Menelaus's Theorem for Hyperbolic Quadrilaterals in The Einstein Relativistic Velocity Model of Hyperbolic Geometry

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Abstract

In this study, we present (i) a proof of the Menelaus theorem for quadrilaterals in hyperbolic geometry, (ii) and a proof for the transversal theorem for triangles, and (iii) the Menelaus's theorem for n-gons

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1. Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. Menelaus of Alexandria was a Greek mathematician and astronomer, the first to recognize geodesics on a curved surface as natural analogs of straight lines. Here, in this study, we give hyperbolic version of Menelaus theorem for quadrilaterals. The well-known Menelaus theorem states that if l is a line not through any vertex of a triangle ABC such that l meets BC in D, CA in E, and AB in F, then $\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$ [1]. F. Smarandache (1983) has generalized the Theorem of Menelaus for any polygon with $n \geq 4$ sides as follows: If a line l intersects the n-gon $A_1A_2...A_n$ sides $A_1A_2, A_2A_3, ...,$ and A_nA_1 respectively in the points $M_1, M_2, ...,$ and M_n, then $\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot ... \cdot \frac{M_nA_n}{M_nA_1} = 1$ [2].

Let D denote the complex unit disc in complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta}\frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyrocommutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of onedimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Definition 1 *Let ABC be a gyrotriangle with sides a, b, c in an Einstein gyrovector space $(V_s, \oplus, \otimes)$, and let h_a, h_b, h_c be three altitudes of ABC drawn from vertices A, B, C perpendicular to their opposite sides a, b, c or their extension, respectively. The number*

$$S_{ABC} = \gamma_a a \gamma_{h_a} h_a = \gamma_b b \gamma_{h_b} h_b = \gamma_c c \gamma_{h_c} h_c$$

is called the gyrotriangle constant of gyrotriangle ABC (here $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{s^2}}}$ is the gamma factor).

(see [3, pp558])

Theorem 1 ***(The Gyrotriangle Constant Principle)*** *Let A_1BC and A_2BC be two gyrotriangles in a Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$, $A_1 \neq A_2$ such that the two gyrosegments A_1A_2 and BC, or their extensions, intersect at a point $P \in \mathbb{R}^2_s$, as shown in Figs 1-2. Then,*

$$\frac{\gamma_{|A_1P|}\,|A_1P|}{\gamma_{|A_2P|}\,|A_2P|} = \frac{S_{A_1BC}}{S_{A_2BC}}$$

(see [3, pp 563])

Theorem 2 ***(The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Space)*** *Let $\mathbf{a}_1, \mathbf{a}_2,$ and $\mathbf{a}_3$ be three non-gyrocollinear points* *in an Einstein gyrovector space* $(V_s, \oplus, \otimes)$. *If a gyroline meets the sides of gyrotriangle* $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ *at points* $\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{23},$ *as in Figure 3, then*

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|}\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}\|}\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}\|} = 1$$

(see [3, pp 463])

For further details we refer to the recent book of A.Ungar [3].

2. Main results

In this section, we prove Menelaus's theorem for hyperbolic quadrilateral.

Theorem 3 *If* l *is a gyroline not through any vertex of a gyroquadrilateral* $ABCD$ *such that* l *meets* AB *in* X, BC *in* Y, CD *in* Z, *and* DA *in* W, *then*

$$\frac{\gamma_{|AX|}|AX|}{\gamma_{|BX|}|BX|}\cdot\frac{\gamma_{|BY|}|BY|}{\gamma_{|CY|}|CY|}\cdot\frac{\gamma_{|CZ|}|CZ|}{\gamma_{|DZ|}|DZ|}\cdot\frac{\gamma_{|DW|}|DW|}{\gamma_{|AW|}|AW|} = 1 \tag{1}$$

Proof. Let T be the intersection point of the gyroline DB and the gyroline XYZ (See Figure 4).

If we use a theorem 3 in the triangles ABD and BCD respectively, then

$$\frac{\gamma_{|AX|}|AX|}{\gamma_{|BX|}|BX|}\cdot\frac{\gamma_{|BT|}|BT|}{\gamma_{|DT|}|DT|}\cdot\frac{\gamma_{|DW|}|DW|}{\gamma_{|AW|}|AW|}=1 \tag{2}$$

and

$$\frac{\gamma_{|DT|}|DT|}{\gamma_{|BT|}|BT|}\cdot\frac{\gamma_{|CZ|}|CZ|}{\gamma_{|DZ|}|DZ|}\cdot\frac{\gamma_{|BY|}|BY|}{\gamma_{|CY|}|CY|}=1. \tag{3}$$

Multiplying relations (2) and (3) member with member, we obtain the conclusion. ■

We have thus obtained in (1) the following:

Theorem 4 *(Transversal theorem for triangles) Let D be on gyroside BC, and l is a gyroline not through any vertex of a gyrotriangle ABC such that l meets AB in M, AC in N, and AD in P, then*

$$\frac{\gamma_{|AM|}|AM|}{\gamma_{|AB|}|AB|}\cdot\frac{\gamma_{|AC|}|AC|}{\gamma_{|AN|}|AN|}\cdot\frac{\gamma_{|PN|}|PN|}{\gamma_{|PM|}|PM|}\cdot\frac{\gamma_{|DB|}|DB|}{\gamma_{|DC|}|DC|}=1$$

Proof. If we use a theorem 4 for gyroquadrilateral $BCNM$ and gyrocollinear points D, A, P, and A (See Figure 5) then the conclusion follows.

■

Theorem 5 *If l is a gyroline not through any vertex of a $n-gyrogon$ $A_1A_2...A_n$ such that l meets A_1A_2 in M_1, A_2A_3 in $M_2, ...,$ and A_nA_1 in M_n, then*

$$\frac{\gamma_{|M_1A_1|}|M_1A_1|}{\gamma_{|M_1A_2|}|M_1A_2|} \cdot \frac{\gamma_{|M_2A_2|}|M_2A_2|}{\gamma_{|M_2A_3|}|M_2A_3|} \cdot ... \cdot \frac{\gamma_{|M_nA_n|}|M_nA_n|}{\gamma_{|M_nA_1|}|M_nA_1|} = 1 \tag{4}$$

Proof. We use mathematical induction. For $n = 3$ the theorem is true (see Theorem 3). Let's suppose by induction upon $k \geq 3$ that the theorem is true for any $k-gyrogon$ with $3 \leq k \leq n-1$, and we need to prove it is also true for $k = n$. Suppose a line l intersect the gyroline A_2A_n into the point M. We consider the $n-gyrogon$ $A_1A_2...A_n$ and we split in a $3-gyrogon$ $A_1A_2A_n$ and $(n-1)-gyrogon$ $A_nA_2A_3...A_{n-1}$ and we can respectively apply the theorem 3 according to our previously hypothesis of induction in each of them, and we respectively get:

$$\frac{\gamma_{|M_1A_1|}|M_1A_1|}{\gamma_{|M_1A_2|}|M_1A_2|} \cdot \frac{\gamma_{|MA_2|}|MA_2|}{\gamma_{|MA_n|}|MA_n|} \cdot \frac{\gamma_{|M_nA_n|}|M_nA_n|}{\gamma_{|M_nA_1|}|M_nA_1|} = 1$$

and

$$\frac{\gamma_{|MA_n|}|MA_n|}{\gamma_{|MA_2|}|MA_2|} \cdot \frac{\gamma_{|M_2A_2|}|M_2A_2|}{\gamma_{|M_2A_3|}|M_2A_3|} \cdot ... \cdot \frac{\gamma_{|M_{n-2}A_{n-2}|}|M_{n-2}A_{n-2}|}{\gamma_{|M_{n-2}A_{n-1}|}|M_{n-2}A_{n-1}|} \cdot \frac{\gamma_{|M_{n-1}A_{n-1}|}|M_{n-1}A_{n-1}|}{\gamma_{|M_{n-1}A_n|}|M_{n-1}A_n|} = 1$$

whence, by multiplying the last two equalities, we get

$$\frac{\gamma_{|M_1A_1|}|M_1A_1|}{\gamma_{|M_1A_2|}|M_1A_2|} \cdot \frac{\gamma_{|M_2A_2|}|M_2A_2|}{\gamma_{|M_2A_3|}|M_2A_3|} \cdot ... \cdot \frac{\gamma_{|M_nA_n|}|M_nA_n|}{\gamma_{|M_nA_1|}|M_nA_1|} = 1.$$

■

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