T_P - \hbar\right)} \right] \tag{2.29}$$

or

$$\lim_{\Delta v_x \to c} \Delta E = k\, m c^2 \left[\frac{2 m c^2 T_P - 2\hbar}{2\left(m c^2 T_P - \hbar\right)} \right] \tag{2.30}$$

Because the value inside the square bracket is 1, we can write

$$\lim_{\Delta v_x \to c} \Delta E = k\, m c^2 \tag{2.31}$$

This result agrees with thousand of annihilation experiments: if the kinetic energy of a given particle before annihilation increases, the energy of the corresponding photon after annihilation increases proportionally. Thus the use of the proportionality constant *k* is properly justified.

According to **Appendix 3** of paper [2] the allowed values of the proportionality constant *k* are 1 and -1. I shall adopt *k*=1 and neglect *k*=-1. This produces the following equation

$$\lim_{\Delta v_x \to c} \Delta E = m c^2 \tag{2.32}$$

Considering the annihilation experiments (See reference [2])

$$\lim_{\Delta v_x \to c} \Delta E = E \tag{2.33}$$

where *E* is the total relativistic energy of the particle. Finally, from equations (2.32) and (2.33) we get the Einstein's equation of equivalence of mass and energy

$$E = m c^2 \tag{2.34}$$

Thus we have derived the formula of equivalence of mass and energy from the universal uncertainty relations.

b) Negative root

I shall show that the negative root is capable of producing an undefined result and therefore has no physical meaning. Let us consider the negative sign of the square root of eq. (2.28)

$$\lim_{\Delta v_x \to c} \Delta E = k\, m c^2 \left[\frac{m c^2 T_P - m c^2 T_P + 2\hbar}{2\left(m c^2 T_P - \hbar\right)} \right] \quad (2.35)$$

$$\lim_{\Delta v_x \to c} \Delta E = k\, m c^2 \frac{\hbar}{m c^2 T_P - \hbar} \quad (2.36)$$

Dividing numerator and denominator by $\hbar$ we get

$$\lim_{\Delta v_x \to c} \Delta E = k\, m c^2 \frac{1}{m c^2 \dfrac{T_P}{\hbar} - 1} \quad (2.37)$$

Now we observe that the ratio $\dfrac{\hbar}{T_P}$ is the Planck energy, E_P

$$\frac{\hbar}{T_P} = E_P \quad (2.38)$$

Thus eq. (2.37) can be written in terms of the Planck energy

$$\lim_{\Delta v_x \to c} \Delta E = k\, m c^2 \left(\frac{1}{\dfrac{m c^2}{E_P} - 1} \right) \quad (2.39)$$

Taking *k*=1 and the limit on the first side as *E,* as we did before, we get

$$E = m c^2 \left(\frac{1}{\dfrac{m c^2}{E_P} - 1} \right) \quad (2.40)$$

Now I shall analyze the particular case where the total energy of the particle equals the Planck energy. Thus we have

$$E = E_P \left(\frac{1}{\dfrac{E_P}{E_P} - 1} \right) \quad (2.41)$$

Because the denominator of this equation is zero (which produces an undefined result) we are forced to discard the negative square root. Therefore the positive square root of eq. (2.28) is the only one with physical meaning.

3. Conclusions

This paper shows that Einstein's equation of equivalence of mass and energy (eq. 3.1) is a special case of the universal uncertainty relations when the uncertainty in the velocity of the particle equals the speed of light. It is worthy to remark that we derived Einstein's equation without using the relativistic mass law: $m = m_0 / \sqrt{1 - v^2/c^2}$. So we have also proved that this formula is sufficient but not necessary to derive the formula of equivalence of mass and energy. The “classical” derivation of this formula is based on the relativistic mass law, the work-energy theorem and on Newton's second law of motion and can be found in most books dealing with special relativity and on the web [3].

One point that could be criticized is that this research is based, at least in part, on annihilation experiments rather than on first principles. However this is also the case of other theories. Einstein's special theory of relativity, for example, is based on the results of experiments (Michelson and Morley) which proved that the speed of light is independent of the motion of the light source (a postulate known as: the invariance of *c*). Einstein adopted this experimental result as one of his postulates to formulate his remarkable and revolutionary theory. In summary, the conclusions of this theory are:

1. the special theory of relativity turned out to be based on quantum mechanical principles (the universal uncertainty relations), which means that two of the most important laws of physics have now been unified.

2. the universal uncertainty relations are correct.

3. the universal uncertainty relations (and therefore the Heisenberg uncertainty relations) underline a deeper truth than that of the Einstein's equation of equivalence of mass and energy.

Finally it is worthy to remark that the introduction of the universal uncertainty principle allows to develop the derivation of the thermodynamic properties of black holes: *temperature* and *entropy* (see reference [4]). Thus the UUP should be one of the cornerstones of quantum gravity.

Appendix 1
Nomenclature

c = speed of light in vacuum
$\hbar$ = reduced Planck's constant ($h/2\pi$)
m = relativistic mass of a particle
m_0 = rest mass of a particle
v = group velocity of the particle
E = total relativistic energy of a particle
Δx = uncertainty in the position of the particle along the *x* axis.
Δy = uncertainty in the position of the particle along the *y* axis.
Δz = uncertainty in the position of the particle along the *z* axis.
Δp_x = uncertainty in the momentum of the particle along the *x* axis
Δp_y = uncertainty in the momentum of the particle along the *y* axis
Δp_z = uncertainty in the momentum of the particle along the *z* axis
ΔE = uncertainty in the energy of the particle
Δt = time uncertainty (time taken by the particle to move a length equal to the uncertainty in the position)
Δv_x = uncertainty in the speed of the particle along the *x* axis

Δv_y = uncertainty in the speed of the particle along the *y* axis
Δv_z = uncertainty in the speed of the particle along the *z* axis
Δv = uncertainty in the speed of the particle along the *x* axis (I used Δv instead of Δv_x to simplify the equations)
L_P = Planck length
T_P = Planck time
E_P = Planck energy
k = proportionality constant (real number)

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