IMPROVEMENT ON OPERATOR AXIOMS AND FUNDAMENTAL OPERATOR FUNCTIONS

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ABSTRACT. The reference [2] contructs the Operator axioms to deduce number systems. In this paper, we slightly improve on the syntax of the Operator axioms and construct a semantics of the Operator axioms. Then on the basis of the improved Operator axioms, we define two fundamental operator functions to study the analytic properties of the Operator axioms. Finally, we prove two theorems about the fundamental operator functions.

1. INTRODUCTION

In [1], we distinguish the limit from the infinite sequence. Then in [2], we define the Operator axioms to extend the traditional real number system. the Operator axioms have shown typical features as follows:

1. The logical calculus provides a uniform frame for arithmetic axioms. Based on the same logical calculus, small number system can import new axioms to produce big number systems.

2. Consistent binary relation are the nature of number systems. The order relation and equivalence relation of each number system is consistent, so all numbers of each number system are layed in fixed positions of number line.

3. The number systems equate number with operation. The number '1' and various operators compose all numbers through operation. Any number except '1' is also an opertion. For example, the number "$[[1+[1+1]]----[1+1]]$" derives from the operation of the number "$[1 + [1 + 1]]$", the number "$[1 + 1]$" and the real operator "$----$". So operator distinguishes different number systems and is the foremost component of number system.

4. The Operator axioms produces new real numbers with new operators. While producing new real numbers in arithmetic, the new operators certainly produce new equations and inequalities in algebra. So the Operator axioms not only extends real number system, but also extends equations and inequalities.

In conclusion, the Operator axioms forms a new arithmetic axiom. The [2, Definition 2.2] defines 'number' on the basis of the logical calculus $\{\Phi, \Psi\}$. The [2, TABLE 2] defines new operators according to the definition of number systems. Real operators naturally produce new equations such as $y = [x++++[1+1]]$, $y = [[1+1]----x]$, $y = [x////[1+1]]$ and so on. In other words, real operators extend the traditional mathematical models which are selected to describe various scientific rules. So real operators help to describe complex scientific rules which are difficult described by traditional equations and have an enormous application potential.

The paper is organized as follows. In Section 2, we construct the syntax and semantics of the improved Operator axioms. In Section 3, we define two fundamental operator functions and prove two theorems about the fundamental operator functions.

2. Operator Axioms

The Operator axioms divides into syntax and semantics. The syntax aims at logical deduction, while the semantics aims at the objects mapped from the syntax deduction.

2.1. **Syntax Of The Operator Axioms[2].** The syntax of the Operator axioms derives from [2] and is revised a little in this section. Table 1 translates the simple part of the syntax to natural language. The complex part of the syntax is difficult to be translated to natural language.

TABLE 1. [2, TABLE 2]Translation From Syntax To Natural Language.

Syntax	Natural Language
$\wedge$	and
$\vee$	or
$\neg$	not
$\rightarrow$	replaced by
$\Rightarrow$	imply
$\Leftrightarrow$	symmetrical imply
{	punctuation
}	punctuation
,	punctuation
(	punctuation
)	punctuation
$\emptyset$	emptiness
$\cdots$	omission
$\Phi\{\cdots\}$	Denote Φ as a set of notations and particular axioms between { and }. Different logical calculus correspond to different notations and particular axioms.
$V\{\cdots\}$	Denote V as a set of variables between { and }.
$C\{\cdots\}$	Denote C as a set of constants between { and }.
$P\{\cdots\}$	Denote P as a set of predicate symbols between { and }.
$V \circ C\{\cdots\}$	Denote $V \circ C$ as a set of concatenations between V and C.
$C \circ C\{\cdots\}$	Denote $C \circ C$ as a set of concatenations between C and C.
$V \circ C \circ P\{\cdots\}$	Denote $V \circ C \circ P$ as a set of concatenations among V, C and P.
$(\cdots \in \cdots)\cdots \Leftrightarrow \cdots$	Define the binary predicate symbol $\in$.
$(\cdots \in (V \circ C))$	Define a variable ranging over $V \circ C$.
$(\cdots \in (V \circ C \circ P))$	Define a variable ranging over $V \circ C \circ P$.
$(\cdots \subseteq \cdots)\cdots \Leftrightarrow \cdots$	Define the binary predicate symbol $\subseteq$.
$\Psi\{\cdots\}$	Denote Ψ as a set of general axioms between { and }. Different logical calculus correspond to the same general axioms.
$(\cdots \rightarrow \cdots) \wedge (\cdots \rightarrow \cdots) \Rightarrow \cdots$	Define the binary predicate symbol $\rightarrow$.
$(\cdots \mid \cdots)\cdots \Rightarrow \cdots$	Define the binary predicate symbol $\mid$.
$(\cdots < \cdots)\cdots \Rightarrow \cdots$	Define the binary predicate symbol $<$.
$(\cdots = \cdots)$	Define the binary predicate symbol $=$.
$(\cdots = \cdots)\cdots \Rightarrow \cdots$	Define the binary predicate symbol $=$.
$(\cdots \leq \cdots)\cdots \Leftrightarrow \cdots$	Define the binary predicate symbol $\leq$.
$(\cdots \leq \cdots)\cdots \Rightarrow \cdots$	Define the binary predicate symbol $\leq$.

Definition 2.1. [2, Definition 2.2] In a logical calculus $\{\Phi, \Psi\}$, if $\bar{a} \equiv true$, then $\bar{a}$ is a number.

Definition 2.2. the Operator axioms is a logical calculus $R\{\Phi, \Psi\}$ such that:

$\Phi\{$

$(OA.1)$ $V\{\emptyset, a, b, c, d, e, f, g, h, i, j\},$

$(OA.2)$ $C\{\emptyset, 1, +, [,], -, /\},$

$(OA.3)$ $P\{\emptyset, \in, \subseteq, \to, |, =, <, \le, \|\},$

$(OA.4)$ $V \circ C\{\emptyset, a, b \cdots, 1, + \cdots, aa, ab \cdots, a1, a + \cdots, ba, bb \cdots, b1, b + \cdots,$
$aaa, aab \cdots, aa1, aa + \cdots, baa, bab \cdots, ba1, ba + \cdots\},$

$(OA.5)$ $C \circ C\{\emptyset, 1, + \cdots, 11, 1 + \cdots, 111, 11 + \cdots\},$

$(OA.6)$ $V \circ C \circ P\{\emptyset, a, b \cdots, 1, + \cdots, \in, \subseteq \cdots, aa, ab \cdots, a1, a + \cdots, a \in, a \subseteq \cdots,$
$ba, bb \cdots, b1, b + \cdots, b \in, b \subseteq \cdots, aaa, aab \cdots, aa1, aa + \cdots, aa \in, aa \subseteq \cdots,$
$baa, bab \cdots, ba1, ba + \cdots, ba \in, ba \subseteq \cdots\},$

$(OA.7)$ $(\hat{a} \in V) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots),$

$(OA.8)$ $(\hat{a} \in C) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots),$

$(OA.9)$ $(\hat{a} \in (V \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv 1) \cdots \vee (\hat{a} \equiv aa)$
$\vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a1) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa1) \cdots),$

$(OA.10)$ $(\hat{a} \in (C \circ C)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv 1) \vee (\hat{a} \equiv +) \cdots \vee (\hat{a} \equiv 11) \vee (\hat{a} \equiv 1+) \cdots$
$\vee (\hat{a} \equiv 111) \vee (\hat{a} \equiv 11+) \cdots),$

$(OA.11)$ $(\hat{a} \in (V \circ C \circ P)) \Leftrightarrow ((\hat{a} \equiv \emptyset) \vee (\hat{a} \equiv a) \vee (\hat{a} \equiv b) \cdots \vee (\hat{a} \equiv \in) \cdots \vee (\hat{a} \equiv aa)$
$\vee (\hat{a} \equiv ab) \cdots \vee (\hat{a} \equiv a \in) \cdots \vee (\hat{a} \equiv aaa) \vee (\hat{a} \equiv aab) \cdots \vee (\hat{a} \equiv aa \in) \cdots),$

$(OA.12)$ $(\bar{a} \in (V \circ C)) \wedge (\bar{b} \in (V \circ C)) \wedge (\bar{c} \in (V \circ C)) \wedge (\bar{d} \in (V \circ C)) \wedge (\bar{e} \in (V \circ C))$
$\wedge (\bar{f} \in (V \circ C)) \wedge (\bar{g} \in (V \circ C)) \wedge (\bar{h} \in (V \circ C)) \wedge (\bar{i} \in (V \circ C)) \wedge$
$(\bar{j} \in (V \circ C)) \wedge (\bar{k} \in (V \circ C)) \wedge (\bar{l} \in (V \circ C)) \wedge (\bar{\bar{a}} \in (V \circ C \circ P)) \wedge$
$(\bar{\bar{b}} \in (V \circ C \circ P)) \wedge (\bar{\bar{c}} \in (V \circ C \circ P)),$

$(OA.13)$ $((\bar{a} \subseteq \{\bar{b}, \bar{c}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee (\bar{a} \subseteq \bar{g}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee (\bar{a} \subseteq \bar{f})$
$\vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}))) \wedge$
$((\bar{a} \subseteq \{\bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{i}, \bar{j}\}) \Leftrightarrow ((\bar{a} \subseteq \bar{b}) \vee (\bar{a} \subseteq \bar{c}) \vee (\bar{a} \subseteq \bar{d}) \vee (\bar{a} \subseteq \bar{e}) \vee$
$(\bar{a} \subseteq \bar{f}) \vee (\bar{a} \subseteq \bar{g}) \vee (\bar{a} \subseteq \bar{h}) \vee (\bar{a} \subseteq \bar{i}) \vee (\bar{a} \subseteq \bar{j}))),$

$(OA.14)$ $(\bar{a}\bar{b}\bar{c} = \bar{d}b\bar{e}\bar{f}\bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \wedge \neg(\bar{f} \subseteq \{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{g}\}) \wedge$
$((\bar{b} \to \bar{h})\|(\bar{f} \to \bar{i})) \Rightarrow (\bar{a}\bar{h}\bar{c} = \bar{d}h\bar{e}\bar{i}\bar{g}),$

$(OA.15)$ $(\bar{a}\bar{b}\bar{c}\bar{d}\bar{e} = \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \wedge \neg(\bar{d} \subseteq \{\bar{a}, \bar{b}, \bar{c}, \bar{e}, \bar{f}\}) \wedge ((\bar{b} \to \bar{g})\|(\bar{d} \to \bar{h}))$
$\Rightarrow (\bar{a}\bar{g}\bar{c}\bar{h}\bar{e} = \bar{f}),$

$(OA.16)$ $a \to 1|[aba],$

$(OA.17)$ $b \to +|-,$

$(OA.18)$ $c|d \to e|f|g,$

$(OA.19)$ $e \to +| + e,$

$(OA.20)$ $f \to -| - f,$

$(OA.21)$ $g \to /|/g,$

$(OA.22)$ $(h \to +)\|(i \to -),$

$(OA.23)$ $(h \to +h)\|(i \to -i),$

$(OA.24)$ $(i \to -)\|(h \to +),$

$(OA.25)$ $(i \to -i)\|(h \to +h),$

$(OA.26)$ $(h \to +)\|(j \to /),$

$(OA.27)$ $(h \to +h)\|(j \to /j),$

$(OA.28)$ $a < [1 + a],$

$(OA.29)$ $(\bar{a} < \bar{b}) \wedge \bar{c} \Rightarrow (([\bar{a} + \bar{c}] < [\bar{b} + \bar{c}]) \wedge ([\bar{a} - \bar{c}] < [\bar{b} - \bar{c}]) \wedge ([\bar{c} - \bar{b}] < [\bar{c} - \bar{a}])),$

$(OA.30)$ $([1 - 1] \leq \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ([\bar{a} - -\bar{c}] < [\bar{b} - -\bar{c}]),$

$(OA.31)$ $([1 - 1] < \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ([\bar{c} - -\bar{b}] < [\bar{c} - -\bar{a}]),$

$(OA.32)$ $(1 < \bar{a}) \wedge ([1 - 1] < \bar{b}) \Rightarrow (1 < [\bar{a} - -f\bar{b}]),$

$(OA.33)$ $(1 < \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([1 - 1] < [\bar{a}/g\bar{b}]),$

$(OA.34)$ $(1 < \bar{a}) \wedge (\bar{a} < \bar{b}) \Rightarrow (1 < [\bar{b}/g\bar{a}]),$

$(OA.35)$ $(1 \leq \bar{a}) \wedge (\bar{a} < \bar{b}) \wedge ([1 - 1] < \bar{c}) \Rightarrow ([\bar{a}e\bar{c}] < [\bar{b}e\bar{c}]),$

$(OA.36)$ $(1 \leq \bar{a}) \wedge (1 \leq \bar{b}) \wedge ([1 - 1] < \bar{c}) \wedge ([\bar{a}e\bar{c}] < [\bar{b}e\bar{c}]) \Rightarrow (\bar{a} < \bar{b}),$

$(OA.37)$ $(1 < \bar{a}) \wedge ([1 - 1] \leq \bar{b}) \wedge (\bar{b} < \bar{c}) \Rightarrow ([\bar{a}e\bar{b}] < [\bar{a}e\bar{c}]),$

$(OA.38)$ $(1 < \bar{a}) \wedge ([1 - 1] \leq \bar{b}) \wedge ([1 - 1] \leq \bar{c}) \wedge ([\bar{a}e\bar{b}] < [\bar{a}e\bar{c}]) \Rightarrow (\bar{b} < \bar{c}),$

$(OA.39)$ $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} - \bar{b}] = [\bar{a}/\bar{b}]),$

$(OA.40)$ $\bar{a} \wedge \neg(\bar{b} = [1 - 1]) \Rightarrow ([\bar{a} - -\bar{b}] = [\bar{a}//\bar{b}]),$

$(OA.41)$ $\bar{a} \Rightarrow ([\bar{a} - \bar{a}] = [1 - 1]),$

$(OA.42)$ $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + \bar{b}] = [\bar{b} + \bar{a}]),$

$(OA.43)$ $\bar{a} \wedge \bar{b} \Rightarrow ([[\bar{a} - \bar{b}] + \bar{b}] = \bar{a}),$

$(OA.44)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([[\bar{a} - \bar{b}] + \bar{c}] = [[\bar{a} + \bar{c}] - \bar{b}]),$

$(OA.45)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} + \bar{c}]] = [[\bar{a} + \bar{b}] + \bar{c}]),$

$(OA.46)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + [\bar{b} - \bar{c}]] = [[\bar{a} + \bar{b}] - \bar{c}]),$

$(OA.47)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} + \bar{c}]] = [[\bar{a} - \bar{b}] - \bar{c}]),$

$(OA.48)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} - [\bar{b} - \bar{c}]] = [[\bar{a} - \bar{b}] + \bar{c}]),$

$(OA.49)$ $\bar{a} \Rightarrow ([\bar{a} + +1] = \bar{a}) \wedge ([\bar{a} - -1] = \bar{a}),$

$(OA.50)$ $\neg(\bar{a} = [1-1]) \Rightarrow ([\bar{a} - -\bar{a}] = 1),$

$(OA.51)$ $\bar{a} \wedge \bar{b} \Rightarrow ([\bar{a} + +\bar{b}] = [\bar{b} + +\bar{a}]),$

$(OA.52)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} + +\bar{c}]] = [[\bar{a} + +\bar{b}] + +\bar{c}]),$

$(OA.53)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} + \bar{c}]] = [[\bar{a} + +\bar{b}] + [\bar{a} + +\bar{c}]]),$

$(OA.54)$ $\bar{a} \wedge \bar{b} \wedge \bar{c} \Rightarrow ([\bar{a} + +[\bar{b} - \bar{c}]] = [[\bar{a} + +\bar{b}] - [\bar{a} + +\bar{c}]]),$

$(OA.55)$ $\bar{a} \wedge \neg(\bar{b} = [1-1]) \Rightarrow ([[\bar{a} - -\bar{b}] + +\bar{b}] = \bar{a}),$

$(OA.56)$ $\bar{a} \wedge \neg(\bar{b} = [1-1]) \wedge \bar{c} \Rightarrow (([[\bar{a} - -\bar{b}] + +\bar{c}] = [[\bar{a} + +\bar{c}] - -\bar{b}]) \wedge$
            $([[\bar{a} + \bar{c}] - -\bar{b}] = [[\bar{a} - -\bar{b}] + [\bar{c} - -\bar{b}]]) \wedge ([[\bar{a} - \bar{c}] - -\bar{b}] =$
            $[[\bar{a} - -\bar{b}] - [\bar{c} - -\bar{b}]])),$

$(OA.57)$ $\bar{a} \wedge \neg(\bar{b} = [1-1]) \wedge \neg(\bar{c} = [1-1]) \Rightarrow (([\bar{a} + +[\bar{b} - -\bar{c}]] = [[\bar{a} + +\bar{b}] - -\bar{c}]) \wedge$
            $([\bar{a} - -[\bar{b} + +\bar{c}]] = [[\bar{a} - -\bar{b}] - -\bar{c}]) \wedge ([\bar{a} - -[\bar{b} - -\bar{c}]] = [[\bar{a} - -\bar{b}] + +\bar{c}])),$

$(OA.58)$ $\bar{a} \Rightarrow ([\bar{a} + + + 1] = \bar{a}) \wedge ([\bar{a} - - - 1] = \bar{a}),$

$(OA.59)$ $\bar{a} \Rightarrow ([1 + + + \bar{a}] = 1),$

$(OA.60)$ $\neg(\bar{a} = [1-1]) \Rightarrow ([\bar{a} + + + [1-1]] = 1),$

$(OA.61)$ $([1-1] < \bar{a}) \Rightarrow ([[1-1] + + + \bar{a}] = [1-1]),$

$(OA.62)$ $([1-1] < \bar{a}) \wedge \neg(\bar{b} = [1-1]) \wedge \bar{c} \Rightarrow (([[\bar{a} - - - \bar{b}] + + + \bar{b}] = \bar{a}) \wedge$
            $([[\bar{a} - - - \bar{b}] + + + \bar{c}] = [[\bar{a} + + + \bar{c}] - - - \bar{b}]) \wedge ([\bar{a} + + + [\bar{c} - -\bar{b}]] =$
            $[[\bar{a} + + + \bar{c}] - - - \bar{b}])),$

$(OA.63)$ $([1-1] < \bar{a}) \wedge ([1-1] < \bar{b}) \wedge \bar{c} \Rightarrow (([\bar{a} + + + [\bar{b}///\bar{a}]] = \bar{b}) \wedge$
            $([[\bar{a} + + + \bar{c}]///\bar{b}] = [\bar{c} + +[\bar{a}///\bar{b}]]) \wedge ([[\bar{a} - -\bar{b}] + + + \bar{c}] =$
            $[[\bar{a} + + + \bar{c}] - -[\bar{b} + + + \bar{c}]])),$

$(OA.64)$ $([1-1] < \bar{a}) \wedge ([1-1] < \bar{b}) \wedge ([1-1] < \bar{c}) \Rightarrow (([[\bar{a}///\bar{c}] - -[\bar{b}///\bar{c}]] =$
            $[\bar{a}///\bar{b}]) \wedge ([[\bar{a} + +\bar{b}]///\bar{c}] = [[\bar{a}///\bar{c}] + [\bar{b}///\bar{c}]]) \wedge ([[\bar{a} - -\bar{b}]///\bar{c}] =$
            $[[\bar{a}///\bar{c}] - [\bar{b}///\bar{c}]])),$

$(OA.65)$ $\neg(\bar{a} = [1-1]) \wedge \neg(\bar{b} = [1-1]) \wedge \neg(\bar{c} = [1-1]) \Rightarrow (([[\bar{a} + +\bar{b}] + + + \bar{c}] =$
            $[[\bar{a} + + + \bar{c}] + +[\bar{b} + + + \bar{c}]]) \wedge ([\bar{a} + + + [\bar{b} + +\bar{c}]] = [[\bar{a} + + + \bar{b}] + + + \bar{c}])$
            $\wedge ([\bar{a} + + + [\bar{b} + \bar{c}]] = [[\bar{a} + + + \bar{b}] + +[\bar{a} + + + \bar{c}]]) \wedge ([\bar{a} + + + [\bar{b} - \bar{c}]] =$
            $[[\bar{a} + + + \bar{b}] - -[\bar{a} + + + \bar{c}]])),$

$(OA.66)$ $([1-1] < \bar{a}) \wedge \neg(\bar{b} = [1-1]) \wedge \neg(\bar{c} = [1-1]) \Rightarrow (([\bar{a} - - - [\bar{b} + +\bar{c}]] =$
            $[[\bar{a} - - - \bar{b}] - - - \bar{c}]) \wedge ([\bar{a} - - - [\bar{b} - -\bar{c}]] = [[\bar{a} - - - \bar{b}] + + + \bar{c}])),$

$(OA.67)$ $([1-1] < \bar{a}) \wedge ([1-1] < \bar{b}) \wedge \neg(\bar{c} = [1-1]) \Rightarrow (([[\bar{a} + +\bar{b}] - - - \bar{c}] =$
            $[[\bar{a} - - - \bar{c}] + +[\bar{b} - - - \bar{c}]]) \wedge ([[\bar{a} - -\bar{b}] - - - \bar{c}] =$
            $[[\bar{a} - - - \bar{c}] - -[\bar{b} - - - \bar{c}]])),$

$(OA.68)$ $(1 \leq \bar{a}) \Rightarrow ([\bar{a} + e1] = \bar{a}),$

$(OA.69)$ $(1 \leq \bar{a}) \Rightarrow ([\bar{a} + +e[1-1]] = 1),$

$(OA.70)$ $(1 \leq \bar{a}) \Rightarrow ([\bar{a} - f1] = \bar{a}),$

$(OA.71)$ $([1-1] \leq \bar{a}) \Rightarrow ([1 + +e\bar{a}] = 1),$

$(OA.72)$ $([1-1] < \bar{a}) \Rightarrow ([1 - -f\bar{a}] = 1),$

$(OA.73)$ $(1 < \bar{a}) \Rightarrow ([1//g\bar{a}] = [1-1]),$

$(OA.74)$ $(1 < \bar{a}) \Rightarrow ([\bar{a}/g\bar{a}] = 1),$

$(OA.75)$ $(1 \leq \bar{a}) \wedge ([1-1] < \bar{b}) \Rightarrow ([[\bar{a}i\bar{b}]h\bar{b}] = \bar{a}),$

$(OA.76)$ $(1 \leq \bar{a}) \wedge ([1-1] < \bar{b}) \Rightarrow ([[\bar{a}h\bar{b}]i\bar{b}] = \bar{a}),$

$(OA.77)$ $(1 \leq \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([\bar{b}h[\bar{a}j\bar{b}]] = \bar{a}),$

$(OA.78)$ $(1 \leq \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([[\bar{b}h\bar{a}]j\bar{b}] = \bar{a}),$

$(OA.79)$ $(1 \leq \bar{a}) \wedge (1 < \bar{b}) \Rightarrow ([\bar{a} + + + e\bar{b}] = [\bar{a} + +e[\bar{a} + + + e[\bar{b} - 1]]]),$

$(OA.80)$ $(1 \leq \bar{a}) \wedge ([1-1] \leq \bar{b}) \wedge (\bar{b} \leq 1) \Rightarrow ([\bar{a} + + + e\bar{b}] = [\bar{a} + + + \bar{b}])$

$\}$,

$\Psi\{$

$(OA.81)$ $(\bar{a} \subseteq \bar{b}) \Leftrightarrow (\bar{b} = \bar{c}\bar{a}\bar{d}),$

$(OA.82)$ $(\bar{a} \to \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \to \bar{e}) \Rightarrow (\bar{a} \to \bar{b}\bar{e}\bar{d}),$

$(OA.83)$ $(\bar{\bar{a}} \to \bar{\bar{b}}|\bar{c}) \Rightarrow ((\bar{\bar{a}} \to \bar{\bar{b}}) \wedge (\bar{\bar{a}} \to \bar{\bar{c}})),$

$(OA.84)$ $(\bar{\bar{a}}|\bar{\bar{b}} \to \bar{\bar{c}}) \Rightarrow ((\bar{\bar{a}} \to \bar{\bar{c}}) \wedge (\bar{\bar{b}} \to \bar{\bar{c}})),$

$(OA.85)$ $(\bar{a} < \bar{b}) \Rightarrow \neg(\bar{b} < \bar{a}),$

$(OA.86)$ $(\bar{a} < \bar{b}) \Rightarrow \neg(\bar{a} = \bar{b}),$

$(OA.87)$ $(\bar{a} < \bar{b}) \wedge (\bar{b} < \bar{c}) \Rightarrow (\bar{a} < \bar{c}),$

$(OA.88)$ $(\bar{a} < \bar{b}) \wedge (\bar{a} \in (C \circ C)) \wedge (\bar{b} \in (C \circ C)) \Rightarrow (\bar{a} \wedge \bar{b}),$

$(OA.89)$ $(\bar{a} < \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} = \bar{e}) \Rightarrow (\bar{a} < \bar{b}\bar{e}\bar{d}),$

$(OA.90)$ $(\bar{a}\bar{b}\bar{c} < \bar{d}) \wedge (\bar{b} = \bar{e}) \Rightarrow (\bar{a}\bar{e}\bar{c} < \bar{d}),$

$(OA.91)$ $(\bar{a} < \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \to \bar{e}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}\}) \Rightarrow (\bar{a} < \bar{b}\bar{e}\bar{d}),$

$(OA.92)$ $(\bar{a} < \bar{b}\bar{c}\bar{d}\bar{c}\bar{e}) \wedge (\bar{c} \to \bar{f}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a} < \bar{b}\bar{f}\bar{d}\bar{f}\bar{e}),$

$(OA.93)$ $(\bar{a}\bar{b}\bar{c} < \bar{d}) \wedge (\bar{b} \to \bar{e}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}\}) \Rightarrow (\bar{a}\bar{e}\bar{c} < \bar{d}),$

$(OA.94)$ $(\bar{a}\bar{b}\bar{c} < \bar{d}\bar{b}\bar{e}) \wedge (\bar{b} \to \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c} < \bar{d}\bar{f}\bar{e}),$

$(OA.95)$ $(\bar{a}\bar{b}\bar{c} < \bar{d}\bar{b}\bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \to \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c} < \bar{d}\bar{g}\bar{e}\bar{g}\bar{f}),$

$(OA.96)$ $(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}) \wedge (\bar{b} \to \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c}\bar{f}\bar{d} < \bar{e}),$

$(OA.97)$ $(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \to \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c}\bar{g}\bar{d} < \bar{e}\bar{g}\bar{f}),$

$(OA.98)$ $(\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} < \bar{e}\bar{b}\bar{f}\bar{b}\bar{g}) \wedge (\bar{b} \to \bar{h}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Rightarrow (\bar{a}\bar{h}\bar{c}\bar{h}\bar{d} < \bar{e}\bar{h}\bar{f}\bar{h}\bar{g}),$

$(OA.99)$ $\bar{a} = \bar{a},$

$(OA.100)$ $(\bar{a} = \bar{b}) \Rightarrow (\bar{b} = \bar{a}),$

$(OA.101)$ $(\bar{a} = \bar{b}) \Rightarrow \neg(\bar{a} < \bar{b}),$

$(OA.102)$ $(\bar{a} = \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} = \bar{e}) \Rightarrow (\bar{a} = \bar{b}\bar{e}\bar{d}),$

$(OA.103)$ $(\bar{a}\bar{b}\bar{c}) \wedge (\bar{b} = \bar{d}) \Rightarrow (\bar{a}\bar{b}\bar{c} = \bar{a}\bar{d}\bar{c}),$

$(OA.104)$ $(\bar{a} = \bar{b}\bar{c}\bar{d}) \wedge (\bar{c} \to \bar{e}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}\}) \Rightarrow (\bar{a} = \bar{b}\bar{e}\bar{d}),$

$(OA.105) \quad (\bar{a} = \bar{b}\bar{c}\bar{d}\bar{c}\bar{e}) \wedge (\bar{c} \rightarrow \bar{f}) \wedge \neg(\bar{c} \subseteq \{\bar{a}, \bar{b}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a} = \bar{b}\bar{f}\bar{d}\bar{f}\bar{e}),$

$(OA.106) \quad (\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}) \wedge (\bar{b} \rightarrow \bar{f}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}\}) \Rightarrow (\bar{a}\bar{f}\bar{c} = \bar{d}\bar{f}\bar{e}),$

$(OA.107) \quad (\bar{a}\bar{b}\bar{c} = \bar{d}\bar{b}\bar{e}\bar{b}\bar{f}) \wedge (\bar{b} \rightarrow \bar{g}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}\}) \Rightarrow (\bar{a}\bar{g}\bar{c} = \bar{d}\bar{g}\bar{e}\bar{g}\bar{f}),$

$(OA.108) \quad (\bar{a}\bar{b}\bar{c}\bar{b}\bar{d} = \bar{e}\bar{b}\bar{f}\bar{b}\bar{g}) \wedge (\bar{b} \rightarrow \bar{h}) \wedge \neg(\bar{b} \subseteq \{\bar{a}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}\}) \Rightarrow (\bar{a}\bar{h}\bar{c}\bar{h}\bar{d} = \bar{e}\bar{h}\bar{f}\bar{h}\bar{g}),$

$(OA.109) \quad (\bar{a} \leq \bar{b}) \Leftrightarrow ((\bar{a} < \bar{b}) \vee (\bar{a} = \bar{b})),$

$(OA.110) \quad (\bar{a} \leq \bar{b}) \wedge (\bar{b} \leq \bar{c}) \Rightarrow (\bar{a} \leq \bar{c})$

$ \}.$

In the following, we will deduce some numbers and equalities as examples.

$(A1)$	$(a \rightarrow 1\|[aba]) \Rightarrow (a \rightarrow 1)$	$by(OA.16), (OA.83)$
$(A2)$	$\Rightarrow (a \rightarrow [aba])$	$by(OA.16), (OA.83)$
$(A3)$	$\Rightarrow (a \rightarrow [1ba])$	$by(A2), (A1), (OA.82)$
$(A4)$	$\Rightarrow (a \rightarrow [1b1])$	$by(A3), (A1), (OA.82)$
$(A5)$	$(b \rightarrow +\|-) \Rightarrow (b \rightarrow -)$	$by(OA.17), (OA.83)$
$(A6)$	$\Rightarrow (a \rightarrow [1 - 1])$	$by(A4), (A5), (OA.82)$
$(A7)$	$(a < [1 + a]) \Rightarrow (1 < [1 + 1])$	$by(OA.28), (A1), (OA.94)$
$(A8)$	$\Rightarrow 1$	$by(OA.88)$
$(A9)$	$\Rightarrow [1 + 1]$	$by(A7), (OA.88)$
$(A10)$	$(a < [1 + a]) \Rightarrow ([1 - 1] < [1 + [1 - 1]])$	$by(OA.28), (A6), (OA.94)$
$(A11)$	$\Rightarrow ([1 - 1] < [[1 - 1] + 1])$	$by(A10), (OA.42), (OA.89)$
$(A12)$	$\Rightarrow ([1 - 1] < 1)$	$by(A11), (OA.43), (OA.89)$
$(A13)$	$\Rightarrow [1 - 1]$	$by(OA.88)$
$(A14)$	$(b \rightarrow +\|-) \Rightarrow (b \rightarrow +)$	$by(OA.17), (OA.83)$
$(A15)$	$\Rightarrow (a \rightarrow [1 + 1])$	$by(A4), (A14), (OA.82)$
$(A16)$	$(a < [1 + a]) \Rightarrow ([1 + 1] < [1 + [1 + 1]])$	$by(OA.28), (A15), (OA.94)$
$(A17)$	$\Rightarrow ([1 - 1] < [1 + 1])$	$by(A12), (A7), (OA.87)$
$(A18)$	$\Rightarrow \neg([1 + 1] = [1 - 1])$	$by(A17), (OA.86)$
$(A19)$	$\Rightarrow ([[1 - 1] - -[1 + 1]] < [1 - -[1 + 1]])$	$by(A12), (A17), (OA.30)$
$(A20)$	$\Rightarrow [1 - -[1 + 1]]$	$by(OA.88)$
$(A21)$	$\Rightarrow ([[1 - 1] - -[1 + 1]] = [[1 - -[1 + 1]] - [1 - -[1 + 1]]])$	$by(A6), (A18), (OA.56)$
$(A22)$	$\Rightarrow ([[1 - -[1 + 1]] - [1 - -[1 + 1]]] = [1 - 1])$	$by(A20), (OA.41)$
$(A23)$	$\Rightarrow ([[1 - 1] - -[1 + 1]] = [1 - 1])$	$by(A21), (A22), (OA.102)$
$(A24)$	$\Rightarrow ([1 - 1] < [1 - -[1 + 1]])$	$by(A19), (A23), (OA.90)$
$(A25)$	$\Rightarrow ([1 - -[1 + 1]] < [[1 + 1] - -[1 + 1]])$	$by(A12), (A7),$ $(A17), (OA.30)$
$(A26)$	$\Rightarrow ([[1 + 1] - -[1 + 1]] = 1)$	$by(A18), (OA.50)$
$(A27)$	$\Rightarrow ([1 - -[1 + 1]] < 1)$	$by(A25), (A26), (OA.89)$

$(A28)$ $\qquad$ $([[1 - 1] + 1] < [[1 - -[1 + 1]] + 1])$ $\qquad$ $by\,(A24), (A8),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (OA.29)$

$(A29)$ $\qquad$ $(1 < [[1 - -[1 + 1]] + 1])$ $\qquad$ $by\,(A28), (OA.43),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (OA.90)$

$$\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots$$

Then according to Definition 2.1, we can deduce from $R\{\Phi, \Psi\}$ the numbers as follows:

$$\{1, [1 + 1], [1 - 1], [1 + [1 + 1]], [1 - [1 + 1]], [1 - -[1 + [1 + 1]]],$$
$$[[1 + [1 + 1]] - - - -[1 + 1]], [[1 + [1 + 1]] - -[1 + 1]] \cdots \}$$

$(B1)$ $\quad$ $([[1 + 1] + + + e[1 - -[1 + 1]]] = [[1 + 1] + + + [1 - -[1 + 1]]])$ $\quad$ $by\,(A7), (A24),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A27), (OA.80)$

$(B2)$ $\quad$ $\Rightarrow ([[1 + 1] + + + [1 - -[1 + 1]]] = [[1 + 1] + + + e[1 - -[1 + 1]]])$ $\quad$ $by\,(OA.100)$

$(B3)$ $\qquad\qquad\qquad$ $(e \rightarrow +| + e) \Rightarrow (e \rightarrow +)$ $\qquad\qquad\qquad$ $(OA.19), (OA.83)$

$(B4)$ $\quad$ $\Rightarrow ([[1 + 1] + + + [1 - -[1 + 1]]] = [[1 + 1] + + + +[1 - -[1 + 1]]])$ $\quad$ $by\,(B2), (B3),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (OA.104)$

$(B5)$ $\quad$ $\Rightarrow ([[1 + 1] + + + +[1 - -[1 + 1]]] = [[1 + 1] + + + [1 - -[1 + 1]]])$ $\quad$ $by\,(OA.100)$

$(B6)$ $\qquad\qquad$ $([[1 - 1] + [1 + 1]] < [[1 - -[1 + 1]] + [1 + 1]])$ $\qquad$ $by\,(A24), (A9),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (OA.29)$

$(B7)$ $\qquad\qquad\qquad$ $([[1 - 1] + [1 + 1]] = [[1 + 1] + [1 - 1]])$ $\qquad$ $by\,(A13), (A9),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (OA.42)$

$(B8)$ $\qquad\qquad\qquad$ $([[1 + 1] + [1 - 1]] = [[[1 + 1] + 1] - 1]])$ $\qquad$ $by\,(OA.46)$

$(B9)$ $\qquad\qquad$ $\Rightarrow ([[1 + 1] + [1 - 1]] = [[[1 + 1] - 1] + 1]])$ $\qquad$ $by\,(OA.44), (OA.100)$

$(B10)$ $\qquad\qquad\qquad$ $\Rightarrow ([[1 + 1] + [1 - 1]] = [1 + 1])$ $\qquad$ $by\,(OA.43)$

$(B11)$ $\qquad\qquad\qquad$ $\Rightarrow ([[1 - 1] + [1 + 1]] = [1 + 1])$ $\qquad$ $by\,(B7), (B10),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (OA.102)$

$(B12)$ $\qquad\qquad$ $\Rightarrow ([1 + 1] < [[1 - -[1 + 1]] + [1 + 1]])$ $\qquad$ $by\,(B6), (B11),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (OA.90)$

$(B13)$ $\qquad\qquad$ $\Rightarrow (1 < [[1 - -[1 + 1]] + [1 + 1]])$ $\qquad$ $by\,(A7), (B12),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (OA.87)$

$(B14)$ $\qquad\qquad$ $([[1 + 1] + + + e[[1 - -[1 + 1]] + [1 + 1]]] =$
$\quad [[1 + 1] + +e[[1 + 1] + + + e[[[1 - -[1 + 1]] + [1 + 1]] - 1]]])$ $\quad$ $by\,(A7), (B13),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (OA.79)$

$(B15)$ $\qquad\qquad$ $\Rightarrow ([[1 + 1] + + + +[[1 - -[1 + 1]] + [1 + 1]]] =$
$\quad [[1 + 1] + + + [[1 + 1] + + + +[[[1 - -[1 + 1]] + [1 + 1]] - 1]]])$ $\quad$ $by\,(B14), (B3),$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (OA.107)$

$(B16)$ $\qquad\qquad\qquad$ $([[1 + 1] - 1] = [[1 - 1] + 1])$ $\qquad$ $by\,(A8), (OA.44)$

$(B17)$ $\qquad\qquad\qquad$ $\Rightarrow ([[1 + 1] - 1] = 1)$ $\qquad$ $by\,(A8), (OA.43)$

$(B18)$ $\quad([[[1 - -[1+1]] + [1+1]] - 1] = [[1 - -[1+1]] + [[1+1] - 1]])$ $\quad by(A20), (A9),$
$(A8), (OA.46)$

$(B19)$ $\quad \Rightarrow ([[[1 - -[1+1]] + [1+1]] - 1] = [[1 - -[1+1]] + 1])$ $\quad by(A18), (A17),$
$(OA.102)$

$(B20)$ $\quad \Rightarrow ([[1+1] + + + +[[1 - -[1+1]] + [1+1]]] =$
$[[1+1] + + + [[1+1] + + + +[[1 - -[1+1]] + 1]]])$ $\quad by(B15), (B19),$
$(OA.102)$

$(B21)$ $\quad ([[1+1] + + + e[[1 - -[1+1]] + 1]] =$
$[[1+1] + +e[[1+1] + + + e[[[1 - -[1+1]] + 1] - 1]]])$ $\quad by(A7), (A29),$
$(OA.79)$

$(B22)$ $\quad \Rightarrow ([[1+1] + + + +[[1 - -[1+1]] + 1]] =$
$[[1+1] + + + [[1+1] + + + +[[[1 - -[1+1]] + 1] - 1]]])$ $\quad by(B21), (B3),$
$(OA.107)$

$(B23)$ $\quad ([[[1 - -[1+1]] + 1] - 1] = [[[1 - -[1+1]] - 1] + 1])$ $\quad by(A20), (A8),$
$(OA.44)$

$(B24)$ $\quad \Rightarrow ([[[1 - -[1+1]] + 1] - 1] = [1 - -[1+1]])$ $\quad by(B23), (OA.43),$
$(OA.102)$

$(B25)$ $\quad \Rightarrow ([[1+1] + + + +[[1 - -[1+1]] + 1]] =$
$[[1+1] + + + [[1+1] + + + +[1 - -[1+1]]]])$ $\quad by(B22), (B24),$
$(OA.102)$

$(B26)$ $\quad \Rightarrow ([[1+1] + + + +[[1 - -[1+1]] + 1]] =$
$[[1+1] + + + [[1+1] + + + [1 - -[1+1]]]])$ $\quad by((B25), (B5),$
$(OA.102)$

$(B27)$ $\quad \Rightarrow ([[1+1] + + + +[[1 - -[1+1]] + [1+1]]] =$
$[[1+1] + + + [[1+1] + + + [[1+1] + + + [1 - -[1+1]]]]])$ $\quad by(B20), (B26),$
$(OA.102)$

$\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$

Then we can deduce from $R\{\Phi, \Psi\}$ the equalities as follows:

$$[[1+1] + +[[1+1] - - - [1+1]]] = [[[1+1] - - - [1+1]] + +[1+1]],$$
$$[[1+1] + + + +[[1 - -[1+1]] + 1]] = [[1+1] + + + [[1+1] + + + [1 - -[1+1]]]],$$
$$\vdots \quad \vdots \quad \vdots.$$

The deduced numbers correspond to real numbers as follows:

$$\vdots \quad \vdots \quad \vdots,$$
$$[[[1 + [1+1]] - - - -[1+1]] + +[1+1]] \equiv$$
$$\vdots \quad \vdots \quad \vdots,$$

$$[1 + [1 + 1]] \equiv 3,$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[[1 + 1] + [1 - -[1 + 1]]] \equiv \frac{5}{2},$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[1 + 1] \equiv 2,$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[[1 + [1 + 1]] - - - -[1 + 1]] \equiv$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[[1 + [1 + 1]]///[1 + 1]] \equiv \log_2 3,$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[1 + [1 - -[1 + 1]]] \equiv \frac{3}{2},$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[[1 + 1] - - -[1 + 1]] \equiv \sqrt[2]{2},$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[[1 + 1] - - -[1 + [1 + 1]]] \equiv \sqrt[3]{2},$$

$$\vdots \quad \vdots \quad \vdots,$$

$$1 \equiv 1,$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[[1 + 1]///[1 + [1 + 1]]] \equiv \log_3 2,$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[1 - -[1 + 1]] \equiv \frac{1}{2},$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[1 - 1] \equiv 0,$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[[1 - 1] - [1 - -[1 + 1]]] \equiv -\frac{1}{2},$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[[1 - 1] - 1] \equiv -1,$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[[1 - 1] - [[1 + 1] - - -[1 + [1 + 1]]]] \equiv -\sqrt[3]{2},$$

$$\vdots \quad \vdots \quad \vdots,$$

$$[[1-1]-[[1+1]---[1+1]]]\ \equiv\ -\sqrt[2]{2},$$

$$\vdots\quad\vdots\quad\vdots,$$

$$[[1-1]-[[1+[1+1]]----[1+1]]]\ \equiv$$

$$\vdots\quad\vdots\quad\vdots.$$

The equalities on deduced numbers correspond to addition, subtraction, multiplication, division, exponentiation operation, root-extraction operation, logarithm operation and more other operations in real number system. These deduced numbers hold the consist order relation and equivalence relation, so the logical calculus $R\{\Phi,\Psi\}$ is a consist axiom.

Since $Q\{\Phi,\Psi\}$ is dense in $R\{\Phi,\Psi\}$(with its standard topology), every deduced number has rational numbers arbitrary close to it. So every deduced number holds only one position in number line.

Note that in the correspondence above, some deduced numbers such as $[[[1+[1+1]]----[1+1]]++[1+1]]$, $[[1+[1+1]]----[1+1]]$ and $[[1-1]-[[1+[1+1]]----[1+1]]]$ do not correspond to any known real numbers. So the logical calculus $R\{\Phi,\Psi\}$ can deduce more real numbers than before.

In fact, the logical calculus $R\{\Phi,\Psi\}$ not only deduces more real numbers than before, but also makes its deduced numbers join in algebraical operations. So the logical calculus $R\{\Phi,\Psi\}$ intuitively and logically denote real number system.

2.2. **Semantics Of The Operator Axioms.** The syntax of the Operator axioms may map to many semantics, but only one semantics is shown here to explain the syntax of the Operator axioms. As shown in Figure 1, one semantics of the Operator axioms is a line. We will explain the semantics according to Figure 1 as follows.

The symbol 'O' in Figure 1 stand for "the origin". The number '1' maps to "a point move up unit length from the origin". The symbol '>' maps to "upon". The symbol '<' maps to "under". The symbol '=' maps to "overlap". The numbers such as "$[1+1]$", "$[[1+[1+1]]----[1+1]]$", "$[[1+1]---[1+[1+1]]$", "$[1-1]$" map to the points arranged in the line in Figure 1. The number "$[1-1]$" overlap "the origin O".

Supposing that '$\check{a}$', '$\check{b}$', "$[\check{a}+\check{b}]$" are numbers, then "$[\check{a}+\check{b}]$" maps to "a point move up $\check{b}$ units from the point $\check{a}$". The symbol "$++$" maps to "iteratively $+$". The symbol "$+++$" maps to "iteratively $++$". And so on, the symbols "$++++$", "$+++++$", "$++++++$", $\cdots$ can map to similar semantics.

Supposing that '$\check{a}$', '$\check{b}$', "$[\check{a}-\check{b}]$" are numbers, then "$[\check{a}-\check{b}]$" maps to "a point can move to the point $\check{a}$ by the operation $[[\check{a}-\check{b}]+\check{b}]$". Supposing that '$\check{a}$', '$\check{b}$', "$[\check{a}--\check{b}]$" are numbers, then "$[\check{a}--\check{b}]$" maps to "a point can move to the point $\check{a}$ by the operation $[[\check{a}--\check{b}]++\check{b}]$". Supposing that '$\check{a}$', '$\check{b}$', "$[\check{a}---\check{b}]$" are numbers, then "$[\check{a}---\check{b}]$" maps to "a point can move to the point $\check{a}$ by the operation $[[\check{a}---\check{b}]+++\check{b}]$". And so on, the symbols "$----$", "$-----$", "$------$", $\cdots$ can map to similar semantics.

Supposing that '$\check{a}$', '$\check{b}$', "$[\check{a}/\check{b}]$" are numbers, then "$[\check{a}/\check{b}]$" maps to "a point can move to the point $\check{a}$ by the operation $[\check{b}+[\check{a}/\check{b}]]$". Supposing that '$\check{a}$', '$\check{b}$', "$[\check{a}//\check{b}]$" are numbers, then "$[\check{a}//\check{b}]$" maps to "a point can move to the point $\check{a}$ by the operation $[\check{b}++[\check{a}//\check{b}]]$". Supposing that '$\check{a}$', '$\check{b}$', "$[\check{a}///\check{b}]$" are numbers, then "$[\check{a}///\check{b}]$" maps to "a point can move

[1+[1+1]]

[1+1]

[[1+[1+1]]−−−−[1+1]]

[[1+1]−−−[1+[1+1]]]

1

[[1+1]///[1+[1+1]]]
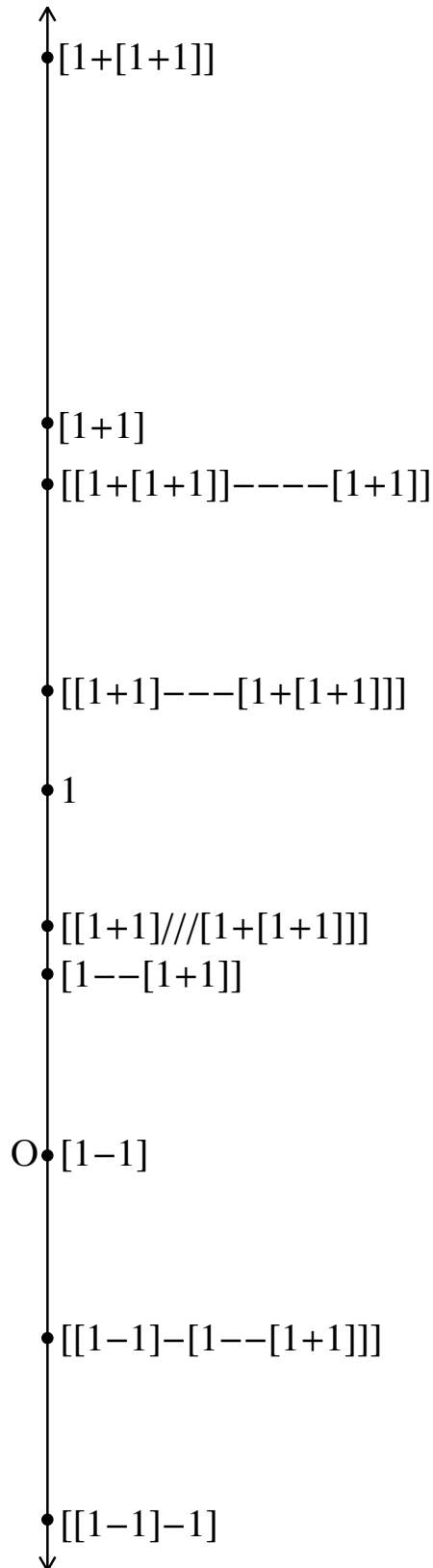
[1−−[1+1]]

O [1−1]

[[1−1]−[1−−[1+1]]]

[[1−1]−1]

FIGURE 1. The semantics of the Operator axioms

to the point $\check{a}$ by the operation $[\check{b}+++[\check{a}///\check{b}]]$". And so on, the symbols "////", "/////", "//////", $\cdots$ can map to similar semantics.

3. Fundamental Operator Functions

On the basis of the Operator axioms, we define two fundamental operator functions in this section to study the analytic properties of the Operator axioms. It is supposed that the constant $\check{d} \in R$ with $[1-1] < \check{d}$. It is supposed that the constant $\check{e} \in R$ with $1 < \check{e}$.

Definition 3.1. *VE Function* is the function $\check{f} : [1, +\infty) \to R$ defined by $\check{f}(\check{x}) = [\check{x} + + + e\check{d}]$.

Definition 3.2. *EV Function* is the function $\check{f} : [[1-1], +\infty) \to R$ defined by $\check{f}(\check{x}) = [\check{e} + + + e\check{x}]$.

Definition 3.3. *Fundamental operator functions* are VE Function and EV Function.

Theorem 3.4. *The VE Function $\check{f}(\check{x}) = [\check{x} + + + e\check{d}]$ is continuous, unbounded and strictly increasing.*

Proof. According to (OA.19), the symbol 'e' represents some successive '+'—"+ $\cdots$ +". According to (OA.20), the symbol 'f' represents some successive '−'—"− $\cdots$ −". According to (OA.21), the symbol 'g' represents some successive '/'—"/ $\cdots$ /".

$(A1)$	Supposing that $\check{x}_1, \check{x}_2 \in [1, +\infty)$ with $\check{x}_1 < \check{x}_2$.	
$(A2)$	$\Rightarrow (1 \leq \check{x}_1) \wedge (\check{x}_1 < \check{x}_2)$	
$(A3)$	$[1-1] < \check{d}$	by (Premise)
$(A4)$	$\Rightarrow [\check{x}_1 + + + e\check{d}] < [\check{x}_2 + + + e\check{d}]$	by (A2),(A3),(OA.35)
$(A5)$	$\check{f}(\check{x}_1) = [\check{x}_1 + + + e\check{d}]$	by (Premise)
$(A6)$	$\Rightarrow \check{f}(\check{x}_1) < [\check{x}_2 + + + e\check{d}]$	by (A4),(A5),(OA.90)
$(A7)$	$\check{f}(\check{x}_2) = [\check{x}_2 + + + e\check{d}]$	by (Premise)
$(A8)$	$\Rightarrow \check{f}(\check{x}_1) < \check{f}(\check{x}_2)$	by (A6),(A7),(OA.89)

(A1)$\sim$(A8) derive that $\check{f}(\check{x})$ is strictly increasing.

For any number $\check{x}_0 \in [1, +\infty)$ and any number $[1-1] < \varepsilon$, we can always construct δ_0 as follows:

$(B1)$	$[1 + + + e\check{d}] \leq [\check{x}_0 + + + e\check{d}]$	by (OA.35),(OA.99)
$(B2)$	$\Rightarrow 1 \leq [\check{x}_0 + + + e\check{d}]$	by (OA.71),(OA.90)
$(B3)$	$\Rightarrow [1-1] \leq [[\check{x}_0 + + + e\check{d}] - 1]$	by (OA.29)
$(B4)$	$[1-1] < \varepsilon$	by (Premise)
$(B5)$	$\Rightarrow [[1-1] - -[1+1]] < [\varepsilon - -[1+1]]$	by (OA.30)
$(B6)$	$\Rightarrow [[1 - -[1+1]] - [1 - -[1+1]]] < [\varepsilon - -[1+1]]$	by (OA.56), (OA.90)
$(B7)$	$\Rightarrow [1-1] < [\varepsilon - -[1+1]]$	by (OA.41), (OA.90)
$(B8)$	$\Rightarrow [[1-1] + [\varepsilon - -[1+1]]] < [[\varepsilon - -[1+1]] + [\varepsilon - -[1+1]]]$	by (OA.29)
$(B9)$	$\Rightarrow [[1-1] + [\varepsilon - -[1+1]]] < [[\varepsilon + \varepsilon] - -[1+1]]$	by (OA.56), (OA.89)

$(B10)$ $\Rightarrow [[\varepsilon - -[1+1]] + [1-1]] < [[\varepsilon + \varepsilon] - -[1+1]]$ by (OA.42), (OA.90)

$(B11)$ $\Rightarrow [[[\varepsilon - -[1+1]] + 1] - 1] < [[\varepsilon + \varepsilon] - -[1+1]]$ by (OA.46), (OA.90)

$(B12)$ $\Rightarrow [[[\varepsilon - -[1+1]] - 1] + 1] < [[\varepsilon + \varepsilon] - -[1+1]]$ by (OA.44), (OA.90)

$(B13)$ $\Rightarrow [\varepsilon - -[1+1]] < [[\varepsilon + \varepsilon] - -[1+1]]$ by (OA.43), (OA.90)

$(B14)$ $\Rightarrow [\varepsilon - -[1+1]] < [[\varepsilon + +[1+1]] - -[1+1]]$ by (OA.53),(OA.49),
(OA.89)

$(B15)$ $\Rightarrow [\varepsilon - -[1+1]] < [\varepsilon + +[[1+1] - -[1+1]]]$ by (OA.57),(OA.89)

$(B16)$ $\Rightarrow [\varepsilon - -[1+1]] < [\varepsilon + +1]$ by (OA.50),(OA.89)

$(B17)$ $\Rightarrow [\varepsilon - -[1+1]] < \varepsilon$ by (OA.49),(OA.89)

$(B18)$ $\delta_0 = [\varepsilon - -[1+1]]$

We construct δ according to $[[\check{x}_0 + + + e\check{d}] - \delta_0]$.

(1) $1 \leq [[\check{x}_0 + + + e\check{d}] - \delta_0]$.

We construct δ as follows:

$(C1)$ $\delta_1 = [\check{x}_0 - [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]]$

$(C2)$ $\delta_2 = [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]$

$(C3)$ $\delta = \min\{\delta_1, \delta_2\}$

$(D1)$ $\delta_0 = [\varepsilon - -[1+1]]$ by (B18)

$(D2)$ $\Rightarrow [1-1] < \delta_0$ by (B7),(D1),
(OA.89)

$(D3)$ $[\varepsilon - -[1+1]] < \varepsilon$ by (B17)

$(D4)$ $\Rightarrow \delta_0 < \varepsilon$ by (D3),(D1),
(OA.90)

$(D5)$ $\delta = \min\{\delta_1, \delta_2\}$ by (C3)

$(D6)$ $\Rightarrow \delta \leq \delta_1$

$(D7)$ $\Rightarrow \delta \leq \delta_2$

$(D8)$ $\delta_1 = [\check{x}_0 - [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]]$ by (C1)

$(D9)$ $\Rightarrow [\check{x}_0 - \delta_1] = [\check{x}_0 - [\check{x}_0 - [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]]]$ by (OA.103)

$(D10)$ $\Rightarrow [\check{x}_0 - \delta_1] = [[\check{x}_0 - \check{x}_0] + [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]]$ by (OA.48),(OA.102)

$(D11)$ $\Rightarrow [\check{x}_0 - \delta_1] = [[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] + [\check{x}_0 - \check{x}_0]]$ by (OA.42),(OA.102)

$(D12)$ $\Rightarrow [\check{x}_0 - \delta_1] = [[[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] + \check{x}_0] - \check{x}_0]$ by (OA.46),(OA.102)

$(D13)$ $\Rightarrow [\check{x}_0 - \delta_1] = [[[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] - \check{x}_0] + \check{x}_0]$ by (OA.44),(OA.102)

$(D14)$ $\Rightarrow [\check{x}_0 - \delta_1] = [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]$ by (OA.43),(OA.102)

$(D15)$ $1 \leq [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]$ by (Premise),(OA.32),
(OA.72)

$(D16)$ $\Rightarrow 1 \leq [\check{x}_0 - \delta_1]$ by (D14),(D15),
(OA.89)

$(D17)$	$\Rightarrow [\check{x}_0 - \delta_1] \leq [\check{x}_0 - \delta]$	by (D6),(OA.29)
$(D18)$	$\Rightarrow [[\check{x}_0 - \delta_1] + + + h\check{d}] \leq [[\check{x}_0 - \delta] + + + h\check{d}]$	by (D17),(OA.35), (OA.103)
$(D19)$	$\Rightarrow [[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] + + + h\check{d}] \leq$ $[[\check{x}_0 - \delta] + + + h\check{d}]$	by (D14),(OA.90)
$(D20)$	$\Rightarrow [[\check{x}_0 + + + h\check{d}] - \delta_0] \leq [[\check{x}_0 - \delta] + + + h\check{d}]$	by (OA.75),(OA.90)
$(D21)$	$\Rightarrow [[\check{x}_0 + + + h\check{d}] - \varepsilon] < [[\check{x}_0 + + + h\check{d}] - \delta_0]$	by (D4),(OA.29)
$(D22)$	$\Rightarrow [[\check{x}_0 + + + e\check{d}] - \varepsilon] < [[\check{x}_0 - \delta] + + + e\check{d}]$	by (D21),(D20), (OA.87)
$(D23)$	$\Rightarrow [\check{f}(\check{x}_0) - \varepsilon] < \check{f}([\check{x}_0 - \delta])$	by (Premise)
$(D24)$	$\delta_2 = [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]$	by (C2)
$(D25)$	$\Rightarrow [\check{x}_0 + \delta_2] = [\check{x}_0 + [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]]$	by (OA.103)
$(D26)$	$\Rightarrow [\check{x}_0 + \delta_2] = [[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0] + \check{x}_0]$	by (OA.42),(OA.102)
$(D27)$	$\Rightarrow [\check{x}_0 + \delta_2] = [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}]$	by (OA.43),(OA.102)
$(D28)$	$[1 - 1] < \delta_0$	by (D2)
$(D29)$	$\Rightarrow [[1 - 1] + 1] < [\delta_0 + 1]$	by (D28),(OA.29)
$(D30)$	$\Rightarrow 1 < [\delta_0 + 1]$	by (OA.43),(OA.90)
$(D31)$	$\Rightarrow 1 < [1 + \delta_0]$	by (OA.42),(OA.89)
$(D32)$	$1 \leq [\check{x}_0 + + + h\check{d}]$	by (B2)
$(D33)$	$\Rightarrow [1 + \delta_0] \leq [[\check{x}_0 + + + h\check{d}] + \delta_0]$	by (OA.29)
$(D34)$	$\Rightarrow 1 < [[\check{x}_0 + + + h\check{d}] + \delta_0]$	by (D31),(D33), (OA.87)
$(D35)$	$\Rightarrow 1 < [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}]$	by (OA.32)
$(D36)$	$\Rightarrow [[\check{x}_0 + \delta_2] + + + h\check{d}] =$ $[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}]$	by (D27),(D35), (OA.103)
$(D37)$	$\Rightarrow [[\check{x}_0 + \delta_2] + + + h\check{d}] = [[\check{x}_0 + + + h\check{d}] + \delta_0]$	by (OA.75),(OA.102)
$(D38)$	$\delta_0 < \varepsilon$	by (D4)
$(D39)$	$\Rightarrow [\delta_0 + [\check{x}_0 + + + h\check{d}]] < [\varepsilon + [\check{x}_0 + + + h\check{d}]]$	by (OA.29)
$(D40)$	$\Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] < [\varepsilon + [\check{x}_0 + + + h\check{d}]]$	by (OA.42),(OA.90)
$(D41)$	$\Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] < [[\check{x}_0 + + + h\check{d}] + \varepsilon]$	by (OA.42),(OA.89)
$(D42)$	$\Rightarrow [[\check{x}_0 + \delta_2] + + + h\check{d}] < [[\check{x}_0 + + + h\check{d}] + \varepsilon]$	by (D41),(D37), (OA.90)
$(D43)$	$\Rightarrow [[\check{x}_0 + + + h\check{d}] - \delta_0] < [[\check{x}_0 + + + h\check{d}] - [1 - 1]]$	by (D2),(OA.29)
$(D44)$	$\Rightarrow [[\check{x}_0 + + + h\check{d}] - \delta_0] < [[[\check{x}_0 + + + h\check{d}] - 1] + 1]$	by (OA.48),(OA.89)

$(D45)$ $\Rightarrow [[\check{x}_0 + + + h\check{d}] - \delta_0] < [\check{x}_0 + + + h\check{d}]$ by (OA.43),(OA.89)

$(D46)$ $\Rightarrow [[\check{x}_0 + + + h\check{d}] - \delta_0] =$

$[[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] + + + h\check{d}]$ by (Premise),(OA.75)

$(D47)$ $\Rightarrow [\check{x}_0 + + + h\check{d}] =$

$[[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}]$ by (B2),(OA.75)

$(D48)$ $\Rightarrow [[[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] + + + h\check{d}] <$

$[[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}]$ by (D45),(D46),

(D47),(OA.89),

(OA.90)

$(D49)$ $\Rightarrow [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] < [[\check{x}_0 + + + h\check{d}] - - - i\check{d}]$ by (D15),(D48),

(OA.32),(OA.36)

$(D50)$ $\Rightarrow [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}] < \check{x}_0$ by (Premise),(OA.76),

(OA.89)

$(D51)$ $\Rightarrow [\check{x}_0 - \check{x}_0] < [\check{x}_0 - [[[\check{x}_0 + + + h\check{d}] - \delta_0] - - - i\check{d}]]$ by (OA.29)

$(D52)$ $\Rightarrow [\check{x}_0 - \check{x}_0] < \delta_1$ by (C1),(OA.89)

$(D53)$ $\Rightarrow [1 - 1] < \delta_1$ by (OA.41),(OA.90)

$(D54)$ $\Rightarrow [[1 - 1] + [\check{x}_0 + + + h\check{d}]] < [\delta_0 + [\check{x}_0 + + + h\check{d}]]$ by (D2),(OA.29)

$(D55)$ $\Rightarrow [[1 - 1] + [\check{x}_0 + + + h\check{d}]] < [[\check{x}_0 + + + h\check{d}] + \delta_0]$ by (OA.42),(OA.89)

$(D56)$ $\Rightarrow [[\check{x}_0 + + + h\check{d}] + [1 - 1]] < [[\check{x}_0 + + + h\check{d}] + \delta_0]$ by (OA.42),(OA.90)

$(D57)$ $\Rightarrow [[[\check{x}_0 + + + h\check{d}] + 1] - 1] < [[\check{x}_0 + + + h\check{d}] + \delta_0]$ by (OA.46),(OA.90)

$(D58)$ $\Rightarrow [[[\check{x}_0 + + + h\check{d}] - 1] + 1] < [[\check{x}_0 + + + h\check{d}] + \delta_0]$ by (OA.44),(OA.90)

$(D59)$ $\Rightarrow [\check{x}_0 + + + h\check{d}] < [[\check{x}_0 + + + h\check{d}] + \delta_0]$ by (OA.43),(OA.90)

$(D60)$ $\Rightarrow [\check{x}_0 + + + h\check{d}] =$

$[[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}]$ by (B2),(OA.75)

$(D61)$ $\Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] =$

$[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}]$ by (D34),(OA.75)

$(D62)$ $\Rightarrow [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}] <$

$[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}]$ by (D59),(D60),

(D61),(OA.89),

(OA.90)

$(D63)$ $\Rightarrow [[\check{x}_0 + + + h\check{d}] - - - i\check{d}] < [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}]$ by (D35),(OA.75),

(D62),(OA.36)

$(D64)$ $\Rightarrow \check{x}_0 < [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}]$ by (Premise),(OA.76),

(OA.90)

$(D65)$ $\Rightarrow [\check{x}_0 - \check{x}_0] < [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]$ by (OA.29)

$(D66)$ $\Rightarrow [1-1] < [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]$ by (OA.41),(OA.90)

$(D67)$ $\Rightarrow [1-1] < \delta_2$ by (C2),(OA.89)

$(D68)$ $[1-1] < \delta$ by (D5),(D53), (D67)

$(D69)$ $\Rightarrow [[1-1]+1] < [\delta+1]$ by (OA.29)

$(D70)$ $\Rightarrow 1 < [\delta+1]$ by (OA.43),(OA.90)

$(D71)$ $\Rightarrow 1 < [1+\delta]$ by (OA.42),(OA.89)

$(D72)$ $1 \le \check{x}_0$ by (Premise)

$(D73)$ $\Rightarrow [1+\delta] \le [\check{x}_0 + \delta]$ by (OA.29)

$(D74)$ $\Rightarrow 1 < [\check{x}_0 + \delta]$ by (D71),(D73), (OA.87)

$(D75)$ $\Rightarrow [\delta + \check{x}_0] \le [\delta_2 + \check{x}_0]$ by (D7),(OA.29)

$(D76)$ $\Rightarrow [\check{x}_0 + \delta] \le [\delta_2 + \check{x}_0]$ by (OA.42),(OA.90)

$(D77)$ $\Rightarrow [\check{x}_0 + \delta] \le [\check{x}_0 + \delta_2]$ by (OA.42),(OA.89)

$(D78)$ $\Rightarrow [[\check{x}_0 + \delta] + + + h\check{d}] \le [[\check{x}_0 + \delta_2] + + + h\check{d}]$ by (D74),(D77), (OA.35)

$(D79)$ $\Rightarrow [[\check{x}_0 + \delta] + + + e\check{d}] < [[\check{x}_0 + + + e\check{d}] + \varepsilon]$ by (D78),(D42), (OA.87)

$(D80)$ $\Rightarrow \check{f}([\check{x}_0 + \delta]) < [\check{f}(\check{x}_0) + \varepsilon]$ by (Premise)

Since $\check{f}(\check{x})$ is strictly increasing, $\check{f}(\check{x}_0 - \delta) < \check{f}(\check{x})$ and $\check{f}(\check{x}) < \check{f}(\check{x}_0 + \delta)$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$. (D23) and (D80) derive that $[\check{f}(\check{x}_0) - \varepsilon] < \check{f}(\check{x})$ and $\check{f}(\check{x}) < [\check{f}(\check{x}_0) + \varepsilon]$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$.

(2) $[[\check{x}_0 + + + e\check{d}] - \delta_0] < 1$.

We construct δ as follows:

$$(E1) \quad \delta = [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]$$

$(F1)$ $a \to 1|[aba]$ by (OA.16)

$(F2)$ $\Rightarrow (a \to 1)$ by (OA.83)

$(F3)$ $\Rightarrow (a \to [aba])$ by (F1),(OA.83)

$(F4)$ $\Rightarrow (a \to [1ba])$ by (F3),(F2),(OA.82)

$(F5)$ $\Rightarrow (a \to [1b1])$ by (F3),(F2),(OA.82)

$(F6)$ $b \to +|-$ by (OA.17)

$(F7)$ $\Rightarrow (b \to -)$ by (OA.83)

$(F8)$ $\Rightarrow (a \to [1-1])$ by (F5),(F7),(OA.82)

$(F9)$ $a < [1+a]$ by (OA.28)

$(F10)$ $\Rightarrow 1 < [1+1]$ by (F9),(F2),(OA.94)

$(F11)$ $\Rightarrow 1$ by (OA.88)

$(F12)$	$\Rightarrow [1-1] < [1 + [1-1]]$	by (F9),(F8),(OA.94)
$(F13)$	$\Rightarrow [1-1] < [[1-1]+1]$	by (OA.42),(OA.89)
$(F14)$	$\Rightarrow [1-1] < 1$	by (F11),(OA.43), (F13),(OA.89)
$(F15)$	$\Rightarrow [1-1] < [\check{x}_0 + + + h\check{d}]$	by (F14),(B2), (OA.89),(OA.87)
$(F16)$	$\Rightarrow [1-1]$	by (OA.88)
$(F17)$	$\Rightarrow [\check{x}_0 + + + h\check{d}]$	by (F15),(OA.88)
$(F18)$	$\delta_0 = [\varepsilon - -[1+1]]$	by (B18)
$(F19)$	$\Rightarrow [1-1] < \delta_0$	by (B7),(F18), (OA.89)
$(F20)$	$\Rightarrow [[1-1] + [\check{x}_0 + + + h\check{d}]] < [\delta_0 + [\check{x}_0 + + + h\check{d}]]$	by (F19),(F17),(OA.29)
$(F21)$	$\Rightarrow [[\check{x}_0 + + + h\check{d}] + [1-1]] < [\delta_0 + [\check{x}_0 + + + h\check{d}]]$	by (F16),(F17), (OA.42),(OA.90)
$(F22)$	$\Rightarrow [[[\check{x}_0 + + + h\check{d}] + 1] - 1] < [\delta_0 + [\check{x}_0 + + + h\check{d}]]$	by (OA.46),(OA.90)
$(F23)$	$\Rightarrow [[[\check{x}_0 + + + h\check{d}] - 1] + 1] < [\delta_0 + [\check{x}_0 + + + h\check{d}]]$	by (OA.44),(OA.90)
$(F24)$	$\Rightarrow [\check{x}_0 + + + h\check{d}] < [\delta_0 + [\check{x}_0 + + + h\check{d}]]$	by (OA.43),(OA.90)
$(F25)$	$\Rightarrow [\check{x}_0 + + + h\check{d}] < [[\check{x}_0 + + + h\check{d}] + \delta_0]$	by (OA.42),(OA.89)
$(F26)$	$\Rightarrow 1 < [[\check{x}_0 + + + h\check{d}] + \delta_0]$	by (B2),(F25), (OA.90),(OA.87)
$(F27)$	$\Rightarrow [\check{x}_0 + + + h\check{d}] = [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}]$	by (B2),(OA.75),(OA.89)
$(F28)$	$\Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] =$ $[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}]$	by (F26),(OA.75),(OA.89)
$(F29)$	$\Rightarrow [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}] <$ $[[\check{x}_0 + + + h\check{d}] + \delta_0]$	by (F25),(F27),(OA.90)
$(F30)$	$\Rightarrow [[[\check{x}_0 + + + h\check{d}] - - - i\check{d}] + + + h\check{d}] <$ $[[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] + + + h\check{d}]$	by (F29),(F28),(OA.89)
$(F31)$	$\Rightarrow 1 \le [[\check{x}_0 + + + h\check{d}] - - - i\check{d}]$	by (B2),(OA.32),(OA.72)
$(F32)$	$\Rightarrow 1 < [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}]$	by (F26),(OA.32)
$(F33)$	$\Rightarrow [[\check{x}_0 + + + h\check{d}] - - - i\check{d}] <$ $[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}]$	by (F31),(F32), (F30),(OA.36)
$(F34)$	$\Rightarrow \check{x}_0 < [[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}]$	by (OA.76),(OA.90)
$(F35)$	$\Rightarrow [\check{x}_0 - \check{x}_0] < [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]$	by (OA.29)
$(F36)$	$\Rightarrow [1-1] < [[[[\check{x}_0 + + + h\check{d}] + \delta_0] - - - i\check{d}] - \check{x}_0]$	by (OA.41),(F35),(OA.90)
$(F37)$	$\Rightarrow [1-1] < \delta$	by (F36),(E1),(OA.89)

$(F38)$ $\qquad [[\check{x}_0+++h\check{d}]-\delta_0]<1$ $\qquad$ by (Premise)

$(F39)$ $\qquad \Rightarrow[[\check{x}_0+++h\check{d}]-[\varepsilon--[1+1]]]<1$ $\qquad$ by (B18),(OA.90)

$(F40)$ $\qquad [1-1]<\varepsilon$ $\qquad$ by (Premise)

$(F41)$ $\qquad \Rightarrow[\varepsilon--[1+1]]<[\varepsilon--1]$ $\qquad$ by (F40),(F10),(F14),(OA.31)

$(F42)$ $\qquad [\varepsilon--1]=\varepsilon$ $\qquad$ by (OA.49)

$(F43)$ $\qquad \Rightarrow[\varepsilon--[1+1]]<\varepsilon$ $\qquad$ by (F41),(F42),(OA.89)

$(F44)$ $\qquad \Rightarrow[[\check{x}_0+++h\check{d}]-\varepsilon]<$
$\qquad\qquad [[\check{x}_0+++h\check{d}]-[\varepsilon--[1+1]]]$ $\qquad$ by (OA.29)

$(F45)$ $\qquad \Rightarrow[[\check{x}_0+++e\check{d}]-\varepsilon]<1$ $\qquad$ by (F44),(F39),(OA.87)

$(F46)$ $\qquad \Rightarrow[\check{f}(\check{x}_0)-\varepsilon]<1$

$(F47)$ $\qquad \check{f}(1)=[1+++e\check{d}]$

$(F48)$ $\qquad \Rightarrow\check{f}(1)=1$ $\qquad$ by (OA.71)

$(F49)$ $\qquad \Rightarrow1\leq\check{f}(\check{x})$ $\qquad$ by (A1)$\sim$(A8),(OA.90)

$(F50)$ $\qquad \Rightarrow[\check{f}(\check{x}_0)-\varepsilon]<\check{f}(\check{x})$ $\qquad$ by (F46),(F49),(OA.87)

$(F51)$ $\qquad \delta_0=[\varepsilon--[1+1]]$ $\qquad$ by (B18)

$(F52)$ $\qquad \Rightarrow[1-1]<\delta_0$ $\qquad$ by (B7),(F51),
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (OA.89)

$(F53)$ $\qquad [\varepsilon--[1+1]]<\varepsilon$ $\qquad$ by (D9)

$(F54)$ $\qquad \Rightarrow\delta_0<\varepsilon$ $\qquad$ by (F53),(F51),
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (OA.90)

$(F55)$ $\qquad \delta=[[[[\check{x}_0+++h\check{d}]+\delta_0]---i\check{d}]-\check{x}_0]$ $\qquad$ by (E1)

$(F56)$ $\Rightarrow[\check{x}_0+\delta]=[\check{x}_0+[[[[\check{x}_0+++h\check{d}]+\delta_0]---i\check{d}]-\check{x}_0]]$ $\qquad$ by (OA.103)

$(F57)$ $\Rightarrow[\check{x}_0+\delta]=[[[[[\check{x}_0+++h\check{d}]+\delta_0]---i\check{d}]-\check{x}_0]+\check{x}_0]$ $\qquad$ by (OA.42),(OA.102)

$(F58)$ $\qquad \Rightarrow[\check{x}_0+\delta]=[[[\check{x}_0+++h\check{d}]+\delta_0]---i\check{d}]$ $\qquad$ by (OA.43),(OA.102)

$(F59)$ $\qquad [1-1]<\delta_0$ $\qquad$ by (F52)

$(F60)$ $\qquad \Rightarrow[[1-1]+1]<[\delta_0+1]$ $\qquad$ by (F59),(OA.29)

$(F61)$ $\qquad \Rightarrow1<[\delta_0+1]$ $\qquad$ by (OA.43),(OA.90)

$(F62)$ $\qquad \Rightarrow1<[1+\delta_0]$ $\qquad$ by (OA.42),(OA.89)

$(F63)$ $\qquad 1\leq[\check{x}_0+++h\check{d}]$ $\qquad$ by (B2)

$(F64)$ $\qquad \Rightarrow[1+\delta_0]\leq[[\check{x}_0+++h\check{d}]+\delta_0]$ $\qquad$ by (OA.29)

$(F65)$ $\qquad \Rightarrow1<[[\check{x}_0+++h\check{d}]+\delta_0]$ $\qquad$ by (F62),(F64),
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (OA.110)

$(F66)$ $\qquad \Rightarrow1<[[[\check{x}_0+++h\check{d}]+\delta_0]---i\check{d}]$ $\qquad$ by (OA.32)

$(F67)$ $\qquad \Rightarrow[[\check{x}_0+\delta]+++h\check{d}]=$
$\qquad\qquad [[[[\check{x}_0+++h\check{d}]+\delta_0]---i\check{d}]+++h\check{d}]$ $\qquad$ by (F58),(F66),
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (OA.103)

$(F68)$ $\quad\quad\quad \Rightarrow [[\check{x}_0 + \delta] + + + h\check{d}] = [[\check{x}_0 + + + h\check{d}] + \delta_0]$ $\quad$ by (OA.75),(OA.102)

$(F69)$ $\quad\quad\quad\quad\quad\quad\quad \delta_0 < \varepsilon$ $\quad$ by (F54)

$(F70)$ $\quad\quad\quad \Rightarrow [\delta_0 + [\check{x}_0 + + + h\check{d}]] < [\varepsilon + [\check{x}_0 + + + h\check{d}]]$ $\quad$ by (OA.29)

$(F71)$ $\quad\quad\quad \Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] < [\varepsilon + [\check{x}_0 + + + h\check{d}]]$ $\quad$ by (OA.42),(OA.90)

$(F72)$ $\quad\quad\quad \Rightarrow [[\check{x}_0 + + + h\check{d}] + \delta_0] < [[\check{x}_0 + + + h\check{d}] + \varepsilon]$ $\quad$ by (OA.42),(OA.89)

$(F73)$ $\quad\quad\quad \Rightarrow [[\check{x}_0 + \delta] + + + e\check{d}] < [[\check{x}_0 + + + e\check{d}] + \varepsilon]$ $\quad$ by (F72),(F68),

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ (OA.90)

$(F74)$ $\quad\quad\quad\quad\quad \Rightarrow \check{f}([\check{x}_0 + \delta]) < [\check{f}(\check{x}_0) + \varepsilon]$ $\quad$ by (Premise)

Since $\check{f}(\check{x})$ is strictly increasing, $\check{f}(1) \leq \check{f}(\check{x})$ and $\check{f}(\check{x}) < \check{f}(\check{x}_0 + \delta)$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$. (F17) and (F41) derive that $[\check{f}(\check{x}_0) - \varepsilon] < \check{f}(\check{x})$ and $\check{f}(\check{x}) < [\check{f}(\check{x}_0) + \varepsilon]$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$.

Items 1~2 derive that $\check{f}(\check{x})$ is continuous.

(F16) derives that $1 \leq \check{f}(\check{x})$ holds on the domain $[1, +\infty)$. For any number $1 \leq \varepsilon$, (OA.32) and (OA.72) always derive that $[[\varepsilon - - - i\check{d}] + 1] \in [1, +\infty)$. So there always exists $\check{x}_0 = [[\varepsilon - - - i\check{d}] + 1]$ on the domain $[1, +\infty)$.

$(G1)$ $\quad\quad\quad\quad\quad\quad [1 - 1] < 1$ $\quad\quad\quad\quad\quad$ by (F14)

$(G2)$ $\quad \Rightarrow [[1 - 1] + [\varepsilon - - - i\check{d}]] < [1 + [\varepsilon - - - i\check{d}]]$ $\quad$ by (OA.29)

$(G3)$ $\quad \Rightarrow [[\varepsilon - - - i\check{d}] + [1 - 1]] < [1 + [\varepsilon - - - i\check{d}]]$ $\quad$ by (OA.42),(OA.90)

$(G4)$ $\quad \Rightarrow [[[\varepsilon - - - i\check{d}] + 1] - 1] < [1 + [\varepsilon - - - i\check{d}]]$ $\quad$ by (OA.46),(OA.90)

$(G5)$ $\quad \Rightarrow [[[\varepsilon - - - i\check{d}] - 1] + 1] < [1 + [\varepsilon - - - i\check{d}]]$ $\quad$ by (OA.44),(OA.90)

$(G6)$ $\quad\quad\quad \Rightarrow [\varepsilon - - - i\check{d}] < [1 + [\varepsilon - - - i\check{d}]]$ $\quad$ by (OA.43),(OA.90)

$(G7)$ $\quad\quad\quad \Rightarrow [\varepsilon - - - i\check{d}] < [[\varepsilon - - - i\check{d}] + 1]$ $\quad$ by (OA.42),(OA.89)

$(G8)$ $\quad\quad\quad\quad \check{x}_0 = [[\varepsilon - - - i\check{d}] + 1]$ $\quad\quad$ by (Premise)

$(G9)$ $\quad\quad\quad\quad \Rightarrow [\varepsilon - - - i\check{d}] < \check{x}_0$ $\quad\quad$ by (G7),(G8),(OA.89)

$(G10)$ $\quad \Rightarrow [[\varepsilon - - - i\check{d}] + + + h\check{d}] < [\check{x}_0 + + + h\check{d}]$ $\quad$ by (OA.32),(OA.72),(OA.35)

$(G11)$ $\quad\quad\quad\quad \Rightarrow \varepsilon < [\check{x}_0 + + + e\check{d}]$ $\quad$ by (Premise),(OA.75),(OA.90)

$(G12)$ $\quad\quad\quad\quad \Rightarrow \varepsilon < \check{f}(\check{x}_0)$ $\quad\quad$ by (Premise)

(G1)~(G12) derive that $\check{f}(\check{x})$ is unbounded. $\quad\quad\quad\quad\quad\quad\quad\quad$ □

Theorem 3.5. *The EV Function* $\check{f}(\check{x}) = [\check{e} + + + e\check{x}]$ *is continuous, unbounded and strictly increasing.*

Proof. According to (OA.19), the symbol 'e' represents some successive '$+$'—"$+ \cdots +$". According to (OA.20), the symbol 'f' represents some successive '$-$'—"$- \cdots -$". According to (OA.21), the symbol 'g' represents some successive '$/$'—"$/ \cdots /$".

$(A1)$ $\quad$ Supposing that $\check{x}_1, \check{x}_2 \in [[1 - 1], +\infty)$ with $\check{x}_1 < \check{x}_2$.

$(A2)$ $\quad\quad\quad\quad \Rightarrow ([1 - 1] \leq \check{x}_1) \wedge (\check{x}_1 < \check{x}_2)$

$(A3)$ $\quad\quad\quad\quad\quad\quad 1 < \check{e}$ $\quad\quad\quad\quad\quad$ by (Premise)

$(A4)$ $\Rightarrow [\check{e}+++e\check{x}_1] < [\check{e}+++e\check{x}_2]$ by (A2),(A3),(OA.37)

$(A5)$ $\check{f}(\check{x}_1) = [\check{e}+++e\check{x}_1]$ by (Premise)

$(A6)$ $\Rightarrow \check{f}(\check{x}_1) < [\check{e}+++e\check{x}_2]$ by (A4),(A5),(OA.90)

$(A7)$ $\check{f}(\check{x}_2) = [\check{e}+++e\check{x}_2]$ by (Premise)

$(A8)$ $\Rightarrow \check{f}(\check{x}_1) < \check{f}(\check{x}_2)$ by (A6),(A7),(OA.89)

(A1)~(A8) derive that $\check{f}(\check{x})$ is strictly increasing.

For any number $\check{x}_0 \in \big[[1-1], +\infty\big)$ and any number $[1-1] < \varepsilon$, we can always construct δ_0 as follows:

$(B1)$ $[\check{e}+++e[1-1]] \leq [\check{e}+++e\check{x}_0]$ by (OA.37),(OA.99)

$(B2)$ $\Rightarrow 1 \leq [\check{e}+++e\check{x}_0]$ by (OA.69),(OA.90)

$(B3)$ $\Rightarrow [1-1] \leq [[\check{e}+++e\check{x}_0]-1]$ by (OA.29)

$(B4)$ $[1-1] < \varepsilon$ by (Premise)

$(B5)$ $\Rightarrow [[1-1]--[1+1]] < [\varepsilon--[1+1]]$ by (OA.30)

$(B6)$ $\Rightarrow [[1--[1+1]]-[1--[1+1]]] < [\varepsilon--[1+1]]$ by (OA.56), (OA.90)

$(B7)$ $\Rightarrow [1-1] < [\varepsilon--[1+1]]$ by (OA.41), (OA.90)

$(B8)$ $\Rightarrow [[1-1]+[\varepsilon--[1+1]]] < [[\varepsilon--[1+1]]+[\varepsilon--[1+1]]]$ by (OA.29)

$(B9)$ $\Rightarrow [[1-1]+[\varepsilon--[1+1]]] < [[\varepsilon+\varepsilon]--[1+1]]$ by (OA.56), (OA.89)

$(B10)$ $\Rightarrow [[\varepsilon--[1+1]]+[1-1]] < [[\varepsilon+\varepsilon]--[1+1]]$ by (OA.42), (OA.90)

$(B11)$ $\Rightarrow [[[\varepsilon--[1+1]]+1]-1] < [[\varepsilon+\varepsilon]--[1+1]]$ by (OA.46), (OA.90)

$(B12)$ $\Rightarrow [[[\varepsilon--[1+1]]-1]+1] < [[\varepsilon+\varepsilon]--[1+1]]$ by (OA.44), (OA.90)

$(B13)$ $\Rightarrow [\varepsilon--[1+1]] < [[\varepsilon+\varepsilon]--[1+1]]$ by (OA.43), (OA.90)

$(B14)$ $\Rightarrow [\varepsilon--[1+1]] < [[\varepsilon++[1+1]]--[1+1]]$ by (OA.53),(OA.49), (OA.89)

$(B15)$ $\Rightarrow [\varepsilon--[1+1]] < [\varepsilon++[[1+1]--[1+1]]]$ by (OA.57),(OA.89)

$(B16)$ $\Rightarrow [\varepsilon--[1+1]] < [\varepsilon++1]$ by (OA.50),(OA.89)

$(B17)$ $\Rightarrow [\varepsilon--[1+1]] < \varepsilon$ by (OA.49),(OA.89)

$(B18)$ $\delta_0 = [\varepsilon--[1+1]]$

We construct δ according to $[[\check{e}+++e\check{x}_0]-\delta_0]$.

(1) $1 \leq [[\check{e}+++e\check{x}_0]-\delta_0]$.
      We construct δ as follows:

$(C1)$ $\delta_1 = [\check{x}_0 - [[[\check{e}+++h\check{x}_0]-\delta_0]///j\check{e}]]$

$(C2)$ $\delta_2 = [[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{e}]-\check{x}_0]$

$(C3)$ $\delta = \min\{\delta_1, \delta_2\}$

$(D1)$	$\delta_0 = [\varepsilon - -[1+1]]$	by (B18)
$(D2)$	$\Rightarrow [1-1] < \delta_0$	by (B7),(D1), (OA.89)
$(D3)$	$[\varepsilon - -[1+1]] < \varepsilon$	by (B17)
$(D4)$	$\Rightarrow \delta_0 < \varepsilon$	by (D3),(D1), (OA.90)
$(D5)$	$\delta = \min\{\delta_1, \delta_2\}$	by (C3)
$(D6)$	$\Rightarrow \delta \leq \delta_1$	
$(D7)$	$\Rightarrow \delta \leq \delta_2$	
$(D8)$	$\delta_1 = [\check{x}_0 - [[[\check{e} +++ h\check{x}_0] - \delta_0]///j\check{h}]]$	by (C1)
$(D9)$	$\Rightarrow [\check{x}_0 - \delta_1] = [\check{x}_0 - [\check{x}_0 - [[[\check{e} +++ h\check{x}_0] - \delta_0]///j\check{h}]]]$	by (OA.103)
$(D10)$	$\Rightarrow [\check{x}_0 - \delta_1] = [[\check{x}_0 - \check{x}_0] + [[[\check{e} +++ h\check{x}_0] - \delta_0]///j\check{h}]]$	by (OA.48),(OA.102)
$(D11)$	$\Rightarrow [\check{x}_0 - \delta_1] = [[[[\check{e} +++ h\check{x}_0] - \delta_0]///j\check{h}] + [\check{x}_0 - \check{x}_0]]$	by (OA.42),(OA.102)
$(D12)$	$\Rightarrow [\check{x}_0 - \delta_1] = [[[[[\check{e} +++ h\check{x}_0] - \delta_0]///j\check{h}] + \check{x}_0] - \check{x}_0]$	by (OA.46),(OA.102)
$(D13)$	$\Rightarrow [\check{x}_0 - \delta_1] = [[[[[\check{e} +++ h\check{x}_0] - \delta_0]///j\check{h}] - \check{x}_0] + \check{x}_0]$	by (OA.44),(OA.102)
$(D14)$	$\Rightarrow [\check{x}_0 - \delta_1] = [[[\check{e} +++ h\check{x}_0] - \delta_0]///j\check{h}]$	by (OA.43),(OA.102)
$(D15)$	$[1-1] \leq [[[\check{e} +++ h\check{x}_0] - \delta_0]///j\check{h}]$	by (Premise),(OA.34)
$(D16)$	$\Rightarrow [1-1] < [\check{x}_0 - \delta_1]$	by (D14),(D15), (OA.89)
$(D17)$	$\Rightarrow [\check{x}_0 - \delta_1] \leq [\check{x}_0 - \delta]$	by (D6),(OA.29)
$(D18)$	$\Rightarrow [\check{e} +++ h[\check{x}_0 - \delta_1]] \leq [\check{e} +++ h[\check{x}_0 - \delta]]$	by (D17),(OA.37), (OA.103)
$(D19)$	$\Rightarrow [\check{e} +++ h[[[\check{e} +++ h\check{x}_0] - \delta_0]///j\check{h}]] \leq$ $[\check{e} +++ h[\check{x}_0 - \delta]]$	by (D14),(OA.90)
$(D20)$	$\Rightarrow [[\check{e} +++ h\check{x}_0] - \delta_0] \leq [\check{e} +++ h[\check{x}_0 - \delta]]$	by (OA.77),(OA.90)
$(D21)$	$\Rightarrow [[\check{e} +++ h\check{x}_0] - \varepsilon] < [[\check{e} +++ h\check{x}_0] - \delta_0]$	by (D4),(OA.29)
$(D22)$	$\Rightarrow [[\check{e} +++ e\check{x}_0] - \varepsilon] < [\check{e} +++ e[\check{x}_0 - \delta]]$	by (D21),(D20), (OA.87)
$(D23)$	$\Rightarrow [\check{f}(\check{x}_0) - \varepsilon] < \check{f}([\check{x}_0 - \delta])$	by (Premise)
$(D24)$	$\delta_2 = [[[[\check{e} +++ h\check{x}_0] + \delta_0]///j\check{h}] - \check{x}_0]$	by (C2)
$(D25)$	$\Rightarrow [\check{x}_0 + \delta_2] = [\check{x}_0 + [[[[\check{e} +++ h\check{x}_0] + \delta_0]///j\check{h}] - \check{x}_0]]$	by (OA.103)
$(D26)$	$\Rightarrow [\check{x}_0 + \delta_2] = [[[[[\check{e} +++ h\check{x}_0] + \delta_0]///j\check{h}] - \check{x}_0] + \check{x}_0]$	by (OA.42),(OA.102)
$(D27)$	$\Rightarrow [\check{x}_0 + \delta_2] = [[[\check{e} +++ h\check{x}_0] + \delta_0]///j\check{h}]$	by (OA.43),(OA.102)
$(D28)$	$[1-1] < \delta_0$	by (D2)
$(D29)$	$\Rightarrow [[1-1] + 1] < [\delta_0 + 1]$	by (D28),(OA.29)
$(D30)$	$\Rightarrow 1 < [\delta_0 + 1]$	by (OA.43),(OA.90)

$(D31)$ $\qquad\Rightarrow 1 < [1 + \delta_0]$ $\qquad$ by (OA.42),(OA.89)

$(D32)$ $\qquad 1 \leq [\check{e} + + + h\check{x}_0]$ $\qquad$ by (B2)

$(D33)$ $\qquad\Rightarrow [1 + \delta_0] \leq [[\check{e} + + + h\check{x}_0] + \delta_0]$ $\qquad$ by (OA.29)

$(D34)$ $\qquad\Rightarrow 1 < [[\check{e} + + + h\check{x}_0] + \delta_0]$ $\qquad$ by (D31),(D33),
(OA.87)

$(D35)$ $\qquad\Rightarrow [1 - 1] < [[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}]$ $\qquad$ by (OA.33)

$(D36)$ $\qquad\Rightarrow [\check{e} + + + h[\check{x}_0 + \delta_2]] =$
$[\check{e} + + + h[[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{h}]]$ $\qquad$ by (D27),(D35),
(OA.103)

$(D37)$ $\qquad\Rightarrow [\check{e} + + + h[\check{x}_0 + \delta_2]] = [[\check{e} + + + h\check{x}_0] + \delta_0]$ $\qquad$ by (OA.77),(OA.102)

$(D38)$ $\qquad \delta_0 < \varepsilon$ $\qquad$ by (D4)

$(D39)$ $\qquad\Rightarrow [\delta_0 + [\check{e} + + + h\check{x}_0]] < [\varepsilon + [\check{e} + + + h\check{x}_0]]$ $\qquad$ by (OA.29)

$(D40)$ $\qquad\Rightarrow [[\check{e} + + + h\check{x}_0] + \delta_0] < [\varepsilon + [\check{e} + + + h\check{x}_0]]$ $\qquad$ by (OA.42),(OA.90)

$(D41)$ $\qquad\Rightarrow [[\check{e} + + + h\check{x}_0] + \delta_0] < [[\check{e} + + + h\check{x}_0] + \varepsilon]$ $\qquad$ by (OA.42),(OA.89)

$(D42)$ $\qquad\Rightarrow [\check{e} + + + h[\check{x}_0 + \delta_2]] < [[\check{e} + + + h\check{x}_0] + \varepsilon]$ $\qquad$ by (D41),(D37),
(OA.90)

$(D43)$ $\qquad\Rightarrow [[\check{e} + + + h\check{x}_0] - \delta_0] < [[\check{e} + + + h\check{x}_0] - [1 - 1]]$ $\qquad$ by (D2),(OA.29)

$(D44)$ $\qquad\Rightarrow [[\check{e} + + + h\check{x}_0] - \delta_0] < [[[\check{e} + + + h\check{x}_0] - 1] + 1]$ $\qquad$ by (OA.48),(OA.89)

$(D45)$ $\qquad\Rightarrow [[\check{e} + + + h\check{x}_0] - \delta_0] < [\check{e} + + + h\check{x}_0]$ $\qquad$ by (OA.43),(OA.89)

$(D46)$ $\qquad\Rightarrow [[\check{e} + + + h\check{x}_0] - \delta_0] =$
$[\check{e} + + + h[[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}]]$ $\qquad$ by (Premise),(OA.77)

$(D47)$ $\qquad\Rightarrow [\check{e} + + + h\check{x}_0] =$
$[\check{e} + + + h[[\check{e} + + + h\check{x}_0]///j\check{h}]]$ $\qquad$ by (B2),(OA.77)

$(D48)$ $\qquad\Rightarrow [\check{e} + + + h[[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}]] <$
$[\check{e} + + + h[[\check{e} + + + h\check{x}_0]///j\check{h}]]$ $\qquad$ by (D45),(D46),
(D47),(OA.89),
(OA.90)

$(D49)$ $\qquad\Rightarrow [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}] < [[\check{e} + + + h\check{x}_0]///j\check{h}]$ $\qquad$ by (D15),(D48),
(OA.33),(OA.38)

$(D50)$ $\qquad\Rightarrow [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}] < \check{x}_0$ $\qquad$ by (Premise),(OA.78),
(OA.89)

$(D51)$ $\qquad\Rightarrow [\check{x}_0 - \check{x}_0] < [\check{x}_0 - [[[\check{e} + + + h\check{x}_0] - \delta_0]///j\check{h}]]$ $\qquad$ by (OA.29)

$(D52)$ $\qquad\Rightarrow [\check{x}_0 - \check{x}_0] < \delta_1$ $\qquad$ by (C1),(OA.89)

$(D53)$ $\qquad\Rightarrow [1 - 1] < \delta_1$ $\qquad$ by (OA.41),(OA.90)

$(D54)$ $\qquad\Rightarrow [[1 - 1] + [\check{e} + + + h\check{x}_0]] < [\delta_0 + [\check{e} + + + h\check{x}_0]]$ $\qquad$ by (D2),(OA.29)

$(D55)$ $\qquad\Rightarrow [[1 - 1] + [\check{e} + + + h\check{x}_0]] < [[\check{e} + + + h\check{x}_0] + \delta_0]$ $\qquad$ by (OA.42),(OA.89)

$(D56)$ $\quad \Rightarrow [[\check{e}+++h\check{x}_0]+[1-1]] < [[\check{e}+++h\check{x}_0]+\delta_0]$ $\qquad$ by (OA.42),(OA.90)

$(D57)$ $\quad \Rightarrow [[[\check{e}+++h\check{x}_0]+1]-1] < [[\check{e}+++h\check{x}_0]+\delta_0]$ $\qquad$ by (OA.46),(OA.90)

$(D58)$ $\quad \Rightarrow [[[\check{e}+++h\check{x}_0]-1]+1] < [[\check{e}+++h\check{x}_0]+\delta_0]$ $\qquad$ by (OA.44),(OA.90)

$(D59)$ $\quad \Rightarrow [\check{e}+++h\check{x}_0] < [[\check{e}+++h\check{x}_0]+\delta_0]$ $\qquad$ by (OA.43),(OA.90)

$(D60)$ $\quad \Rightarrow [\check{e}+++h\check{x}_0] =$
$$[\check{e}+++h[[\check{e}+++h\check{x}_0]///j\check{h}]]$$ $\qquad$ by (B2),(OA.77)

$(D61)$ $\quad \Rightarrow [[\check{e}+++h\check{x}_0]+\delta_0] =$
$$[\check{e}+++h[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]]$$ $\qquad$ by (D34),(OA.77)

$(D62)$ $\quad \Rightarrow [\check{e}+++h[[\check{e}+++h\check{x}_0]///j\check{h}]] <$
$$[\check{e}+++h[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]]$$ $\qquad$ by (D59),(D60), (D61),(OA.89), (OA.90)

$(D63)$ $\quad \Rightarrow [[\check{e}+++h\check{x}_0]///j\check{h}] < [[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]$ $\quad$ by (D35),(D62), (OA.33),(OA.38)

$(D64)$ $\quad \Rightarrow \check{x}_0 < [[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]$ $\qquad$ by (Premise),(OA.78), (OA.90)

$(D65)$ $\quad \Rightarrow [\check{x}_0-\check{x}_0] < [[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]-\check{x}_0]$ $\qquad$ by (OA.29)

$(D66)$ $\quad \Rightarrow [1-1] < [[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]-\check{x}_0]$ $\qquad$ by (OA.41),(OA.90)

$(D67)$ $\quad \Rightarrow [1-1] < \delta_2$ $\qquad$ by (C2),(OA.89)

$(D68)$ $\quad [1-1] < \delta$ $\qquad$ by (D5),(D53), (D67)

$(D69)$ $\quad [1-1] \leq \check{x}_0$ $\qquad$ by (Premise)

$(D70)$ $\quad \Rightarrow [[1-1]+\delta] \leq [\check{x}_0+\delta]$ $\qquad$ by (OA.29)

$(D71)$ $\quad \Rightarrow [\delta+[1-1]] \leq [\check{x}_0+\delta]$ $\qquad$ by (OA.42),(OA.90)

$(D72)$ $\quad \Rightarrow [[\delta+1]-1] \leq [\check{x}_0+\delta]$ $\qquad$ by (OA.46),(OA.90)

$(D73)$ $\quad \Rightarrow [[\delta-1]+1] \leq [\check{x}_0+\delta]$ $\qquad$ by (OA.44),(OA.90)

$(D74)$ $\quad \Rightarrow \delta \leq [\check{x}_0+\delta]$ $\qquad$ by (OA.43),(OA.90)

$(D75)$ $\quad \Rightarrow [1-1] < [\check{x}_0+\delta]$ $\qquad$ by (D68),(D74), (OA.87)

$(D76)$ $\quad \Rightarrow [\delta+\check{x}_0] \leq [\delta_2+\check{x}_0]$ $\qquad$ by (D7),(OA.29)

$(D77)$ $\quad \Rightarrow [\check{x}_0+\delta] \leq [\delta_2+\check{x}_0]$ $\qquad$ by (OA.42),(OA.90)

$(D78)$ $\quad \Rightarrow [\check{x}_0+\delta] \leq [\check{x}_0+\delta_2]$ $\qquad$ by (OA.42),(OA.89)

$(D79)$ $\quad \Rightarrow [\check{e}+++h[\check{x}_0+\delta]] \leq [\check{e}+++h[\check{x}_0+\delta_2]]$ $\qquad$ by (D75),(D78), (OA.37)

$(D80)$ $\quad \Rightarrow [\check{e}+++e[\check{x}_0+\delta]] < [[\check{e}+++e\check{x}_0]+\varepsilon]$ $\qquad$ by (D79),(D42), (OA.87)

$(D81)$ $\quad\quad\quad\quad\quad\quad\quad \Rightarrow \check{f}([\check{x}_0 + \delta]) < [\check{f}(\check{x}_0) + \varepsilon]$ $\quad\quad\quad\quad$ by (Premise)

Since $\check{f}(\check{x})$ is strictly increasing, $\check{f}(\check{x}_0 - \delta) < \check{f}(\check{x})$ and $\check{f}(\check{x}) < \check{f}(\check{x}_0 + \delta)$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$. (D23) and (D81) derive that $[\check{f}(\check{x}_0) - \varepsilon] < \check{f}(\check{x})$ and $\check{f}(\check{x}) < [\check{f}(\check{x}_0) + \varepsilon]$ hold for all $\check{x} \in (\check{x}_0 - \delta, \check{x}_0 + \delta)$.

(2) $[[\check{e} + + + e\check{x}_0] - \delta_0] < 1$.

We construct δ as follows:

$(E1) \quad \delta = [[[[\check{e} + + + h\check{x}_0] + \delta_0]///j\check{e}] - \check{x}_0]$

$(F1)$	$a \to 1\vert[aba]$	by (OA.16)
$(F2)$	$\Rightarrow (a \to 1)$	by (OA.83)
$(F3)$	$\Rightarrow (a \to [aba])$	by (F1),(OA.83)
$(F4)$	$\Rightarrow (a \to [1ba])$	by (F3),(F2),(OA.82)
$(F5)$	$\Rightarrow (a \to [1b1])$	by (F3),(F2),(OA.82)
$(F6)$	$b \to +\vert-$	by (OA.17)
$(F7)$	$\Rightarrow (b \to -)$	by (OA.83)
$(F8)$	$\Rightarrow (a \to [1 - 1])$	by (F5),(F7),(OA.82)
$(F9)$	$a < [1 + a]$	by (OA.28)
$(F10)$	$\Rightarrow 1 < [1 + 1]$	by (F9),(F2),(OA.94)
$(F11)$	$\Rightarrow 1$	by (OA.88)
$(F12)$	$\Rightarrow [1 - 1] < [1 + [1 - 1]]$	by (F9),(F8),(OA.94)
$(F13)$	$\Rightarrow [1 - 1] < [[1 - 1] + 1]$	by (OA.42),(OA.89)
$(F14)$	$\Rightarrow [1 - 1] < 1$	by (F11),(OA.43), (F13),(OA.89)
$(F15)$	$\Rightarrow [1 - 1] < [\check{e} + + + h\check{x}_0]$	by (F14),(B2), (OA.89),(OA.87)
$(F16)$	$\Rightarrow [1 - 1]$	by (OA.88)
$(F17)$	$\Rightarrow [\check{e} + + + h\check{x}_0]$	by (F15),(OA.88)
$(F18)$	$\delta_0 = [\varepsilon - -[1 + 1]]$	by (B18)
$(F19)$	$\Rightarrow [1 - 1] < \delta_0$	by (B7),(F18), (OA.89)
$(F20)$	$\Rightarrow [[1 - 1] + [\check{e} + + + h\check{x}_0]] < [\delta_0 + [\check{e} + + + h\check{x}_0]]$	by (F19),(F17),(OA.29)
$(F21)$	$\Rightarrow [[\check{e} + + + h\check{x}_0] + [1 - 1]] < [\delta_0 + [\check{e} + + + h\check{x}_0]]$	by (F16),(F17), (OA.42),(OA.90)
$(F22)$	$\Rightarrow [[[\check{e} + + + h\check{x}_0] + 1] - 1] < [\delta_0 + [\check{e} + + + h\check{x}_0]]$	by (OA.46),(OA.90)
$(F23)$	$\Rightarrow [[[\check{e} + + + h\check{x}_0] - 1] + 1] < [\delta_0 + [\check{e} + + + h\check{x}_0]]$	by (OA.44),(OA.90)
$(F24)$	$\Rightarrow [\check{e} + + + h\check{x}_0] < [\delta_0 + [\check{e} + + + h\check{x}_0]]$	by (OA.43),(OA.90)
$(F25)$	$\Rightarrow [\check{e} + + + h\check{x}_0] < [[\check{e} + + + h\check{x}_0] + \delta_0]$	by (OA.42),(OA.89)
$(F26)$	$\Rightarrow 1 < [[\check{e} + + + h\check{x}_0] + \delta_0]$	by (B2),(F25),

$$\text{(OA.90),(OA.87)}$$

$(F27) \quad \Rightarrow [\check{e}+++h\check{x}_0] = [\check{e}+++h[[\check{e}+++h\check{x}_0]///j\check{e}]] \quad$ by (B2),(OA.77),(OA.89)

$(F28) \quad \Rightarrow [[\check{e}+++h\check{x}_0]+\delta_0] =$

$\qquad\qquad [\check{e}+++h[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{e}]] \quad$ by (F26),(OA.77),(OA.89)

$(F29) \quad \Rightarrow [\check{e}+++h[[\check{e}+++h\check{x}_0]///j\check{e}]] < [[\check{e}+++h\check{x}_0]+\delta_0] \quad$ by (F25),(F27),(OA.90)

$(F30) \quad \Rightarrow [\check{e}+++h[[\check{e}+++h\check{x}_0]///j\check{e}]] <$

$\qquad\qquad [\check{e}+++h[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{e}]] \quad$ by (F29),(F28),(OA.89)

$(F31) \quad \Rightarrow [1-1] \le [[\check{e}+++h\check{x}_0]///j\check{e}] \quad$ by (B2),(OA.33),(OA.73)

$(F32) \quad \Rightarrow [1-1] < [[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{e}] \quad$ by (F26),(OA.33)

$(F33) \quad \Rightarrow [[\check{e}+++h\check{x}_0]///j\check{e}] <$

$\qquad\qquad [[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{e}] \quad$ by (F31),(F32),

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (F30),(OA.38)

$(F34) \quad \Rightarrow \check{x}_0 < [[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{e}] \quad$ by (OA.78),(OA.90)

$(F35) \quad \Rightarrow [\check{x}_0-\check{x}_0] < [[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{e}]-\check{x}_0] \quad$ by (OA.29)

$(F36) \quad \Rightarrow [1-1] < [[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{e}]-\check{x}_0] \quad$ by (OA.41),(F35),(OA.90)

$(F37) \quad \Rightarrow [1-1] < \delta \quad$ by (F36),(E1),(OA.89)

$(F38) \quad [[\check{e}+++h\check{x}_0]-\delta_0] < 1 \quad$ by (Premise)

$(F39) \quad \Rightarrow [[\check{e}+++h\check{x}_0]-[\varepsilon--[1+1]]] < 1 \quad$ by (B18),(OA.90)

$(F40) \quad [1-1] < \varepsilon \quad$ by (Premise)

$(F41) \quad \Rightarrow [\varepsilon--[1+1]] < [\varepsilon--1] \quad$ by (F40),(F10),(F14),(OA.31)

$(F42) \quad [\varepsilon--1] = \varepsilon \quad$ by (OA.49)

$(F43) \quad \Rightarrow [\varepsilon--[1+1]] < \varepsilon \quad$ by (F41),(F42),(OA.89)

$(F44) \quad \Rightarrow [[\check{e}+++h\check{x}_0]-\varepsilon] <$

$\qquad\qquad [[\check{e}+++h\check{x}_0]-[\varepsilon--[1+1]]] \quad$ by (OA.29)

$(F45) \quad \Rightarrow [[\check{e}+++e\check{x}_0]-\varepsilon] < 1 \quad$ by (F44),(F39),(OA.87)

$(F46) \quad \Rightarrow [\check{f}(\check{x}_0)-\varepsilon] < 1$

$(F47) \quad \check{f}([1-1]) = [\check{e}+++e[1-1]]$

$(F48) \quad \Rightarrow \check{f}([1-1]) = 1 \quad$ by (OA.69)

$(F49) \quad \Rightarrow 1 \le \check{f}(\check{x}) \quad$ by (A1)$\sim$(A8),(OA.90)

$(F50) \quad \Rightarrow [\check{f}(\check{x}_0)-\varepsilon] < \check{f}(\check{x}) \quad$ by (F46),(F49),(OA.87)

$(F51) \quad \delta_0 = [\varepsilon--[1+1]] \quad$ by (B18)

$(F52) \quad \Rightarrow [1-1] < \delta_0 \quad$ by (B7),(F51),

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (OA.89)

$(F53) \quad [\varepsilon--[1+1]] < \varepsilon \quad$ by (D9)

$(F54) \quad \Rightarrow \delta_0 < \varepsilon \quad$ by (F53),(F51),

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (OA.90)

$(F55)$ $\qquad \delta = [[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]-\check{x}_0]$ $\qquad$ by (E1)

$(F56)$ $\quad \Rightarrow [\check{x}_0+\delta] = [\check{x}_0+[[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]-\check{x}_0]]$ $\quad$ by (OA.103)

$(F57)$ $\quad \Rightarrow [\check{x}_0+\delta] = [[[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]-\check{x}_0]+\check{x}_0]$ $\quad$ by (OA.42),(OA.102)

$(F58)$ $\quad \Rightarrow [\check{x}_0+\delta] = [[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]$ $\quad$ by (OA.43),(OA.102)

$(F59)$ $\qquad [1-1] < \delta_0$ $\qquad$ by (F52)

$(F60)$ $\quad \Rightarrow [[1-1]+1] < [\delta_0+1]$ $\qquad$ by (F59),(OA.29)

$(F61)$ $\quad \Rightarrow 1 < [\delta_0+1]$ $\qquad$ by (OA.43),(OA.90)

$(F62)$ $\quad \Rightarrow 1 < [1+\delta_0]$ $\qquad$ by (OA.42),(OA.89)

$(F63)$ $\qquad 1 \le [\check{e}+++h\check{x}_0]$ $\qquad$ by (B2)

$(F64)$ $\quad \Rightarrow [1+\delta_0] \le [[\check{e}+++h\check{x}_0]+\delta_0]$ $\qquad$ by (OA.29)

$(F65)$ $\quad \Rightarrow 1 < [[\check{e}+++h\check{x}_0]+\delta_0]$ $\qquad$ by (F62),(F64),

$\qquad$ (OA.87)

$(F66)$ $\quad \Rightarrow [1-1] < [[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]$ $\quad$ by (OA.33)

$(F67)$ $\quad \Rightarrow [\check{e}+++h[\check{x}_0+\delta]] =$

$\qquad [\check{e}+++h[[[\check{e}+++h\check{x}_0]+\delta_0]///j\check{h}]]$ $\qquad$ by (F58),(F66),

$\qquad$ (OA.103)

$(F68)$ $\quad \Rightarrow [\check{e}+++h[\check{x}_0+\delta]] = [[\check{e}+++h\check{x}_0]+\delta_0]$ $\quad$ by (OA.77),(OA.102)

$(F69)$ $\qquad \delta_0 < \varepsilon$ $\qquad$ by (F54)

$(F70)$ $\quad \Rightarrow [\delta_0+[\check{e}+++h\check{x}_0]] < [\varepsilon+[\check{e}+++h\check{x}_0]]$ $\quad$ by (OA.29)

$(F71)$ $\quad \Rightarrow [[\check{e}+++h\check{x}_0]+\delta_0] < [\varepsilon+[\check{e}+++h\check{x}_0]]$ $\quad$ by (OA.42),(OA.90)

$(F72)$ $\quad \Rightarrow [[\check{e}+++h\check{x}_0]+\delta_0] < [[\check{e}+++h\check{x}_0]+\varepsilon]$ $\quad$ by (OA.42),(OA.89)

$(F73)$ $\quad \Rightarrow [\check{e}+++e[\check{x}_0+\delta]] < [[\check{e}+++e\check{x}_0]+\varepsilon]$ $\quad$ by (F72),(F68),

$\qquad$ (OA.90)

$(F74)$ $\qquad \Rightarrow \check{f}([\check{x}_0+\delta]) < [\check{f}(\check{x}_0)+\varepsilon]$ $\qquad$ by (Premise)

$\qquad$ Since $\check{f}(\check{x})$ is strictly increasing, $\check{f}(1) \le \check{f}(\check{x})$ and $\check{f}(\check{x}) < \check{f}(\check{x}_0+\delta)$ hold for all $\check{x} \in \left(\check{x}_0-\delta, \check{x}_0+\delta\right)$. (F17) and (F41) derive that $[\check{f}(\check{x}_0)-\varepsilon] < \check{f}(\check{x})$ and $\check{f}(\check{x}) < [\check{f}(\check{x}_0)+\varepsilon]$ hold for all $\check{x} \in \left(\check{x}_0-\delta, \check{x}_0+\delta\right)$.

Items 1~2 derive that $\check{f}(\check{x})$ is continuous.

(F16) derives that $1 \le \check{f}(\check{x})$ holds on the domain $\left[1, +\infty\right)$. For any number $1 \le \varepsilon$, (OA.33) and (OA.73) always derive that $[\varepsilon///j\check{e}] \in \left[[1-1], +\infty\right)$. So there always exists $\check{x}_0 = [[\varepsilon///j\check{e}]+1]$ on the domain $\left[1, +\infty\right)$.

$(G1)$ $\qquad [1-1] < 1$ $\qquad$ by (F14)

$(G2)$ $\quad \Rightarrow [[1-1]+[\varepsilon///j\check{e}]] < [1+[\varepsilon///j\check{e}]]$ $\quad$ by (OA.29)

$(G3)$ $\quad \Rightarrow [[\varepsilon///j\check{e}]+[1-1]] < [1+[\varepsilon///j\check{e}]]$ $\quad$ by (OA.42),(OA.90)

$(G4)$ $\quad \Rightarrow [[[\varepsilon///j\check{e}]+1]-1] < [1+[\varepsilon///j\check{e}]]$ $\quad$ by (OA.46),(OA.90)

$(G5)$ $\quad \Rightarrow [[[\varepsilon///j\check{e}]-1]+1] < [1+[\varepsilon///j\check{e}]]$ $\quad$ by (OA.44),(OA.90)

$(G6)$ $\quad \Rightarrow [\varepsilon///j\check{e}] < [1+[\varepsilon///j\check{e}]]$ $\qquad$ by (OA.43),(OA.90)

$(G7)$ $\qquad \Rightarrow [\varepsilon///j\check{e}] < [[\varepsilon///j\check{e}] + 1]$ $\qquad$ by (OA.42),(OA.89)

$(G8)$ $\qquad \check{x}_0 = [[\varepsilon///j\check{e}] + 1]$ $\qquad$ by (Premise)

$(G9)$ $\qquad \Rightarrow [\varepsilon///j\check{e}] < \check{x}_0$ $\qquad$ by (G7),(G8),(OA.89)

$(G10)$ $\quad \Rightarrow [\check{e}+++h[\varepsilon///j\check{e}]] < [\check{e}+++h\check{x}_0]$ $\quad$ by (OA.33),(OA.73),(OA.37)

$(G11)$ $\qquad \Rightarrow \varepsilon < [\check{e}+++e\check{x}_0]$ $\qquad$ by (Premise),(OA.77),(OA.90)

$(G12)$ $\qquad \Rightarrow \varepsilon < \check{f}(\check{x}_0)$ $\qquad$ by (Premise)

(G1)~(G12) derive that $\check{f}(\check{x})$ is unbounded. $\qquad\qquad\qquad\qquad\qquad$ □

After further studies with Theorem 3.4 and Theorem 3.5, we pose the conjectures as follows.

Conjecture 3.6. *For a function $\check{f} : (1, +\infty) \to R$ defined by $\check{f}(\check{x}) = [\check{x}+++e\check{d}]$, it is smooth.*

Conjecture 3.7. *For a function $\check{f} : (1, +\infty) \to R$ defined by $\check{f}(\check{x}) = [\check{e}+++e\check{x}]$, it is smooth.*

References

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