‘Spooky action at a distance’ in the Micropolar Electromagnetic Theory

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Abstract

Still now there are no theoretical background for explanation of physical phenomena of ‘spooky action at a distance’ as a quantum superposition of quantum particles. Several experiments shows that speed of this phenomena is at least four orders of magnitude of light speed in vacuum. The classical electromagnetic field theory is based on similarity to the classic dynamic of solid continuum media. The today’s experimental data of spin of photon is not reflected reasonable manner in Maxwell’s equations of EM. So, new proposed micropolar extensions of electromagnetic field equations cloud explain observed rotational speed of electromagnetic field, which experimentally exceed speed of light at least in four order of magnitude.

Introduction

In order to test the speed of ‘spooky action at a distance’ (Einstein, Podolsky, & Rosen, 1935), Eberhard proposed (Eberhard, 1989) a 12-hour continuous space-like Bell inequality (Bell, 1964; Clauser, Horne, Shimony, & Holt, 1969) measurement over a long east-west oriented distance. Benefited from the Earth self rotation, the measurement would be ergodic over all possible translation frames and as a result, the bound of the speed would be universal(Eberhard, 1989; Salart Daniel, Baas Augustin, Branciard Cyril, Gisin Nicolas, & Zbinden Hugo, 2008). Other authors (Salart Daniel et al., 2008; Yin et al., 2013) recently report to have achieved the lower bound of ‘spooky action’ through an experiment using Eberhard’s proposal at least four orders of magnitude of light speed in vacuum.

Still now there are no theoretical background for explanation of this physical phenomena. The aim of this article is to propose the useful theoretical explanation based on linear micropolar elasticity of continuum media.

1 Linear Elasticity

It is good known that dynamic linear elasticity is derivable form third Newton low for density of continuum

$$\rho\ddot{u}_i = \frac{\partial\sigma_{ji}}{\partial x_j} + F_i \tag{1}$$

$$\sigma_{ij} = C_{ij}^{kl}\varepsilon_{kl} \textit{ for isotropic media} \Rightarrow \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

where u_i are displacements in direction i, ε_{ij} are strain-displacements, $\sigma_{ij} = \sigma_{ji}$ symmetric force stress tensor, $F_i = \frac{\partial P}{\partial x_i}$ are body force components per unit volume and could be expressed on base of Gauss-Ostrogradsky theorem as gradient of pressure on boundaries, λ, μ are Lame’s constants. After inserting of expression of stress and strain into equation of motion we obtain

$$
\begin{aligned}
\rho\ddot{\mathbf{u}} &= \mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\nabla P &(4)\\
\mathbf{u} &= \nabla\phi+\nabla\times\psi &(5)\\
\rho\nabla\ddot{\phi} &= \mu\nabla^2\nabla\phi+\nabla P &(6)\\
\rho\nabla\times\ddot{\psi} &= (2\mu+\lambda)\nabla^2\nabla\times\psi &(7)
\end{aligned}
$$

where ϕ irrotational and ψ_i rotational or shear wave potentials could be found separately. So, linear elasticity could be described by two waves potentials: irrotational scalar potential for translational motion and by irrotational vector potential for shear motion.

2 Linear Micropolar Elasticity

Authors (Cosserat & Cosserat, 1909) proposed extension for dynamic linear elasticity by adding rotational motion

$$
\begin{aligned}
\rho\ddot{u}_i &= \frac{\partial\sigma_{ji}}{\partial x_j}+F_i &(8)\\
J\ddot{\phi}_i &= \varepsilon_{ijk}\sigma_{jk}+\frac{\partial\mu_{ji}}{\partial x_j}+M_i &(9)\\
\gamma_{ji} &= \frac{\partial u_i}{\partial x_j}-\varepsilon_{kji}\phi_k,\ \kappa_{ji}=\frac{\partial\phi_i}{\partial x_j} &(10)\\
\sigma_{ji} &= (\mu+\alpha)\gamma_{ji}+(\mu-\alpha)\gamma_{ij}+\lambda\delta_{ij}\gamma_{kk} &(11)\\
\mu_{ji} &= (\gamma+\varepsilon)\kappa_{ji}+(\gamma-\varepsilon)\kappa_{ij}+\beta\delta_{ij}\kappa_{kk} &(12)
\end{aligned}
$$

where ε_{ijk} is three dimensions Levi-Civita symbol is defined as follows:

$$
\varepsilon_{ijk}=\begin{cases}+1 & \text{if } (i,j,k) \text{ is } (1,2,3),(2,3,1) \text{ or } (3,1,2),\\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1),(1,3,2) \text{ or } (2,1,3),\\ 0 & \text{if } i=j \text{ or } j=k \text{ or } k=i\end{cases} \tag{13}
$$

J is rotational inertia density, ϕ_i is angular displacement, μ_{ji} is moment stress, F_i is body force density, M_i is body inertia moment density.

In the vector form the equations are as follow (Nowacki, 1974)

$$
\begin{aligned}
\Box_2\mathbf{u}+(\lambda+\mu-\alpha)\nabla(\nabla\cdot\mathbf{u})+2\alpha\nabla\times\phi+\mathbf{F} &= 0, &(14)\\
\Box_4\phi+(\beta+\gamma-\varepsilon)\nabla(\nabla\cdot\phi)+2\alpha\nabla\times\mathbf{u}+\mathbf{M} &= 0 &(15)
\end{aligned}
$$

where $\Box_2$ and $\Box_4$ are D'Alembert operators

$$
\Box_2=(\mu+\alpha)\nabla^2-\rho\partial_t^2, \Box_4=(\gamma+\varepsilon)\nabla^2-4\alpha-J\partial_t^2 \tag{16}
$$

Linear and angular displacements $\mathbf{u}$, ϕ , forces $\mathbf{F}$ and moments $\mathbf{M}$ could by decomposed by Helmhotz decomposition

$$
\begin{aligned}
\mathbf{u} &= \nabla\Phi+\nabla\times\Psi,\ \mathbf{F}=\rho(\nabla\vartheta+\nabla\times\chi),\ \nabla\cdot\chi=0, &(17)\\
\phi &= \nabla\Gamma+\nabla\times\mathbf{H},\ \mathbf{M}=J(\nabla\sigma+\nabla\times\eta),\ \nabla\cdot\eta=0 &(18)
\end{aligned}
$$

The results wave equations

$$
\begin{aligned}
\Box_1\Phi+\rho\vartheta &= 0, &(19)\\
\Box_3\Gamma+J\sigma &= 0, &(20)\\
\Box_2\Psi+2\alpha\nabla\times H+\rho\chi &= 0, &(21)\\
\Box_4 H+2\alpha\nabla\times\Psi+J\eta &= 0 &(22)
\end{aligned}
$$

where

$$\Box_1 = (\lambda + 2\mu)\nabla^2 - \rho\partial_t^2,\ \Box_3 = (\beta + 2\gamma)\nabla^2 - 4\alpha - J\partial_t^2 \tag{23}$$

Now, linear micropolar elasticity could be described by four waves potentials: irrotational scalar potential for translational motion, by irrotational vector potential for shear motion, by scalar potential for rotational motion and by vector potential for rotational motion.

3 Maxwell stress tensor

Starting with the Lorentz force law (Griffiths, 2008; Jackson, 1999; Becker, 1964)

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{24}$$

the force per unit volume for an unknown charge distribution is

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{25}$$

Next, ρ and $\mathbf{J}$ can be replaced by the fields $\mathbf{E}$ and $\mathbf{B}$, using Gauss's law and Ampère's circuital law:

$$\mathbf{f} = \varepsilon_0 \left(\nabla \cdot \mathbf{E}\right) \mathbf{E} + \frac{1}{\mu_0} \left(\nabla \times \mathbf{B}\right) \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} \tag{26}$$

The time derivative can be rewritten to something that can be interpreted physically, namely the Poynting vector. Using the product rule and Faraday's law of induction gives

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} - \mathbf{E} \times (\nabla \times \mathbf{E}) \tag{27}$$

and we can now rewrite $\mathbf{f}$ as

$$\mathbf{f} = \varepsilon_0 \left(\nabla \cdot \mathbf{E}\right) \mathbf{E} + \frac{1}{\mu_0} \left(\nabla \times \mathbf{B}\right) \times \mathbf{B} - \varepsilon_0 \frac{\partial}{\partial t} \left(\mathbf{E} \times \mathbf{B}\right) - \varepsilon_0 \mathbf{E} \times \left(\nabla \times \mathbf{E}\right) \tag{28}$$

then collecting terms with $\mathbf{E}$ and $\mathbf{B}$ gives

$$\mathbf{f} = \varepsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})\right] + \frac{1}{\mu_0} \left[-\mathbf{B} \times (\nabla \times \mathbf{B})\right] - \varepsilon_0 \frac{\partial}{\partial t} \left(\mathbf{E} \times \mathbf{B}\right) \tag{29}$$

A term seems to be "missing" from the symmetry in $\mathbf{E}$ and $\mathbf{B}$, which can be achieved by inserting $(\nabla \cdot \mathbf{B})\mathbf{B}$ because of Gauss' law for magnetism:

$$\mathbf{f} = \varepsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})\right] + \frac{1}{\mu_0} \left[(\nabla \cdot \mathbf{B})\mathbf{B} - \mathbf{B} \times (\nabla \times \mathbf{B})\right] - \varepsilon_0 \frac{\partial}{\partial t} \left(\mathbf{E} \times \mathbf{B}\right) \tag{30}$$

. Eliminating the curls (which are fairly complicated to calculate), using the vector calculus identity

$$\tfrac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A}) = \mathbf{A} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{A} \tag{31}$$

, leads to:

$$\mathbf{f} = \varepsilon_0 \left[(\nabla \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \nabla)\mathbf{E}\right] + \frac{1}{\mu_0} \left[(\nabla \cdot \mathbf{B})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}\right] - \frac{1}{2}\nabla \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) - \varepsilon_0 \frac{\partial}{\partial t} \left(\mathbf{E} \times \mathbf{B}\right) \tag{32}$$

This expression contains every aspect of electromagnetism and momentum and is relatively easy to compute. It can be written more compactly by introducing the Maxwell stress tensor,

$$\sigma_{ij} \equiv \varepsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right) \tag{33}$$

and notice that all but the last term of the above can be written as the divergence of this:

$$\mathbf{f}+\varepsilon_0\mu_0\frac{\partial \mathbf{S}}{\partial t} = \nabla\cdot\sigma \tag{34}$$

As in the Poynting's theorem, the second term in the left side of above equation can be interpreted as time derivative of EM field's momentum density and this way, the above equation will be the law of conservation of momentum in classical electrodynamics.

where we have finally introduced the Poynting vector,

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{35}$$

in the above relation for conservation of momentum, $\nabla\cdot\sigma$ is the momentum flux density and plays a role similar to $\mathbf{S}$ in Poynting's theorem.

In physics, the Maxwell stress tensor is the stress tensor of an electromagnetic field. As derived above in SI units, it is given by:

$$\sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j - \frac{1}{2}\left(\varepsilon_0 E^2 + \tfrac{1}{\mu_0}B^2\right)\delta_{ij}, \tag{36}$$

where ε_0 is the electric constant and μ_0 is the magnetic constant, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field and δ_{ij} is Kronecker's delta.

The element ij of the Maxwell stress tensor has units of momentum per unit of area times time and gives the flux of momentum parallel to the ith axis crossing a surface normal to the jth axis (in the negative direction) per unit of time.

These units can also be seen as units of force per unit of area (negative pressure), and the ij element of the tensor can also be interpreted as the force parallel to the ith axis suffered by a surface normal to the jth axis per unit of area. Indeed the diagonal elements give the tension (pulling) acting on a differential area element normal to the corresponding axis. Unlike forces due to the pressure of an ideal gas, an area element in the electromagnetic field also feels a force in a direction that is not normal to the element. This shear is given by the off-diagonal elements of the stress tensor.

4 Micropolar Electromagnetic field

Let decide, that coefficient $\alpha = 0$. We obtain independent rotational gradient ant curl wave equations (20),(22). Now, we could use analogy of eq. (12), the stress tensor of inertia momentum could be expressed as follow

$$\mu_{ji} = (\gamma+\varepsilon)C_j C_i + (\gamma-\varepsilon)C_i C_j + \beta\delta_{ij}C_k C_k \tag{37}$$

On other hand this tensor could be expressed using analogy of eq. (36) as follow

$$\mu_{ij} = \gamma_0 C_i C_j + \frac{1}{\beta_0}G_i G_j - \frac{1}{2}\left(\gamma_0 C^2 + \tfrac{1}{\beta_0}G^2\right)\delta_{ij}, \tag{38}$$

which could be derived by using formulas of seq. 3 in which $\mathbf{E}$ is replaced to $\mathbf{C}$ and $\mathbf{B}$ is replaced to $\mathbf{G}$. Now we could write force balance equalities for motion of micropolar electromagnetic continuum

$$\mathbf{f}_i + \varepsilon_0\mu_0\frac{\partial S_i}{\partial t} = \frac{\partial \sigma_{ji}}{\partial x_j} \tag{39}$$

$$\mathbf{f}_i^m + \beta_0\gamma_0\frac{\partial \Sigma_i}{\partial t} = \varepsilon_{ijk}\sigma_{jk} + \frac{\partial \mu_{ji}}{\partial x_j} \tag{40}$$

$$\sigma_{ji} = \varepsilon_0 E_j E_i + \frac{1}{\mu_0}B_j B_i - \frac{1}{2}\left(\varepsilon_0 E^2 + \tfrac{1}{\mu_0}B^2\right)\delta_{ji} \tag{41}$$

$$\mu_{ji} = \gamma_0 C_j C_i + \frac{1}{\beta_0}G_j G_i - \frac{1}{2}\left(\gamma_0 C^2 + \tfrac{1}{\beta_0}G^2\right)\delta_{ji} \tag{42}$$

where vector Σ is rotational Pointing's vector of micropolar electromagnetic field and equals to

$$\Sigma = \frac{1}{\beta_0}\mathbf{C}\times\mathbf{G} \tag{43}$$

The same way, **C** and **G** vectors are gradient of scalar rotational electromagnetic field and curl of vector rotational electromagnetic field as follow

$$\mathbf{C} = \nabla\phi_C \tag{44}$$
$$\mathbf{G} = \nabla\times\mathbf{A}_G \tag{45}$$

5 Micropolar Maxwell equations

In 1895 it was proposed by author (Maxwell, 1865) dynamic electromagnetic field theory which today is known as system of four differential equations.

- Gauss's law
$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{46}$$
- Gauss's law for magnetism
$$\nabla\cdot\mathbf{B} = 0 \tag{47}$$
- Maxwell–Faraday equation (Faraday's law of induction)
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{48}$$
- Ampère's circuital law (with Maxwell's addition)
$$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \tag{49}$$

The same way, we could write Maxwell equations for rotational components using micropolar elasticity analogy for coefficient $\alpha = 0$ as follow

- Gauss's law for micropolar rotational electric field
$$\nabla\cdot\mathbf{C} = \frac{\rho_C}{\gamma_0} \tag{50}$$
- Gauss's law for for micropolar rotational magnetic field
$$\nabla\cdot\mathbf{G} = 0 \tag{51}$$
- Micropolar Maxwell–Faraday equation (Faraday's law of induction)
$$\nabla\times\mathbf{C} = -\frac{\partial\mathbf{G}}{\partial t} \tag{52}$$
- Micropolar Ampère's circuital law (with Maxwell's addition)
$$\nabla\times\mathbf{G} = \beta_0\left(\mathbf{J}_G+\gamma_0\frac{\partial\mathbf{C}}{\partial t}\right) \tag{53}$$

So, proposed equations describe than the translational but also rotational motion of micropolar electromagnetic field.

6 Micropolar Electromagnetic Waves

In a region with no charges ($\rho = 0$) and no currents ($J = 0$), such as in a vacuum, rotational part of Maxwell's equations reduce to:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 & (54)\\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & (55)\\ \nabla \cdot \mathbf{B} &= 0 & (56)\\ \nabla \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. & (57) \end{aligned}$$

Taking the curl ($\nabla\times$) of the curl equations, and using the curl of the curl identity $\nabla \times (\nabla \times \mathbf{X}) = \nabla(\nabla \cdot \mathbf{X}) - \nabla^2 \mathbf{X}$ we obtain the wave equations

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} &= 0, & (58)\\ \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} &= 0, & (59) \end{aligned}$$

which identify

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99792458 \times 10^8 \,\mathrm{m\ s^{-1}} \tag{60}$$

with the speed of light in free space.

The same way, in a region with no charges ($\rho_C = 0$) and no currents ($J_G = 0$), such as in a vacuum, rotational part of micropolar Maxwell's equations reduce to:

$$\begin{aligned} \nabla \cdot \mathbf{C} &= 0 & (61)\\ \nabla \times \mathbf{C} &= -\frac{\partial \mathbf{G}}{\partial t}, & (62)\\ \nabla \cdot \mathbf{G} &= 0 & (63)\\ \nabla \times \mathbf{B} &= \frac{1}{c_R^2}\frac{\partial \mathbf{C}}{\partial t}. & (64) \end{aligned}$$

Taking the curl ($\nabla\times$) of the curl equations, and using the curl of the curl identity $\nabla \times (\nabla \times \mathbf{X}) = \nabla(\nabla \cdot \mathbf{X}) - \nabla^2 \mathbf{X}$ we obtain the wave equations

$$\begin{aligned} \frac{1}{c_R^2}\frac{\partial^2 \mathbf{C}}{\partial t^2} - \nabla^2 \mathbf{C} &= 0, & (65)\\ \frac{1}{c_R^2}\frac{\partial^2 \mathbf{G}}{\partial t^2} - \nabla^2 \mathbf{G} &= 0, & (66) \end{aligned}$$

which identify

$$c_R = \frac{1}{\sqrt{\beta_0 \gamma_0}} >= 1.38 \times 10^4 c = 4.13713592 \times 10^{12} \,\mathrm{m\ s^{-1}} \tag{67}$$

with the rotational speed of light in free space according experiments made by physicists (Salart Daniel et al., 2008; Yin et al., 2013).

7 Discussion

According to special relativity, the energy of an object with rest mass m and speed v is given by γmc^2, where γ is the Lorentz factor defined above. When v is zero, γ is equal to one, giving rise to the famous $E = mc^2$ formula for mass -energy equivalence. The γ factor approaches infinity as v approaches c, and it would take an infinite amount of energy to accelerate an object with mass to the speed of light. The speed of light is the upper limit for the speeds of objects with positive rest mass. This is experimentally established in many tests of relativistic energy and momentum (Fowler, March 2008). This theory was originally proposed in 1905 by Albert Einstein in the paper "On the Electrodynamics of Moving Bodies" (Einstein et al., 1935). But this derivation is based on translational motion. The same way upper limit for translational speed follows from Lorentz transformation invariant of translational wave functions (see Appendix). If we have rotational waves equations, they are invariant on Lorentz transform too or in other words exist upper limit for wave motion speed for such kind of wave motion. But this does not mean that translational wave speed must be equal to rotational wave speed. The nature will be too sample if so happens. So, what is the speed of rotational wave motion could say just experiment. If polarisation of light could propagate as rotational waves in vacuum, the speed of this propagation is $c_R = 1.38 \times 10^4 c$, where $c = 2.99792458 \times 10^8\,\mathrm{m\,s^{-1}}$ is translational wave propagation speed in vacuum. That is keynote statement of this article.

Conclusions

It was proposed micropolar extensions of electromagnetic field equations. This equations could be reasonable explanation of observed rotational speed of electromagnetic field waves, which motion experimentally exceed speed of light at least in four order of magnitude.

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Appendix - Derivation of Lorentz transformations

Requiring the form of the wave equation to remain invariant under linear transformation of coordinates forces this linear transformation to match the Einstein-Lorentz-Minkowski transformation of special relativity (Feinstein, 2014).

Begin by considering a general transformation form one coordinate system to another

$$\begin{aligned} x' &= x'(x,t) \\ t' &= t'(x,t) \end{aligned}$$

The multivariate chain rule relates derivatives on one set of variables to derivatives on the other set of variables. Thus

$$\begin{aligned}\partial_x &= \partial_x x' \partial_{x'} + \partial_x t' \partial_{t'} \\ \partial_t &= \partial_t x' \partial_{x'} + \partial_t t' \partial_{t'}\end{aligned}$$

The second derivative is

$$\begin{aligned}\partial^2_{xx} &= \partial_x(\partial_x x' \partial_{x'} + \partial_x t' \partial_{t'}) \\ \partial^2_{tt} &= \partial_t(\partial_t x' \partial_{x'} + \partial_t t' \partial_{t'})\end{aligned}$$

The law governing the derivative of a product facilitates expansion of expression. Thus

$$\begin{aligned}\partial^2_{xx} &= \partial^2_{xx}\partial_{x'} + \partial_x x'(\partial_x x' \partial^2_{x'x'} + \partial_x t' \partial^2_{t'x'}) + \partial^2_{xx}\partial t' \partial'_t + \partial_x t'(\partial_x x' \partial^2_{x't'} + \partial_x t' \partial^2_{t't'}) \\ \partial^2_{tt} &= \end{aligned}$$

Combining and rearranging terms in this expression leads to a simpler and more compact result:

$$\partial^2_{xx} = \partial^2_{xx} x' \partial_{x'} + \partial^2_{xx} t' \partial_{t'} + (\partial_x x')^2 \partial^2_{x'x'} + 2\partial_x x' \partial_x t' \partial^2_{x't'} + (\partial_x t')^2 \partial^2_{t't'}$$

and

$$\partial^2_{tt} = \partial^2_{tt} x' \partial_{x'} + \partial^2_{tt} t' \partial_{t'} + (\partial_t x')^2 \partial^2_{x'x'} + 2\partial_t x' \partial_t t' \partial^2_{x't'} + (\partial_t t')^2 \partial^2_{t't'}$$

on account of the complete symmetry of x and t in the transformation functions x' and t'. Notice that when x' and t' are linear combination of x and t, the first two terms on the right are zero because these terms all involve second derivatives of the transformations.

Consider the wave equation with driving term Φ:

$$\partial^2_{tt}\Psi - c^2 \partial^2_{xx} = \Phi$$

How does the wave equation appear in the transformed variables x' and $t^{|}prime$ in the special case where these are linear combinations of x and t? Using the transformations derived above and collecting terms appropriately yields:

$$((\partial_t t')^2 - c^2(\partial_x t')^2)\partial^2_{tt}\Psi - ((\partial_x x')^2 - (\partial_t x')^2/c^2)c^2\partial^2_{x'x'}\Psi = 2(c^2\partial_x x' \partial_x t' - \partial_t x' \partial_t t')\partial^2_{x't'} + \Phi$$

Suppose in particular that

$$\begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} \alpha & \beta c \\ \gamma c^{-1} & \delta \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$

, than the transformed wave equation becomes

$$(\delta^2 - \gamma^2)\partial^2_{t't'}\Psi - (\alpha^2 - \beta^2)c^2\partial^2_{x'x'} = 2c(\alpha\gamma - \beta\delta)\partial^2_{x't'} + \Phi$$

When $\delta^2 - \gamma^2 = 1$ and $\alpha^2 - \beta^2 = 1$. the parenthesized terms on the left are unity, and when $\alpha\gamma - \beta\delta = 0$, the parametrized term on the right is 0, and only when these 3 constrains hold does the form of the wave equation in the transformed coordinates precisely match its form in original coordinates.

The constraint $\delta^2 - \gamma^2 = 1$ requires $\delta = \cosh(u)$, $\gamma = \sinh(u)$ for some u, similarly, the constraint $\alpha^2 - \beta^2 = 1$ requires $\alpha = \cosh(u)$, $\beta = \sinh(v)$ for some v. In terms of u and v, the constraint $\alpha\gamma - \beta\delta = 0$ becomes $\cosh(v)\sinh(u) - \sinh(v)\cosh(u) = 0$, of, expressed compactly, $\sinh(u - v) = 0$; and this uniquely requires $v = u + in\pi$, where n is an integer.

Thus, the 1-dimensional family of transformations,

$$\begin{bmatrix} x' \\ \pm t' \end{bmatrix} = \begin{bmatrix} \cosh(u) & c\sinh(u) \\ c^{-1}\sinh(u) & \cosh(u) \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$

parametrized by parameter u, are the complete set of linear transformations under which the wave equation remains invariant form. But this is none other than the Lorentz transformation in elegant guise.

To find what u actually is, from the standard configuration the origin of the primed frame $x' = 0$ is measured in the unprimed frame to be $x = vt$ (or the equivalent and opposite way round; the origin of the unprimed frame is $x = 0$ and in the primed frame it is at $x' = -vt$):

$$0 = \cosh(u)vt - \sinh(u)ct \Rightarrow \tanh(u) = \frac{v}{c} = \beta$$

and manipulation of hyperbolic identities leads to

$$\cosh(u) = \gamma, \sinh(u) = \beta\gamma$$

so the transformations are also:

$$\begin{aligned} x' &= \gamma x - \frac{\gamma v}{c} ct \Rightarrow x' = \gamma(x - vt) \\ ct' &= -\frac{\gamma v}{c} x + \gamma ct \Rightarrow t' = \gamma\left(t - \frac{vx}{c^2}\right) \\ \gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \end{aligned}$$