Some Generalized Neutrosophic Number Hamacher Aggregation Operators and Their Application to Group Decision Making

Peide Liu, Yanchang Chu, Yanwei Li, and Yubao Chen

Abstract

The neutrosophic set (NS) can be better to express the incomplete, indeterminate and inconsistent information, and Hamacher aggregation operators extended the Algebraic and Einstein aggregation operators and the generalized aggregation operators can generalize the arithmetic, geometric, and quadratic aggregation operators. In this paper, we combined Hamacher operations and generalized aggregation operators to NS, and proposed some new aggregation operators. Firstly, we presented some new operational laws for neutrosophic numbers (NNs) based on Hamacher operations and studied their properties. Then, we proposed the generalized neutrosophic number Hamacher weighted averaging (GNNHWA) operator, generalized neutrosophic number Hamacher ordered weighted averaging (GNNHOWA) operator, and generalized neutrosophic number Hamacher hybrid averaging (GNNHHA) operator, and explored some properties of these operators and analyzed some special cases of them. Furthermore, we gave a new method based on these operators for multiple attribute group decision making problems with neutrosophic numbers, and the operational steps were illustrated in detail. Finally, an application example is given to verify the proposed method and to demonstrate its effectiveness.

Keywords: Group decision-making, neutrosophic set (NS), Hamacher aggregation operator, generalized aggregation operator.

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1. Introduction

In real decision making, the decision information is often incomplete, indeterminate and inconsistent information. Zadeh [1] firstly proposed the fuzzy set theory which is a very useful tool to process fuzzy information. However, it has a shortcoming that it only has a membership function, and cannot express non-membership function. Then Atanassov [2, 3] proposed the intuitionistic fuzzy set (IFS) by adding a non-membership function, i.e., the intuitionistic fuzzy sets consider both membership (or called truth-membership) $T_A(x)$ and non-membership (or called falsity-membership) $F_A(x)$ and satisfy the conditions $T_A(x), F_A(x) \in [0,1]$ and $0 \le T_A(x) + F_A(x) \le 1$. IFSs can only handle incomplete information not the indeterminate information and inconsistent information. In IFSs, the indeterminacy (or called Hesitation degree) is $1\text{-}T_A(x)\text{-}F_A(x)$ by default. Further, on the basis of IFS, Smarandache [4] proposed the neutrosophic set (NS) by adding an independent indeterminacy-membership $I_A(x)$, which is a generalization of IFSs. When $I_A(x) = 1\text{-}T_A(x)\text{-}F_A(x)$, NS will become the IFS, i.e., IFS is a special case of NS. Because the indeterminacy is quantified explicitly, and truth-membership, indeterminacy membership, and false-membership are completely independent, NSs can handle the incomplete, indeterminate and inconsistent information. When $T_A(x) + I_A(x) + F_A(x) < 1$, it shows this is indeterminate information, and when $T_A(x) + I_A(x) + F_A(x) > 1$, it is inconsistent information.

Recently, NSs have caused the widespread concerns and made some applications. Wang et al. [5] proposed a single valued neutrosophic set (SVNS) by adding the conditions $T_A(x), I_A(x), F_A(x) \in [0,1]$, and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$. Obviously, the SVNS is an instance of the neutrosophic set. Ye [6] proposed the correlation coefficient and weighted correlation coefficient of SVNSs, and proved that the cosine similarity degree is a special case of the correlation coefficient in SVNS. Further, a comparison method for SVNSs based on the correlation coefficient was proposed. Similar to extension from IFS to interval-valued intuitionistic fuzzy set (IVIFS) [7, 8], Wang et al. [9] defined interval neutrosophic sets (INSs) in which the truth-membership, indeterminacy-membership, and false-membership were extended to interval numbers, and discussed various properties of INSs. INSs can easily express the incomplete, indeterminate and inconsistent information. Ye [10] defined the similarity measures between INSs on the basis of the Hamming and Euclidean distances, and proposed a multiple attribute decision-making method based on the similarity degree. Guo et al. [11], Guo and Cheng [12] applied neutrosophic sets to process the images with noise and proposed a new neutrosophic approach for image segmentation. However, so far, there has been no research on aggregation operators for INSs.

The information aggregation operators are the important research areas, which are receiving wide attentions [13-29]. Yager [13] proposed the ordered weighted average (OWA) operator which weighted the inputs according to the ranking position of them. Xu [14], Xu and Yager [15] proposed some arithmetic aggregation operators and geometric aggregation operators for intuitionistic fuzzy information. Zhao [16] extended generalized aggregation operators to intuitionistic fuzzy sets and proposed generalized weighted operator, generalized ordered weighted operator, and generalized hybrid operator for intuitionistic fuzzy information. The generalized aggregation operators can generalize arithmetic and geometric aggregation operators. All above aggregation operators are based on the algebraic operational rules of intuitionistic fuzzy numbers (IFNs) which are one type of operations that can be chosen to model the intersection and union of IFNs. In general, a general t-norm and t-conorm can always be used to model the intersection and union of IFNs [17, 18]. Wang and Liu [19] proposed the some Einstein aggregation operators, which have the same smooth approximations as the algebraic operators. Further, they were extended to intuitionistic fuzzy sets.

Hamacher t-conorm and t-norm are the generalization of algebraic and Einstein t-conorm and t-norm [20], Liu [21] proposed some Hamacher aggregation operators for the interval-valued intuitionistic fuzzy numbers. However, until to now, there is no research about neutrosophic number aggregation operators based on Hamacher t-conorm and t-norm. Because the NS can be better to express the incomplete, indeterminate and inconsistent information, and Hamacher aggregation operators extend the Algebraic and Einstein aggregation operators and the generalized aggregation operators can generalize the arithmetic, geometric and quadratic aggregation operators. So, it is meaningful to research the aggregation operators based on Frank operations and the generalized aggregation operators for SVNSs, and apply them to MAGDM problems with neutrosophic information.

The remainder of this paper is shown as follows. In Section 2, we briefly review some basic concepts and operational rules of SVNS, propose some new operational laws for neutrosophic numbers based on Hamacher t-conorm and t-norm and discuss some properties. In Section 3, we propose the generalized neutrosophic number Hamacher weighted averaging (GNNHWA) operator, generalized neutrosophic number Hamacher ordered weighted averaging (GNNHOWA) operator, and generalized neutrosophic number Hamacher hybrid averaging (GNNHHA) operator, discussed various desirable properties of these operators and analyze some special cases of them. In Section 4, we propose a new method based on these operators for multi-attribute group decision making with SVNNs. In Section 5, we give an example to show the application of proposed method, and compare the developed method with the existing methods. In Section 6, we end this paper with some conclusions.

2. Preliminaries

A. *The single valued neutrosophic set*

Definition 1 [4]: Let X be a universe of discourse, with a generic element in X denoted by x. A neutrosophic set A in X is

$$A = \{x(T_A(x), I_A(x), F_A(x)) \mid x \in X\} \quad (1)$$

where, T_A is the truth-membership function, I_A is the indeterminacy-membership function, and F_A is the falsity-membership function. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$ which is proposed by Abraham Robinson in 1960s [30].

There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

Because neutrosophic set was difficult to apply in the real applications. Wang [5] further proposed the single valued neutrosophic set (SVNS) from scientific or engineering point of view, which is a generalization of classical set, fuzzy set, intuitionistic fuzzy set and paraconsistent sets etc., and it can be defined as follows.

Definition 2 [5]: Let X be a universe of discourse, with a generic element in X denoted by x. A single valued neutrosophic set A in X is

$$A = \{x(T_A(x), I_A(x), F_A(x)) \mid x \in X\} \quad (2)$$

where, T_A is the truth-membership function, I_A is the indeterminacy-membership function, and F_A is the falsity-membership function. For each point x in X, we have $T_A(x), I_A(x), F_A(x) \in [0,1]$, and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

For convenience, we can simply use $x = (T, I, F)$ to represent an element x in SVNS, and the element x can be called a single valued neutrosophic number (SVNN).

Definition 3 [31]: Suppose $x = (T_1, I_1, F_1)$ is a SVNN, and then

(1) $sc(x)=T_1+1-I_1+1-F_1$;

(2) $ac(x)=T_1-F_1$

where $sc(x)$ and $ac(x)$ represent the score function and accuracy function of the SVNN, respectively.

Definition 4 [31]: Let $x=(T_1,I_1,F_1)$ and $y=(T_2,I_2,F_2)$ be two SVNNs, The comparison approach can be defined as follows.

(1) if $sc(x)>sc(y)$, then x is greater than y and denoted by $x\succ y$.

(2) if $sc(x)=sc(y)$ and $ac(x)>ac(y)$, then x is greater than y and denoted by $x\succ y$.

(3) if $sc(x)=sc(y)$ and $ac(x)=ac(y)$, then x is equal to y, and denoted by $x\sim y$.

B. 2.2 GHWA operator

Definition 5 [16]: A GWA operator of dimension n is a mapping GWA: $\left(R^+\right)^n\to R^+$. Such that,

$$\mathrm{GWA}(a_1,a_2,\cdots,a_n)=\left(\sum_{j=1}^{n}w_ja_j^{\lambda}\right)^{1/\lambda} \tag{3}$$

where $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $(a_1,a_2,\cdots,a_n)$ with the conditions $w_j\in[0,1](j=1,2,\cdots,n)$ and $\sum_{j=1}^{n}w_j=1$. w can be obtained by AHP method proposed Saaty [32]. λ is a parameter such that $\lambda\in(-\infty,0)\cup(0,+\infty)$, and R^+ is the set of all nonnegative real numbers.

Definition 6 [16]: A GOWA operator of dimension n is a mapping GOWA: $\left(R^+\right)^n\to R^+$. Such that,

$$\mathrm{GOWA}(a_1,a_2,\cdots,a_n)=\left(\sum_{j=1}^{n}\omega_jb_j^{\lambda}\right)^{1/\lambda} \tag{4}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ is the weight vector which is associated with GOWA, and with the conditions $\omega_j\in[0,1](j=1,2,\cdots,n)$ and $\sum_{j=1}^{n}\omega_j=1$; b_j is the jth largest of real numbers $a_k(k=1,2,\cdots,n)$. λ is a parameter such that $\lambda\in(-\infty,0)\cup(0,+\infty)$, and R^+ is the set of all nonnegative real numbers.

Definition 7 [16]: A GHWA operator of dimension n is a mapping GHWA: $\left(R^+\right)^n\to R^+$. Such that,

$$\mathrm{GHWA}(a_1,a_2,\cdots,a_n)=\left(\sum_{j=1}^{n}\omega_jb_j^{\lambda}\right)^{1/\lambda} \tag{5}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ is the weight vector which is associated with GOWA, and satisfying $\omega_j\in[0,1](j=1,2,\cdots,n)$ and $\sum_{j=1}^{n}\omega_j=1$; b_j is the jth largest of real numbers $\left(nw_ka_k\right)(k=1,2,\cdots,n)$, $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of $(a_1,a_2,\cdots,a_n)$ with the conditions $w_k\in[0,1](k=1,2,\cdots,n)$ and $\sum_{k=1}^{n}w_k=1$. λ is a parameter such that $\lambda\in(-\infty,0)\cup(0,+\infty)$, and R^+ is the set of all nonnegative real numbers.

C. Hamacher operators

The t-operators are in fact Union and Intersection operators in fuzzy set theory which are symbolized by T-conorm (Γ^*) and T-norm (Γ), respectively [33]. Based on the T-norm and T-conorm, a generalized union and a generalized intersection for intuitionistic fuzzy sets were introduced by Deschrijver and Kerre [34], and a generalized union and a generalized intersection of the single valued neutrosophic numbers were introduced by Smarandache and Vlâdâreanu [35].

Definition 8 [35]: Let $x=(T_1,I_1,F_1)$ and $y=(T_2,I_2,F_2)$ are any two single valued neutrosophic numbers, then, the generalized intersection and union are defined as follows:

$$x\cap_{\Gamma,\Gamma^*}y=\left(\Gamma(T_1,T_2),\Gamma^*(I_1,I_2),\Gamma^*(F_1,F_2)\right) \tag{6}$$

$$x\cup_{\Gamma,\Gamma^*}y=\left(\Gamma^*(T_1,T_2),\Gamma(I_1,I_2),\Gamma(F_1,F_2)\right) \tag{7}$$

where Γ denotes a T-norm and Γ^* a T-conorm.

Some special examples of T-norms and T-conorms are listed as follows [19]:

(1) Algebraic T-norm and T-conorm

$$\Gamma(x,y)=x\times y \quad\text{and}\quad \Gamma^*(x,y)=x+y-x\times y \tag{8}$$

(2) Einstein T-norm and T-conorm [19]

$$\Gamma(x,y)=\frac{x\times y}{1+(1-x)\times(1-y)} \quad\text{and}\quad \Gamma^*(x,y)=\frac{x+y}{1+x\times y} \tag{9}$$

(3) Hamacher T-norm and T-conorm [36].

$$\Gamma_\gamma(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)},\gamma>0 \tag{10}$$

$$\Gamma_\gamma^*(x,y)=\frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy},\gamma>0 \tag{11}$$

Especially, when $\gamma=1$, then Hamacher T-norm and T-conorm will reduce to $\Gamma(x,y)=xy$ and $\Gamma^*(x,y)=x+y-xy$ which are the Algebraic T-norm and T-conorm respectively; when $\gamma=2$, then Hamacher T-norm and T-conorm will reduce to $\Gamma(x,y)=\frac{xy}{1+(1-x)(1-y)}$, and $\Gamma^*(x,y)=\frac{x+y}{1+xy}$ which are the Einstein T-norm and T-conorm respectively [19].

D. *The operational rules of SVNNs based on Hamacher T-norm and T-conorm*

Based on the Definition 8, we can establish the operational rules of SVNNs.

Let $\tilde{a}_1=(T_1,I_1,F_1)$ and $\tilde{a}_2=(T_2,I_2,F_2)$ be two SVNNs, and $\gamma,n>0$, then the operational rules based on Hamacher T-norm and T-conorm are defined as follows.

$$\tilde{a}_1\oplus_h\tilde{a}_2=\left(\frac{T_1+T_2-T_1T_2-(1-\gamma)T_1T_2}{1-(1-\gamma)T_1T_2},\frac{I_1I_2}{\gamma+(1-\gamma)(I_1+I_2-I_1I_2)},\frac{F_1F_2}{\gamma+(1-\gamma)(F_1+F_2-F_1F_2)}\right)\tag{12}$$

$$\tilde{a}_1\otimes_h\tilde{a}_2=\left(\frac{T_1T_2}{\gamma+(1-\gamma)(T_1+T_2-T_1T_2)},\frac{I_1+I_2-I_1I_2-(1-\gamma)I_1I_2}{1-(1-\gamma)I_1I_2},\frac{F_1+F_2-F_1F_2-(1-\gamma)F_1F_2}{1-(1-\gamma)F_1F_2}\right)\tag{13}$$

$$n\tilde{a}_1=\left(\frac{\left(1+(\gamma-1)T_1\right)^n-(1-T_1)^n}{\left(1+(\gamma-1)T_1\right)^n+(\gamma-1)(1-T_1)^n},\frac{\gamma I_1^n}{\left(1+(\gamma-1)(1-I_1)\right)^n+(\gamma-1)I_1^n},\frac{\gamma F_1^n}{\left(1+(\gamma-1)(1-F_1)\right)^n+(\gamma-1)F_1^n}\right)\tag{14}$$

$$\tilde{a}_1^n=\left(\frac{\gamma T_1^n}{\left(1+(\gamma-1)(1-T_1)\right)^n+(\gamma-1)T_1^n},\frac{\left(1+(\gamma-1)I_1\right)^n-(1-I_1)^n}{\left(1+(\gamma-1)I_1\right)^n+(\gamma-1)(1-I_1)^n},\frac{\left(1+(\gamma-1)F_1\right)^n-(1-F_1)^n}{\left(1+(\gamma-1)F_1\right)^n+(\gamma-1)(1-F_1)^n}\right)\tag{15}$$

Theorem 1: Let $\tilde{a}_1=(T_1,I_1,F_1)$ and $\tilde{a}_2=(T_2,I_2,F_2)$ be any two SVNNs, and $\gamma>0$, then

(1) $\tilde{a}_1\oplus_h\tilde{a}_2=\tilde{a}_2\oplus_h\tilde{a}_1$ (16)

(2) $\tilde{a}_1\otimes_h\tilde{a}_2=\tilde{a}_2\otimes_h\tilde{a}_1$ (17)

(3) $\eta(\tilde{a}_1\oplus_h\tilde{a}_2)=\eta\tilde{a}_1\oplus_h\eta\tilde{a}_2,\eta\ge 0$ (18)

(4) $\eta_1\tilde{a}_1\oplus_h\eta_2\tilde{a}_1=(\eta_1+\eta_2)\tilde{a}_1,\ \eta_1,\eta_2\ \ge 0$ (19)

(5) $\tilde{a}_1^{\eta_1}\otimes_h\tilde{a}_1^{\eta_2}=(\tilde{a}_1)^{\eta_1+\eta_2},\ \eta_1,\eta_2\ \ge 0$ (20)

(6) $\tilde{a}_1^{\eta}\otimes_h\tilde{a}_2^{\eta}=(\tilde{a}_1\otimes_h\tilde{a}_2)^{\eta},\ \eta\ge 0$ (21)

It is easy to prove the formulas in Theorem 1, omitted in here.

3. Some Generalized Neutrosophic Bumber Hamacher Weighted Aggregation Operators

In this section, we will combine Hamacher operations and the generalized aggregation operators to SVNSs, and develop some generalized neutrosophic number Hamacher weighted aggregation operators.

Definition 9: Let $\tilde{a}_j=(T_j,I_j,F_j)$ $(j=1,2\cdots,n)$ be a collection of the SVNNs, and $GNNHWA:\Omega^n\to\Omega$, if

$$GNNHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\bigoplus_{j=1}^{n}{}_h\left(w_j\tilde{a}_j^{\lambda}\right)\right)^{1/\lambda}\tag{22}$$

where Ω is the set of all SVNNs, and $\lambda>0$, $w=\left(w_1,w_2,\cdots,w_n\right)^T$ is weight vector of $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$, such that $w_j\ge 0$ and $\sum_{j=1}^{n}w_j=1$. Then GNNHWA is called the generalized neutrosophic number Hamacher weighted averaging operator.

Based on the Hamacher operational rules of the SVNNs, we can get the result aggregated from Definition 9 shown as theorem 2.

Theorem 2: Let $\tilde{a}_j=(T_j,I_j,F_j)$ $(j=1,2\cdots,n)$ be a collection of the SVNNs and $\lambda>0$, then the result aggregated from Definition 9 is still an SVNN, and even

$$GNNHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\frac{\gamma\left(\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}x_j^{w_j}+(\gamma^2-1)\prod_{j=1}^{n}y_j^{w_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}\right)^{1/\lambda}},\right.$$
$$\frac{\left(\prod_{j=1}^{n}z_j^{w_j}+(\gamma^2-1)\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}z_j^{w_j}-\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}z_j^{w_j}+(\gamma^2-1)\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}z_j^{w_j}-\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}},\tag{23}$$
$$\left.\frac{\left(\prod_{j=1}^{n}u_j^{w_j}+(\gamma^2-1)\prod_{j=1}^{n}v_j^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}u_j^{w_j}-\prod_{j=1}^{n}v_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}u_j^{w_j}+(\gamma^2-1)\prod_{j=1}^{n}v_j^{w_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}u_j^{w_j}-\prod_{j=1}^{n}v_j^{w_j}\right)^{1/\lambda}}\right)$$

where $x_j=\left(1+(\gamma-1)(1-T_j)\right)^{\lambda}+(\gamma^2-1)T_j^{\lambda}$,

$y_j=\left(1+(\gamma-1)(1-T_j)\right)^{\lambda}-T_j^{\lambda}$

$z_j=\left(1+(\gamma-1)I_j\right)^{\lambda}+(\gamma^2-1)(1-I_j)^{\lambda}$,

$t_j=\left(1+(\gamma-1)I_j\right)^{\lambda}-(1-I_j)^{\lambda}$,

$u_j=\left(1+(\gamma-1)F_j\right)^{\lambda}+(\gamma^2-1)(1-F_j)^{\lambda}$,

$v_j=\left(1+(\gamma-1)F_j\right)^{\lambda}-(1-F_j)^{\lambda}$, here $\gamma>0$.

Proof:

(1) From (22), we can calculate $\tilde{a}_j^{\lambda}$ firstly, and get

$$\tilde{a}_j^{\lambda}=\left(\frac{\gamma T_j^{\lambda}}{\left(1+(\gamma-1)(1-T_j)\right)^{\lambda}+(\gamma-1)T_j^{\lambda}},\frac{\left(1+(\gamma-1)I_j\right)^{\lambda}-(1-I_j)^{\lambda}}{\left(1+(\gamma-1)I_j\right)^{\lambda}+(\gamma-1)(1-I_j)^{\lambda}},\frac{\left(1+(\gamma-1)F_j\right)^{\lambda}-(1-F_j)^{\lambda}}{\left(1+(\gamma-1)F_j\right)^{\lambda}+(\gamma-1)(1-F_j)^{\lambda}}\right)$$

(2) Calculate $w_j\tilde{a}_j^{\lambda}$, and get

$$w_j\tilde{a}_j^{\lambda} =$$

$$\left(\frac{\left(\left(1+(\gamma-1)(1-T_j)\right)^{\lambda}+(\gamma^2-1)T_j^{\lambda}\right)^{w_j}-\left(\left(1+(\gamma-1)(1-T_j)\right)^{\lambda}-T_j^{\lambda}\right)^{w_j}}{\left(\left(1+(\gamma-1)(1-T_j)\right)^{\lambda}+(\gamma^2-1)T_j^{\lambda}\right)^{w_j}+((\gamma-1))\left(\left(1+(\gamma-1)(1-T_j)\right)^{\lambda}-T_j^{\lambda}\right)^{w_j}},\right.$$

$$\frac{\gamma\left(\left(1+(\gamma-1)I_j\right)^{\lambda}-(1-I_j)^{\lambda}\right)^{w_j}}{\left(\left(1+(\gamma-1)I_j\right)^{\lambda}+(\gamma^2-1)(1-I_j)^{\lambda}\right)^{w_j}+(\gamma-1)\left(\left(1+(\gamma-1)I_j\right)^{\lambda}-(1-I_j)^{\lambda}\right)^{w_j}},$$

$$\left.\frac{\gamma\left(\left(1+(\gamma-1)F_j\right)^{\lambda}-(1-F_j)^{\lambda}\right)^{w_j}}{\left(\left(1+(\gamma-1)F_j\right)^{\lambda}+(\gamma^2-1)(1-F_j)^{\lambda}\right)^{w_j}+(\gamma-1)\left(\left(1+(\gamma-1)F_j\right)^{\lambda}-(1-F_j)^{\lambda}\right)^{w_j}}\right)$$

(3) Calculate $\bigoplus_{h\,j=1}^{n}\left(w_j\tilde{a}_j^{\lambda}\right)$.

For convenience, let $x_j=\left(1+(\gamma-1)(1-T_j)\right)^{\lambda}+(\gamma^2-1)T_j^{\lambda}$,

$y_j=\left(1+(\gamma-1)(1-T_j)\right)^{\lambda}-T_j^{\lambda}$

$z_j=\left(1+(\gamma-1)I_j\right)^{\lambda}+(\gamma^2-1)(1-I_j)^{\lambda}$, $t_j=\left(1+(\gamma-1)I_j\right)^{\lambda}-(1-I_j)^{\lambda}$,

$u_j=\left(1+(\gamma-1)F_j\right)^{\lambda}+(\gamma^2-1)(1-F_j)^{\lambda}$, $v_j=\left(1+(\gamma-1)F_j\right)^{\lambda}-(1-F_j)^{\lambda}$,

Then

$$w_j\tilde{a}_j^{\lambda}=\left(\frac{x_j^{w_j}-y_j^{w_j}}{x_j^{w_j}+(\gamma-1)y_j^{w_j}},\frac{\gamma t_j^{w_j}}{z_j^{w_j}+(\gamma-1)t_j^{w_j}},\frac{\gamma v_j^{w_j}}{u_j^{w_j}+(\gamma-1)v_j^{w_j}}\right).$$

In the following, by Mathematical induction, we can prove

$$\bigoplus_{h\,j=1}^{n}\left(w_j\tilde{a}_j^{\lambda}\right)=\left(\frac{\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}}{\prod_{j=1}^{n}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}y_j^{w_j}},\frac{\gamma\prod_{j=1}^{n}t_j^{w_j}}{\prod_{j=1}^{n}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}t_j^{w_j}},\frac{\gamma\prod_{j=1}^{n}v_j^{w_j}}{\prod_{j=1}^{n}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}v_j^{w_j}}\right) \quad (24)$$

(i) When $n=1$,

$\because w_1=1$,

For the left-hand side of the equation (24),

$$\bigoplus_{h\,j=1}^{n}\left(w_j\tilde{a}_j^{\lambda}\right)=\tilde{a}_1^{\lambda}=\left(\frac{\gamma T_1^{\lambda}}{\left(1+(\gamma-1)(1-T_1)\right)^{\lambda}+(\gamma-1)T_1^{\lambda}},\frac{\left(1+(\gamma-1)I_1\right)^{\lambda}-(1-I_1)^{\lambda}}{\left(1+(\gamma-1)I_1\right)^{\lambda}+(\gamma-1)(1-I_1)^{\lambda}},\frac{\left(1+(\gamma-1)F_1\right)^{\lambda}-(1-F_1)^{\lambda}}{\left(1+(\gamma-1)F_1\right)^{\lambda}+(\gamma-1)(1-F_1)^{\lambda}}\right)$$

and for the right-hand side of the equation (24), we have

$$\left(\frac{\prod_{j=1}^{1}x_j^{w_j}-\prod_{j=1}^{1}y_j^{w_j}}{\prod_{j=1}^{1}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{1}y_j^{w_j}},\frac{\gamma\prod_{j=1}^{1}t_j^{w_j}}{\prod_{j=1}^{1}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{1}t_j^{w_j}},\frac{\gamma\prod_{j=1}^{1}v_j^{w_j}}{\prod_{j=1}^{1}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{1}v_j^{w_j}}\right)$$

$$=\left(\frac{x_1-y_1}{x_1+(\gamma-1)y_1},\frac{\gamma t_1}{z_1+(\gamma-1)t_1},\frac{\gamma v_1}{u_1+(\gamma-1)v_1}\right)$$

$$=\left(\frac{\gamma T_1^{\lambda}}{\left(1+(\gamma-1)(1-T_1)\right)^{\lambda}+(\gamma-1)T_1^{\lambda}},\frac{\left(1+(\gamma-1)I_1\right)^{\lambda}-(1-I_1)^{\lambda}}{\left(1+(\gamma-1)I_1\right)^{\lambda}+(\gamma-1)(1-I_1)^{\lambda}},\frac{\left(1+(\gamma-1)F_1\right)^{\lambda}-(1-F_1)^{\lambda}}{\left(1+(\gamma-1)F_1\right)^{\lambda}+(\gamma-1)(1-F_1)^{\lambda}}\right)$$

So, Equation (24) holds for $n=1$.

(ii) Assume Equation (24) holds for $n=k$, we have

$$\bigoplus_{h\,j=1}^{k}\left(w_j\tilde{a}_j^{\lambda}\right)=\left(\frac{\prod_{j=1}^{k}x_j^{w_j}-\prod_{j=1}^{k}y_j^{w_j}}{\prod_{j=1}^{k}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}y_j^{w_j}},\frac{\gamma\prod_{j=1}^{k}t_j^{w_j}}{\prod_{j=1}^{k}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}t_j^{w_j}},\frac{\gamma\prod_{j=1}^{k}v_j^{w_j}}{\prod_{j=1}^{k}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}v_j^{w_j}}\right)$$

When $n=k+1$,

$$\bigoplus_{h\,j=1}^{k+1}\left(w_j\tilde{a}_j^{\lambda}\right)=\bigoplus_{h\,j=1}^{k+1}\left(w_j\tilde{a}_j^{\lambda}\right)\oplus_h\left(w_{k+1}\tilde{a}_{k+1}^{\lambda}\right)$$

$$=\left(\frac{\prod_{j=1}^{k}x_j^{w_j}-\prod_{j=1}^{k}y_j^{w_j}}{\prod_{j=1}^{k}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}y_j^{w_j}},\frac{\gamma\prod_{j=1}^{k}t_j^{w_j}}{\prod_{j=1}^{k}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}t_j^{w_j}},\frac{\gamma\prod_{j=1}^{k}v_j^{w_j}}{\prod_{j=1}^{k}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}v_j^{w_j}}\right)$$

$$\oplus_h\left(\frac{x_{k+1}^{w_{k+1}}-y_{k+1}^{w_{k+1}}}{x_{k+1}^{w_{k+1}}+(\gamma-1)y_{k+1}^{w_{k+1}}},\frac{\gamma t_{k+1}^{w_{k+1}}}{z_{k+1}^{w_{k+1}}+(\gamma-1)t_{k+1}^{w_{k+1}}},\frac{\gamma v_{k+1}^{w_{k+1}}}{u_{k+1}^{w_{k+1}}+(\gamma-1)v_{k+1}^{w_{k+1}}}\right)$$

$$=\left(\frac{\frac{\prod_{j=1}^{k}x_j^{w_j}-\prod_{j=1}^{k}y_j^{w_j}}{\prod_{j=1}^{k}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}y_j^{w_j}}+\frac{x_{k+1}^{w_{k+1}}-y_{k+1}^{w_{k+1}}}{x_{k+1}^{w_{k+1}}+(\gamma-1)y_{k+1}^{w_{k+1}}}-\frac{\prod_{j=1}^{k}x_j^{w_j}-\prod_{j=1}^{k}y_j^{w_j}}{\prod_{j=1}^{k}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}y_j^{w_j}}\times\frac{x_{k+1}^{w_{k+1}}-y_{k+1}^{w_{k+1}}}{x_{k+1}^{w_{k+1}}+(\gamma-1)y_{k+1}^{w_{k+1}}}(2-\gamma)}{1-(1-\gamma)\frac{\prod_{j=1}^{k}x_j^{w_j}-\prod_{j=1}^{k}y_j^{w_j}}{\prod_{j=1}^{k}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}y_j^{w_j}}\times\frac{x_{k+1}^{w_{k+1}}-y_{k+1}^{w_{k+1}}}{x_{k+1}^{w_{k+1}}+(\gamma-1)y_{k+1}^{w_{k+1}}}},\right.$$

$$\frac{\frac{\gamma\prod_{j=1}^{k}t_j^{w_j}}{\prod_{j=1}^{k}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}t_j^{w_j}}\frac{\gamma t_{k+1}^{w_{k+1}}}{z_{k+1}^{w_{k+1}}+(\gamma-1)t_{k+1}^{w_{k+1}}}}{\gamma+(1-\gamma)\left(\frac{\gamma\prod_{j=1}^{k}t_j^{w_j}}{\prod_{j=1}^{k}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}t_j^{w_j}}+\frac{\gamma t_{k+1}^{w_{k+1}}}{z_{k+1}^{w_{k+1}}+(\gamma-1)t_{k+1}^{w_{k+1}}}-\frac{\gamma\prod_{j=1}^{k}t_j^{w_j}}{\prod_{j=1}^{k}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}t_j^{w_j}}\frac{\gamma t_{k+1}^{w_{k+1}}}{z_{k+1}^{w_{k+1}}+(\gamma-1)t_{k+1}^{w_{k+1}}}\right)},$$

$$\left.\frac{\frac{\gamma\prod_{j=1}^{k}v_j^{w_j}}{\prod_{j=1}^{k}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}v_j^{w_j}}\frac{\gamma v_{k+1}^{w_{k+1}}}{u_{k+1}^{w_{k+1}}+(\gamma-1)v_{k+1}^{w_{k+1}}}}{\gamma+(1-\gamma)\left(\frac{\gamma\prod_{j=1}^{k}v_j^{w_j}}{\prod_{j=1}^{k}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}v_j^{w_j}}+\frac{\gamma v_{k+1}^{w_{k+1}}}{u_{k+1}^{w_{k+1}}+(\gamma-1)v_{k+1}^{w_{k+1}}}-\frac{\gamma\prod_{j=1}^{k}v_j^{w_j}}{\prod_{j=1}^{k}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{k}v_j^{w_j}}\times\frac{\gamma v_{k+1}^{w_{k+1}}}{u_{k+1}^{w_{k+1}}+(\gamma-1)v_{k+1}^{w_{k+1}}}\right)}\right)$$

$$=\left(\frac{\prod_{j=1}^{k+1}x_j^{w_j}-\prod_{j=1}^{k+1}y_j^{w_j}}{\prod_{j=1}^{k+1}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{k+1}y_j^{w_j}},\frac{\gamma\prod_{j=1}^{k+1}t_j^{w_j}}{\prod_{j=1}^{k+1}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{k+1}t_j^{w_j}},\frac{\gamma\prod_{j=1}^{k+1}v_j^{w_j}}{\prod_{j=1}^{k+1}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{k+1}v_j^{w_j}}\right)$$

So, when $n=k+1$, Equation (24) holds.

(iii) According to steps (i) and (ii), we can get Equation (24) holds for any n.

So, $\bigoplus_{h\,j=1}^{n}\left(w_j\tilde{a}_j^{\lambda}\right)=$

$$\left(\frac{\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}}{\prod_{j=1}^{n}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}y_j^{w_j}},\frac{\gamma\prod_{j=1}^{n}t_j^{w_j}}{\prod_{j=1}^{n}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}t_j^{w_j}},\frac{\gamma\prod_{j=1}^{n}v_j^{w_j}}{\prod_{j=1}^{n}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}v_j^{w_j}}\right)$$

(4) Calculate $\left(\bigoplus_{h\, j=1}^{n}\left(w_j\tilde{a}_j^{\lambda}\right)\right)^{1/\lambda}$, we can get

$$\left(\bigoplus_{h\, j=1}^{n}\left(w_j\tilde{a}_j^{\lambda}\right)\right)^{1/\lambda}=$$

$$\left(\frac{\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}}{\prod_{j=1}^{n}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}y_j^{w_j}},\frac{\gamma\prod_{j=1}^{n}t_j^{w_j}}{\prod_{j=1}^{n}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}t_j^{w_j}},\frac{\gamma\prod_{j=1}^{n}v_j^{w_j}}{\prod_{j=1}^{n}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}v_j^{w_j}}\right)^{1/\lambda}$$

$$=\left(\frac{\gamma\left(\frac{\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}}{\prod_{j=1}^{n}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}y_j^{w_j}}\right)^{1/\lambda}}{\left(1+(\gamma-1)\left(1-\frac{\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}}{\prod_{j=1}^{n}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}y_j^{w_j}}\right)\right)^{1/\lambda}+(\gamma-1)\left(\frac{\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}}{\prod_{j=1}^{n}x_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}y_j^{w_j}}\right)^{1/\lambda}},\right.$$

$$\frac{\left(1+(\gamma-1)\frac{\gamma\prod_{j=1}^{n}t_j^{w_j}}{\prod_{j=1}^{n}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}t_j^{w_j}}\right)^{1/\lambda}-\left(1-\frac{\gamma\prod_{j=1}^{n}t_j^{w_j}}{\prod_{j=1}^{n}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}t_j^{w_j}}\right)^{1/\lambda}}{\left(1+(\gamma-1)\frac{\gamma\prod_{j=1}^{n}t_j^{w_j}}{\prod_{j=1}^{n}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}t_j^{w_j}}\right)^{1/\lambda}+(\gamma-1)\left(1-\frac{\gamma\prod_{j=1}^{n}t_j^{w_j}}{\prod_{j=1}^{n}z_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}t_j^{w_j}}\right)^{1/\lambda}},$$

$$\left.\frac{\left(1+(\gamma-1)\frac{\gamma\prod_{j=1}^{n}v_j^{w_j}}{\prod_{j=1}^{n}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}v_j^{w_j}}\right)^{1/\lambda}-\left(1-\frac{\gamma\prod_{j=1}^{n}v_j^{w_j}}{\prod_{j=1}^{n}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}v_j^{w_j}}\right)^{1/\lambda}}{\left(1+(\gamma-1)\frac{\gamma\prod_{j=1}^{n}v_j^{w_j}}{\prod_{j=1}^{n}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}v_j^{w_j}}\right)^{1/\lambda}+(\gamma-1)\left(1-\frac{\gamma\prod_{j=1}^{n}v_j^{w_j}}{\prod_{j=1}^{n}u_j^{w_j}+(\gamma-1)\prod_{j=1}^{n}v_j^{w_j}}\right)^{1/\lambda}}\right)$$

$$=\left(\frac{\gamma\left(\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}x_j^{w_j}+(\gamma^2-1)\prod_{j=1}^{n}y_j^{w_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}\right)^{1/\lambda}},\right.$$

$$\frac{\left(\prod_{j=1}^{n}z_j^{w_j}+(\gamma^2-1)\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}z_j^{w_j}-\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}z_j^{w_j}+(\gamma^2-1)\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}z_j^{w_j}-\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}},$$

$$\left.\frac{\left(\prod_{j=1}^{n}u_j^{w_j}+(\gamma^2-1)\prod_{j=1}^{n}v_j^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}u_j^{w_j}-\prod_{j=1}^{n}v_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}u_j^{w_j}+(\gamma^2-1)\prod_{j=1}^{n}v_j^{w_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}u_j^{w_j}-\prod_{j=1}^{n}v_j^{w_j}\right)^{1/\lambda}}\right)$$

The proof ends.

It is easy to prove that the GNNHWA operator has the following properties.

(1) *Theorem 3 (Monotonicity):*

Let $(\tilde{a}_1^{'},\tilde{a}_2^{'},\cdots,\tilde{a}_n^{'})$ and $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$ be two collections of SVNNs, if $\tilde{a}_j^{'}\le\tilde{a}_j$ for all $j=1,2,\cdots,n$, then

$$GNNHWA(\tilde{a}_1^{'},\tilde{a}_2^{'},\cdots,\tilde{a}_n^{'})\le GNNHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n).$$

(2) *Theorem 4 (Idempotency):*

Let $\tilde{a}_j=\tilde{a},\ j=1,2,\cdots,n$, hen $GNNHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\tilde{a}$.

(3) *Theorem 5 (Bounded):*

Let $\tilde{a}_j=\tilde{a},\ j=1,2,\cdots,n$, then the *GNNHWA* operator lies between the maximum and minimum of $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$. i.e.,

$$\min(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)\le GNNHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)\le\max(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$$

where min and max represent the maximum and minimum of $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$, respectively.

In the following, we will discuss some special cases of the GNNHWA operator with respect to the parameters λ and γ.

(1) If $\lambda=1$, then the GNNHWA operator defined by (22) will be reduced to the neutrosophic number Hamacher weighted averaging (NNHWA) operator which is defined as follows:

$$NNHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=w_1\tilde{a}_1\oplus_h w_2\tilde{a}_2\oplus_h\cdots\oplus_h w_n\tilde{a}_n \tag{25}$$

According to (23), we can get

$$NNHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)T_j\right)^{w_j}-\prod_{j=1}^{n}(1-T_j)^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)T_j\right)^{w_j}+(\gamma-1)\prod_{j=1}^{n}(1-T_j)^{w_j}},\frac{\gamma\prod_{j=1}^{n}I_j^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(1-I_j)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{n}I_j^{w_j}},\frac{\gamma\prod_{j=1}^{n}F_j^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(1-F_j)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{n}F_j^{w_j}}\right) \tag{26}$$

Further,

(i) When $\gamma=1$, the formula (26) will be reduced to the neutrosophic number weighted averaging (NNWA) operator which is shown as follows:

$$NNWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(1-\prod_{j=1}^{n}(1-T_j)^{w_j},\prod_{j=1}^{n}I_j^{w_j},\prod_{j=1}^{n}F_j^{w_j}\right) \tag{27}$$

(ii) When $\gamma=2$, the formula (26) will be reduced to the neutrosophic number Einstein weighted averaging (NNEWA) operator which is shown as follows:

$$NNEWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\frac{\prod_{j=1}^{n}\left(1+T_j\right)^{w_j}-\prod_{j=1}^{n}(1-T_j)^{w_j}}{\prod_{j=1}^{n}\left(1+T_j\right)^{w_j}+\prod_{j=1}^{n}(1-T_j)^{w_j}},\frac{2\prod_{j=1}^{n}I_j^{w_j}}{\prod_{j=1}^{n}\left(2-I_j\right)^{w_j}+\prod_{j=1}^{n}I_j^{w_j}},\frac{2\prod_{j=1}^{n}F_j^{w_j}}{\prod_{j=1}^{n}\left(2-F_j\right)^{w_j}+\prod_{j=1}^{n}F_j^{w_j}}\right) \tag{28}$$

(2) If $\lambda \to 0$, then the GNNHWA operator defined by (22) will be reduced to the neutrosophic number Hamacher weighted geometric (NNHWG) operator which is defined as follows:

$$NNHWG(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\tilde{a}_1^{w_1}\otimes_h \tilde{a}_2^{w_2}\otimes_h\cdots\otimes_h \tilde{a}_n^{w_n} \tag{29}$$

According to (23), we can get

$$NNHWG(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)= \left(\frac{\gamma\prod_{j=1}^n T_j^{w_j}}{\prod_{j=1}^n\left(1+(\gamma-1)(1-T_j)\right)^{w_j}+(\gamma-1)\prod_{j=1}^n T_j^{w_j}}, \frac{\prod_{j=1}^n\left(1+(\gamma-1)I_j\right)^{w_j}-\prod_{j=1}^n(1-I_j)^{w_j}}{\prod_{j=1}^n\left(1+(\gamma-1)I_j\right)^{w_j}+(\gamma-1)\prod_{j=1}^n(1-I_j)^{w_j}}, \frac{\prod_{j=1}^n\left(1+(\gamma-1)F_j\right)^{w_j}-\prod_{j=1}^n(1-F_j)^{w_j}}{\prod_{j=1}^n\left(1+(\gamma-1)F_j\right)^{w_j}+(\gamma-1)\prod_{j=1}^n(1-F_j)^{w_j}}\right) \tag{30}$$

Further,

(i) When $\gamma=1$, the formula (30) will be reduced to the neutrosophic number weighted geometric (NNWG) operator which is defined as follows:

$$NNWG(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\prod_{j=1}^n T_j^{w_j},1-\prod_{j=1}^n(1-I_j)^{w_j},1-\prod_{j=1}^n(1-F_j)^{w_j}\right) \tag{31}$$

(ii) When $\gamma=2$, the formula (30) will be reduced to the neutrosophic number Einstein weighted geometric (NNEWG) operator which is defined as follows:

$$NNEWG(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\frac{2\prod_{j=1}^n T_j^{w_j}}{\prod_{j=1}^n\left(2-T_j\right)^{w_j}+\prod_{j=1}^n T_j^{w_j}}, \frac{\prod_{j=1}^n\left(1+I_j\right)^{w_j}-\prod_{j=1}^n(1-I_j)^{w_j}}{\prod_{j=1}^n\left(1+I_j\right)^{w_j}+\prod_{j=1}^n(1-I_j)^{w_j}}, \frac{\prod_{j=1}^n\left(1+F_j\right)^{w_j}-\prod_{j=1}^n(1-F_j)^{w_j}}{\prod_{j=1}^n\left(1+F_j\right)^{w_j}+\prod_{j=1}^n(1-F_j)^{w_j}}\right) \tag{32}$$

(3) If $\gamma=1$, then the GNNHWA operator defined by (22) will be reduced to the generalized neutrosophic number weighted averaging operator (GNNWA) which is defined as follows

$$GNNWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(w_1\tilde{a}_1^{\lambda}\oplus w_2\tilde{a}_2^{\lambda}\oplus\cdots\oplus w_n\tilde{a}_n^{\lambda}\right)^{1/n} \tag{33}$$

According to (23), we can get

$$GNNWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\left(1-\prod_{j=1}^n\left(1-T_j^{\lambda}\right)^{w_j}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^n\left(1-\left(1-I_j\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}, 1-\left(1-\prod_{j=1}^n\left(1-\left(1-F_j\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right) \tag{34}$$

(4) If $\gamma=2$, then then the GNNHWA operator defined by (22) will be reduced to the generalized neutrosophic number Einstein weighted averaging operator (GNNE-WA) which is defined as follows

$$GNNEWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(w_1\tilde{a}_1^{\lambda}\oplus_E w_2\tilde{a}_2^{\lambda}\oplus_E\cdots\oplus_E w_n\tilde{a}_n^{\lambda}\right)^{1/n} \tag{35}$$

According to (23), we can get

$$GNNEWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)= \left(\frac{2\left(\prod_{j=1}^n x_j^{w_j}-\prod_{j=1}^n y_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n x_j^{w_j}+3\prod_{j=1}^n y_j^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n x_j^{w_j}-\prod_{j=1}^n y_j^{w_j}\right)^{1/\lambda}}, \frac{\left(\prod_{j=1}^n z_j^{w_j}+3\prod_{j=1}^n t_j^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^n z_j^{w_j}-\prod_{j=1}^n t_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n z_j^{w_j}+3\prod_{j=1}^n t_j^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n z_j^{w_j}-\prod_{j=1}^n t_j^{w_j}\right)^{1/\lambda}}, \frac{\left(\prod_{j=1}^n u_j^{w_j}+3\prod_{j=1}^n v_j^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^n u_j^{w_j}-\prod_{j=1}^n v_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^n u_j^{w_j}+3\prod_{j=1}^n v_j^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^n u_j^{w_j}-\prod_{j=1}^n v_j^{w_j}\right)^{1/\lambda}}\right) \tag{36}$$

where $x_j=\left(2-T_j\right)^{\lambda}+3T_j^{\lambda}$, $y_j=\left(2-T_j\right)^{\lambda}-T_j^{\lambda}$,

$z_j=\left(1+I_j\right)^{\lambda}+3(1-I_j)^{\lambda}$, $t_j=\left(1+I_j\right)^{\lambda}-(1-I_j)^{\lambda}$,

$u_j=\left(1+F_j\right)^{\lambda}+3(1-F_j)^{\lambda}$, $v_j=\left(1+F_j\right)^{\lambda}-(1-F_j)^{\lambda}$

From the above descriptions, we can know GNNHWA operator is more generalized.

In the following, we will discuss another operator which can weigh the inputs by the ordering positions.

Definition 10: Let $\tilde{a}_j=(T_j,I_j,F_j)$ $(j=1,2\cdots,n)$ be a collection of the SVNNs, and $GNNHOWA:\Omega^n\to\Omega$, if

$$GNNHOWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\bigoplus_{h\, j=1}^{n}\left(\omega_j\tilde{a}_{\sigma(j)}^{\lambda}\right)\right)^{1/\lambda} \tag{37}$$

where Ω is the set of all SVNNs, and $\lambda>0$. $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ is the weighted vector associated with GNNHOWA, such that $\omega_j\geq 0$ and $\sum_{j=1}^n\omega_j=1$. $(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $\tilde{a}_{\sigma(j-1)}\geq\tilde{a}_{\sigma(j)}$ for any j Then GNNHOWA is called the generalized neutrosophic numbe Hamacher ordered weighted averaging (GNNHOWA) operator.

Based on the Hamacher operational rules of the SVNNs, we can derive the result aggregated from Definition 10 shown as theorem 6.

Theorem 6: Let $\tilde{a}_j=(T_j,I_j,F_j)$ $(j=1,2\cdots,n)$ be a collection of the SVNNs, then, the result aggregated from Definition 10 is still an SVNN, and even

$$GNNHOWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) = \left(\frac{\gamma\left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} + (\gamma^2-1)\prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} + (\gamma-1)\left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}, \right.$$
$$\frac{\left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} + (\gamma^2-1)\prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} + (\gamma^2-1)\prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} + (\gamma-1)\left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}, \quad (38)$$
$$\left. \frac{\left(\prod_{j=1}^{n} u_{\sigma(j)}^{\omega_j} + (\gamma^2-1)\prod_{j=1}^{n} v_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} u_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} v_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} u_{\sigma(j)}^{\omega_j} + (\gamma^2-1)\prod_{j=1}^{n} v_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} + (\gamma-1)\left(\prod_{j=1}^{n} u_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} v_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}} \right)$$

where

$$x_j = \left(1+(\gamma-1)(1-T_j)\right)^{\lambda} + (\gamma^2-1)T_j^{\lambda},$$
$$y_j = \left(1+(\gamma-1)(1-T_j)\right)^{\lambda} - T_j^{\lambda}$$
$$z_j = \left(1+(\gamma-1)I_j\right)^{\lambda} + (\gamma^2-1)(1-I_j)^{\lambda},$$
$$t_j = \left(1+(\gamma-1)I_j\right)^{\lambda} - (1-I_j)^{\lambda},$$
$$u_j = \left(1+(\gamma-1)F_j\right)^{\lambda} + (\gamma^2-1)(1-F_j)^{\lambda},$$
$$v_j = \left(1+(\gamma-1)F_j\right)^{\lambda} - (1-F_j)^{\lambda}, \text{ here } \gamma>0$$

$(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $\tilde{a}_{\sigma(j-1)} \ge \tilde{a}_{\sigma(j)}$ for any j.

The proof is similar to Theorem 2, and it is omitted here.

An important characteristic of the GNNHOWA operator is that it can weigh the input data according to these data's ordering positions from largest to smallest. So, ω can also be called the position weighted vector. In general, the position weighted vector ω can be determined according to actual needs of decision making problems. For example, in the Olympic diving competition, there are 7 referees, and the scoring rules are the average of the other 5 scores after removing the most high and low scores. i.e., $\omega=(0,0.2,0.2,0.2,0.2,0.2,0)$. In some special cases, it can also be determined by some mathematical methods. For example, Xu [37] proposed a method shown as follows

$$\omega_j = \frac{e^{-\frac{(j-\theta_{n-1})^2}{2o_{n-1}^2}}}{\sum_{k=1}^{n-1} e^{-\frac{(j-\theta_{n-1})^2}{2o_{n-1}^2}}} \quad (j=1,2,\cdots,n-1) \tag{39}$$

where θ_{n-1} and o_{n-1} are the mean value and the standard deviation of the collection of $1,2,\cdots,n-1$, respectively. θ_{n-1} and o_{n-1} was calculated by the following formulas, respectively.

$$\theta_{n-1} = \frac{n}{2} \tag{40}$$

$$o_{n-1} = \sqrt{\frac{1}{n-1}\sum_{j=1}^{n-1}\left(j-\theta_{n-1}\right)^2} \tag{41}$$

The GNNHOWA operator has the following properties:

(1) *Theorem 7 (Monotonicity):*
Let $(\tilde{a}_1',\tilde{a}_2',\cdots,\tilde{a}_n')$ and $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$ be two collections of SVNNs, if $\tilde{a}_j' \le \tilde{a}_j$ for all $j=1,2,\cdots,n$, then $GNNHOWA(\tilde{a}_1',\tilde{a}_2',\cdots,\tilde{a}_n') \le GNNHOWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$.

(2) *Theorem 8 (Idempotency):*
Let $\tilde{a}_j=\tilde{a}, j=1,2,\cdots,n$, then $GNNHOWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\tilde{a}$.

(3) *Theorem 9 (Bounded):*
Let $\tilde{a}_j=\tilde{a}, j=1,2,\cdots,n$, then the $GNNHOWA$ operator lies between the maximum and minimum of $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$. i.e.,

$$\min(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) \le GNNHOWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) \le \max(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$$

where min and max represent the maximum and minimum of $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$, respectively.

(4) *Theorem 10 (Commutativity):*
Let $(\tilde{a}_1',\tilde{a}_2',\cdots,\tilde{a}_n')$ and $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$ be two collections of SVNNs, and $(\tilde{a}_1',\tilde{a}_2',\cdots,\tilde{a}_n')$ is any permutation of $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$, then

$$GNNHOWA(\tilde{a}_1',\tilde{a}_2',\cdots,\tilde{a}_n') = GNNHOWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n).$$

Similar to GNNHWA operator, some special cases of the GNNHOWA operator with respect to the parameters λ and γ can be discussed as follows.

(1) If $\lambda=1$, then the GNNHOWA operator defined by (37) will be reduced to the neutrosophic number Hamacher ordered weighted averaging (NNHOWA) operator which is defined as follows:

$$NNHOWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) = \omega_1\tilde{a}_{\sigma(1)} \oplus_h \omega_2\tilde{a}_{\sigma(2)} \oplus_h \cdots \oplus_h \omega_n\tilde{a}_{\sigma(n)} \tag{42}$$

According to (38), we can get

$$NNHOWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n) = \left(\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)T_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{n}(1-T_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)T_{\sigma(j)}\right)^{\omega_j} + (\gamma-1)\prod_{j=1}^{n}(1-T_{\sigma(j)})^{\omega_j}}, \right.$$
$$\frac{\gamma\prod_{j=1}^{n} I_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(1-I_{\sigma(j)})\right)^{\omega_j} + (\gamma-1)\prod_{j=1}^{n} I_{\sigma(j)}^{\omega_j}}, \quad (43)$$
$$\left. \frac{\gamma\prod_{j=1}^{n} F_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(1-F_{\sigma(j)})\right)^{\omega_j} + (\gamma-1)\prod_{j=1}^{n} F_{\sigma(j)}^{\omega_j}} \right)$$

Further,

(i) When $\gamma = 1$, the formula (43) will be reduced to the neutrosophic number ordered weighted averaging (NNOWA) operator which is defined as follows:

$$NNOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(1 - \prod_{j=1}^{n}(1 - T_{\sigma(j)})^{\omega_j}, \prod_{j=1}^{n} I_{\sigma(j)}^{\omega_j}, \prod_{j=1}^{n} F_{\sigma(j)}^{\omega_j}\right) \quad (44)$$

(ii) When $\gamma = 2$, the formula (43) will be reduced to the neutrosophic number Einstein ordered weighted averaging (NNEOWA) operator which is defined as follows:

$$NNEOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\frac{\prod_{j=1}^{n}(1+T_{\sigma(j)})^{\omega_j} - \prod_{j=1}^{n}(1-T_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}(1+T_{\sigma(j)})^{\omega_j} + \prod_{j=1}^{n}(1-T_{\sigma(j)})^{\omega_j}}, \frac{2\prod_{j=1}^{n} I_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}(2-I_{\sigma(j)})^{\omega_j} + \prod_{j=1}^{n} I_{\sigma(j)}^{\omega_j}}, \frac{2\prod_{j=1}^{n} F_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}(2-F_{\sigma(j)})^{\omega_j} + \prod_{j=1}^{n} F_{\sigma(j)}^{\omega_j}}\right) \quad (45)$$

(2) If $\lambda \to 0$, then the GNNHOWA operator defined by (37) will be reduced to the neutrosophic number Hamacher ordered weighted geometric (NNHOWG) operator which is defined as follows:

$$NNHOWG(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a}_{\sigma(1)}^{\omega_1} \otimes_h \tilde{a}_{\sigma(2)}^{\omega_2} \otimes_h \cdots \otimes_h \tilde{a}_{\sigma(n)}^{\omega_n} \quad (46)$$

According to (38), we can get

$$NNHOWG(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\frac{\gamma\prod_{j=1}^{n} T_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}(1+(\gamma-1)(1-T_{\sigma(j)}))^{\omega_j} + (\gamma-1)\prod_{j=1}^{n} T_{\sigma(j)}^{\omega_j}}, \frac{\prod_{j=1}^{n}(1+(\gamma-1)I_{\sigma(j)})^{\omega_j} - \prod_{j=1}^{n}(1-I_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}(1+(\gamma-1)I_{\sigma(j)})^{\omega_j} + (\gamma-1)\prod_{j=1}^{n}(1-I_{\sigma(j)})^{\omega_j}}, \frac{\prod_{j=1}^{n}(1+(\gamma-1)F_{\sigma(j)})^{\omega_j} - \prod_{j=1}^{n}(1-F_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}(1+(\gamma-1)F_{\sigma(j)})^{\omega_j} + (\gamma-1)\prod_{j=1}^{n}(1-F_{\sigma(j)})^{\omega_j}}\right) \quad (47)$$

Further,

(i) When $\gamma = 1$, the formula (47) will be reduced to the neutrosophic number ordered weighted geometric (NNOWG) operator which is defined as follows:

$$NNOWG(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\prod_{j=1}^{n} T_{\sigma(j)}^{\omega_j}, 1 - \prod_{j=1}^{n}(1-I_{\sigma(j)})^{\omega_j}, 1 - \prod_{j=1}^{n}(1-F_{\sigma(j)})^{\omega_j}\right) \quad (48)$$

(ii) When $\gamma = 2$, the formula (47) will be reduced to the neutrosophic number Einstein ordered weighted geometric (NNEOWG) operator which is defined as follows:

$$NNEOWG(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\frac{2\prod_{j=1}^{n} T_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}(2-T_{\sigma(j)})^{\omega_j} + \prod_{j=1}^{n} T_{\sigma(j)}^{\omega_j}}, \frac{\prod_{j=1}^{n}(1+I_{\sigma(j)})^{\omega_j} - \prod_{j=1}^{n}(1-I_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}(1+I_{\sigma(j)})^{\omega_j} + \prod_{j=1}^{n}(1-I_{\sigma(j)})^{\omega_j}}, \frac{\prod_{j=1}^{n}(1+F_{\sigma(j)})^{\omega_j} - \prod_{j=1}^{n}(1-F_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}(1+F_{\sigma(j)})^{\omega_j} + \prod_{j=1}^{n}(1-F_{\sigma(j)})^{\omega_j}}\right) \quad (49)$$

(3) If $\gamma = 1$, then the GNNHOWA operator defined by (37) will be reduced to the generalized neutrosophic number ordered weighted averaging (GNNOWA) operator which is defined as follows:

$$GNNOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\omega_1 \tilde{a}_{\sigma(1)}^{\lambda} \oplus \omega_2 \tilde{a}_{\sigma(2)}^{\lambda} \oplus \cdots \oplus \omega_n \tilde{a}_{\sigma(n)}^{\lambda}\right)^{1/n} \quad (50)$$

According to (38), we can get

$$GNNOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\left(1 - \prod_{j=1}^{n}\left(1 - T_{\sigma(j)}^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - I_{\sigma(j)}\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - F_{\sigma(j)}\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right) \quad (51)$$

(4) If $\gamma = 2$, then the GNNHOWA operator defined by (37) will be reduced to the generalized neutrosophic number Einstein ordered weighted averaging (GNNEOWA) operator which is defined as follows:

$$GNNEOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\omega_1 \tilde{a}_{\sigma(1)}^{\lambda} \oplus_E \omega_2 \tilde{a}_{\sigma(2)}^{\lambda} \oplus_E \cdots \oplus_E \omega_n \tilde{a}_{\sigma(n)}^{\lambda}\right)^{1/n} \quad (52)$$

According to (38), we can get

$$GNNEOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\frac{2\left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} + 3\prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}\right)}, \frac{\left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} + 3\prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} + 3\prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}, \frac{\left(\prod_{j=1}^{n} u_{\sigma(j)}^{\omega_j} + 3\prod_{j=1}^{n} v_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} u_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} v_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} u_{\sigma(j)}^{\omega_j} + 3\prod_{j=1}^{n} v_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} u_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} v_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}\right) \quad (53)$$

where $x_j = \left(2 - T_j\right)^{\lambda} + 3T_j^{\lambda}$, $y_j = \left(2 - T_j\right)^{\lambda} - T_j^{\lambda}$, $z_j = \left(1 + I_j\right)^{\lambda} + 3(1 - I_j)^{\lambda}$, $t_j = \left(1 + I_j\right)^{\lambda} - (1 - I_j)^{\lambda}$,

$u_j=\left(1+F_j\right)^{\lambda}+3(1-F_j)^{\lambda}$, $v_j=\left(1+F_j\right)^{\lambda}-(1-F_j)^{\lambda}$.

As GNNHWA operator only emphasizes the self-importance of each SVNN, and GNNHOWA operator only emphasizes the ordering position importance of all SVNNs. However, in many practical applications, we need consider these two weights together because they represent different aspects of decision making problems. Obviously, two operators have shortcomings. In order to overcome these defects, a generalized hybrid averaging operator for SVNNs based on Hamacher T-norm and T-conorm is given as follows.

Definition 11: Let $\tilde{a}_j=(T_j,I_j,F_j)$ $(j=1,2\cdots,n)$ be a collection of the SVNNs, and $GNNHHWA:\Omega^n\rightarrow\Omega$, if

$$GNNHHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\bigoplus_{j=1}^{n}{}_{h}\left(\omega_j\tilde{b}_{\sigma(j)}\right) \tag{54}$$

where Ω is the set of all SVNNs, and $\omega=\left(\omega_1,\omega_2,\cdots,\omega_n\right)^T$ is the weighted vector associated with GNNHHWA, such that $\omega_j\geq 0$ and $\sum_{j=1}^{n}\omega_j=1$. $w=(w_1,w_2,\cdots,w_n)$ is the weight vector of $\tilde{a}_j(j=1,2,\cdots,n)$, and $w_j\in[0,1],\sum_{j=1}^{n}w_j=1$. Let $\tilde{b}_j=nw_j\tilde{a}_j=(\dot{T}_j,\dot{I}_j,\dot{F}_j)$, n is the adjustment factor. Suppose $(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $\tilde{b}_{\sigma(j-1)}\geq\tilde{b}_{\sigma(j)}$ for any j, and then function GNNHHWA is called the generalized neutrosophic number Hamacher hybrid weighted averaging (GNNHHWA) operator.

Based on the Hamacher operational rules of the ISVNNs, we can derive the result aggregated from Definition 11 shown as theorem 11.

Theorem 11: Let $\tilde{a}_j=(T_j,I_j,F_j)$ $(j=1,2\cdots,n)$ be a collection of the SVNNs, then, the result aggregated from Definition 11 is still an SVNN, and even

$$GNNHHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$$
$$=\left(\frac{\gamma\left(\prod_{j=1}^{n}\dot{x}_{\sigma(j)}^{\omega_j}-\prod_{j=1}^{n}\dot{y}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}\dot{x}_{\sigma(j)}^{\omega_j}+(\gamma^2-1)\prod_{j=1}^{n}\dot{y}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}\dot{x}_{\sigma(j)}^{\omega_j}-\prod_{j=1}^{n}\dot{y}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}},\right.$$
$$\frac{\left(\prod_{j=1}^{n}\dot{z}_{\sigma(j)}^{\omega_j}+(\gamma^2-1)\prod_{j=1}^{n}\dot{t}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}\dot{z}_{\sigma(j)}^{\omega_j}-\prod_{j=1}^{n}\dot{t}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}\dot{z}_{\sigma(j)}^{\omega_j}+(\gamma^2-1)\prod_{j=1}^{n}\dot{t}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}\dot{z}_{\sigma(j)}^{\omega_j}-\prod_{j=1}^{n}\dot{t}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}, \tag{55}$$
$$\left.\frac{\left(\prod_{j=1}^{n}\dot{u}_{\sigma(j)}^{\omega_j}+(\gamma^2-1)\prod_{j=1}^{n}\dot{v}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}\dot{u}_{\sigma(j)}^{\omega_j}-\prod_{j=1}^{n}\dot{v}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}\dot{u}_{\sigma(j)}^{\omega_j}+(\gamma^2-1)\prod_{j=1}^{n}\dot{v}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}\dot{u}_{\sigma(j)}^{\omega_j}-\prod_{j=1}^{n}\dot{v}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}\right)$$

where $\dot{x}_j=\left(1+(\gamma-1)(1-\dot{T}_j)\right)^{\lambda}+(\gamma^2-1)\dot{T}_j^{\lambda}$,

$$\dot{y}_j=\left(1+(\gamma-1)(1-\dot{T}_j)\right)^{\lambda}-\dot{T}_j^{\lambda}$$

$$\dot{z}_j=\left(1+(\gamma-1)\dot{I}_j\right)^{\lambda}+(\gamma^2-1)(1-\dot{I}_j)^{\lambda},\ \dot{t}_j=\left(1+(\gamma-1)\dot{I}_j\right)^{\lambda}-(1-\dot{I}_j)^{\lambda}.$$

$$\dot{u}_j=\left(1+(\gamma-1)\dot{F}_j\right)^{\lambda}+(\gamma^2-1)(1-\dot{F}_j)^{\lambda},$$

$$\dot{v}_j=\left(1+(\gamma-1)\dot{F}_j\right)^{\lambda}-(1-\dot{F}_j)^{\lambda}.$$

$$\dot{T}_j=\frac{\gamma T_j^{nw_j}}{\left(1+(\gamma-1)(1-T_j)\right)^{nw_j}+(\gamma-1)T_j^{nw_j}},$$

$$\dot{I}_j=\frac{\left(1+(\gamma-1)I_j\right)^{nw_j}-(1-I_j)^{nw_j}}{\left(1+(\gamma-1)I_j\right)^{nw_j}+(\gamma-1)(1-I_j)^{nw_j}},$$

$$\dot{F}_j=\frac{\left(1+(\gamma-1)F_j\right)^{nw_j}-(1-F_j)^{nw_j}}{\left(1+(\gamma-1)F_j\right)^{nw_j}+(\gamma-1)(1-F_j)^{nw_j}}.$$

$(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$, such that $\tilde{b}_{\sigma(j-1)}\geq\tilde{b}_{\sigma(j)}$ for any j, $\tilde{b}_j=nw_j\tilde{a}_j=(\dot{T}_j,\dot{I}_j,\dot{F}_j)$.

The proof is similar to Theorem 2, and it is omitted here.

Theorem 12: The GNNHWA and GNNHOWA operators are the special cases of the GNNHHWA operator.

It is easy to prove that when $W=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, the GNNHHWA operator will reduce to GNNHOWA operator, and when $\omega=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, the GNNHHWA operator will reduce to GNNHWA operator.

From definition 11, we can know that the GNNHHWA operator firstly weights the input arguments, and then reorders the weighted values in descending order and weights them. So, the GNNHHWA operator can consider the importance degrees of both the input arguments and their weighted ordered positions.

4. The Multiple Attribute Decision Making Methods Based on the Generalized Neutrosophic Number Hamacher Aggregation Operators

In this section, we will use the generalized neutrosophic number Hamacher aggregation operators to the multiple attribute group decision making problems in which the attribute weights take the form of crisp numbers and attribute values take the form of SVNNs.

For a multiple attribute group decision making problem, let $E=\{e_1,e_2,\cdots,e_q\}$ be the collection of decision makers, $S=\{S_1,S_2,\cdots,S_m\}$ be the collection of alternatives, and $C=\{C_1,C_2,\cdots,C_n\}$ be the collection of attributes. Suppose that $r_{ij}^k=\left(T_{ij}^k,I_{ij}^k,F_{ij}^k\right)$ is an attribute value given by the decision maker e_k for the alternative S_i with respect to the attribute C_j which is expressed by a SVNN, $w=(w_1,w_2,\cdots,w_n)$ is the weight vector of attribute set $C=\{C_1,C_2,\cdots,C_n\}$, and $w_j\in[0,1],\sum_{j=1}^{n}w_j=1$. Let $\xi=(\xi_1,\xi_2,\cdots,\xi_q)$ be the vector of decision makers $\{e_1,e_2,\cdots,e_q\}$, and $\xi_k\in[0,1],\sum_{k=1}^{q}\xi_k=1$. Then we use the attribute weights, the decision makers' weights, and the attribute values to rank the order of the alternatives.

In group decision making, we need aggregate the different attribute values to the comprehensive values and the comprehensive values of different decision makers to collective overall values. As mentioned above, we don't need to consider the position weight in aggregating the different attribute values to the comprehensive values, so we can select the GNNHWA operator. However, in aggregating the comprehensive values of different decision makers to collective overall values, we can use the GNNHHWA operator so that the weights of decision makers and ordering position of the comprehensive values can be considered together. The steps are shown as follows.

Step 1: Utilize the GNNHWA operator

$$r_i^k=\left(T_i^k,I_i^k,F_i^k\right)=GNNHWA(r_{i1}^k,r_{i2}^k,\cdots,r_{in}^k) \quad (56)$$

to derive the comprehensive values $r_i^k\,(i=1,2,\cdots,m;k=1,2,\cdots,q)$ of each decision maker.

Step 2: Utilize the GNNHHWA operator

$$r_i=\left(T_i,I_i,F_i\right)=GNNHHWA(r_i^1,r_i^2,\cdots,r_i^q) \quad (57)$$

to derive the collective overall values $r_i\ (i=1,2,\cdots,m)$.

Step 3: Calculate the score function $sc(r_i)$ and accuracy function $ac(r_i)\ (i=1,2,\cdots,m)$ by definition 3.

Step 4: Rank all the alternatives $\{S_1,S_2,\cdots,S_m\}$ by definition 4.

Step 5: End.

5. An Application Example

In order to demonstrate the application of the proposed methods to multi-attribute group decision making problems, we will cite an example about the air quality evaluation (adapted from [38]). To evaluate the air quality of Guangzhou for the 16th Asian Olympic Games, the air quality in Guangzhou for the Novembers of 2006, 2007, 2008 and 2009 were collected in order to find out the trends and to forecast the situation in 2010. There are 3 air-quality monitoring stations expressed by (e_1,e_2,e_3) which can be seen as decision makers, and their weight $\xi=(0.314,0.355,0.331)^T$. There are 3 measured indexes, namely, SO_2 (C_1), NO_2(C_2) and PM_{10}(C_3), and their weight $W=(0.40,0.20,0.40)^T$. The measured values from air-quality monitoring stations under these indexes are shown in tables 1, 2 and 3, and they can be expressed by SVNNs (Note: the original data take the form of intuitionistic fuzzy numbers, we can get SVNNs by $I=1-T-F$). Let (S_1,S_2,S_3,S_4) ={November of 2006, November of 2007, November of 2008, November of 2009} be the set of alternatives, please give the rank of air quality from 2006 to 2009.

Table 1. Air quality data from station e_1.

	C_1	C_2	C_3
A_1	(0.265,0.350,0.385)	(0.330,0.390,0.280)	(0.245,0.275,0.480)
A_2	(0.345,0.245,0.410)	(0.430,0.290,0.280)	(0.245,0.375,0.380)
A_3	(0.365,0.300,0.335)	(0.480,0.315,0.205)	(0.340,0.370,0.290)
A_4	(0.430,0.300,0.270)	(0.460,0.245,0.295)	(0.310,0.520,0.170)

Table 2. Air quality data from station e_2.

	C_1	C_2	C_3
A_1	(0.125,0.470,0.405)	(0.220,0.420,0.360)	(0.345,0.490,0.165)
A_2	(0.355,0.315,0.330)	(0.300,0.370,0.330)	(0.205,0.630,0.165)
A_3	(0.315,0.380,0.305)	(0.330,0.565,0.105)	(0.280,0.520,0.200)
A_4	(0.365,0.365,0.270)	(0.355,0.320,0.325)	(0.425,0.485,0.090)

Table 3. Air quality data from station e_3.

	C_1	C_2	C_3
A_1	(0.260,0.425,0.315)	(0.220,0.450,0.330)	(0.255,0.500,0.245)
A_2	(0.270,0.370,0.360)	(0.320,0.215,0.465)	(0.135,0.575,0.290)
A_3	(0.245,0.465,0.290)	(0.250,0.570,0.180)	(0.175,0.660,0.165)
A_4	(0.390,0.340,0.270)	(0.305,0.475,0.220)	(0.465,0.485,0.050)

To get the best alternative(s), the following steps are involved:

Step 1: Utilize the GNNHWA operator expressed by (57) to derive the comprehensive values $r_i^k\,(i=1,2,\cdots,m;k=1,2,\cdots,q)$ of each decision maker (suppose $\lambda=1$, $\gamma=1$), we can get

$r_1^1=(0.319,0.321,0.350)$, $r_2^1=(0.384,0.284,0.332)$, $r_3^1=(0.434,0.311,0.256)$, $r_4^1=(0.436,0.316,0.228)$, $r_1^2=(0.266,0.435,0.277)$, $r_2^2=(0.322,0.395,0.251)$, $r_3^2=(0.342,0.465,0.175)$, $r_4^2=(0.407,0.363,0.192)$, $r_1^3=(0.272,0.436,0.282)$, $r_2^3=(0.291,0.333,0.352)$, $r_3^3=(0.261,0.534,0.195)$, $r_4^3=(0.408,0.410,0.138)$.

Step 2: Utilize the GNNHHWA operator expressed by (58) to derive the collective overall values $r_i\ (i=1,2,\cdots,m)$ (suppose $\lambda=1$, $\gamma=1$, $\omega=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$), we can get $r_1=(0.288,0.396,0.304)$, $r_2=(0.329,0.344,0.310)$,

$r_3 = (0.345, 0.430, 0.210)$, $r_4 = (0.417, 0.367, 0.183)$

Step 3: Calculate the score function $sc(r_i)$ of the comprehensive values $r_i (i = 1,2,3,4)$, we can get

$$sc(r_1) = 1.587, \quad sc(r_2) = 1.675,$$
$$sc(r_3) = 1.706, \quad sc(r_4) = 1.866$$

Step 4: Rank the alternatives

According to definition 4, we can rank the alternatives $\{S_1, S_2, S_3, S_4\}$ shown as follows

$$S_4 \succ S_3 \succ S_2 \succ S_1.$$

So, the best alternative is S_4, i.e., the best air quality in Guangzhou is November of 2009 among the Novembers of 2006, 2007, 2008, and 2009.

Obviously, this ranking result is the same as that in [38].

In step 2, we suppose the position weight is equal because 3 air-quality monitoring stations are parage.

In the following, we will discuss the influences of the parameters γ and λ on decision making results of this example, we use the different values λ and γ in steps 1and 2 to rank the alternatives. The ranking results are shown in Table 4.

Table 4. Ordering of the alternatives by utilizing the different γ, λ.

λ	$\gamma=1$ score function $sc(\tilde{r}_i)$	$\gamma=1$ Ranking	$\gamma=2$ score function $sc(\tilde{r}_i)$	$\gamma=2$ Ranking
$\lambda=0.01$	$sc(r_1)=1.561$ $sc(r_2)=1.644$ $sc(r_3)=1.680$ $sc(r_4)=1.853$	$S_4 \succ S_3 \succ S_2 \succ S_1$	$sc(r_1)=1.570$ $sc(r_2)=1.655$ $sc(r_3)=1.690$ $sc(r_4)=1.863$	$S_4 \succ S_3 \succ S_2 \succ S_1$
$\lambda=0.5$	$sc(r_1)=1.577$ $sc(r_2)=1.668$ $sc(r_3)=1.699$ $sc(r_4)=1.864$	$S_4 \succ S_3 \succ S_2 \succ S_1$	$sc(r_1)=1.569$ $sc(r_2)=1.652$ $sc(r_3)=1.687$ $sc(r_4)=1.859$	$S_4 \succ S_3 \succ S_2 \succ S_1$
$\lambda=1.0$	$sc(r_1)=1.595$ $sc(r_2)=1.693$ $sc(r_3)=1.721$ $sc(r_4)=1.876$	$S_4 \succ S_3 \succ S_2 \succ S_1$	$sc(r_1)=1.587$ $sc(r_2)=1.675$ $sc(r_3)=1.706$ $sc(r_4)=1.866$	$S_4 \succ S_3 \succ S_2 \succ S_1$
$\lambda=2.0$	$sc(r_1)=1.633$ $sc(r_2)=1.745$ $sc(r_3)=1.768$ $sc(r_4)=1.902$	$S_4 \succ S_3 \succ S_2 \succ S_1$	$sc(r_1)=1.654$ $sc(r_2)=1.760$ $sc(r_3)=1.781$ $sc(r_4)=1.907$	$S_4 \succ S_3 \succ S_2 \succ S_1$
$\lambda=5$	$sc(r_1)=1.738$ $sc(r_2)=1.873$ $sc(r_3)=1.898$ $sc(r_4)=1.987$	$S_4 \succ S_3 \succ S_2 \succ S_1$	$sc(r_1)=1.830$ $sc(r_2)=1.962$ $sc(r_3)=1.988$ $sc(r_4)=2.059$	$S_4 \succ S_3 \succ S_2 \succ S_1$
$\lambda=10.0$	$sc(r_1)=1.848$ $sc(r_2)=1.997$ $sc(r_3)=2.042$ $sc(r_4)=2.100$	$S_4 \succ S_3 \succ S_2 \succ S_1$	$sc(r_1)=1.958$ $sc(r_2)=2.111$ $sc(r_3)=2.164$ $sc(r_4)=2.206$	$S_4 \succ S_3 \succ S_2 \succ S_1$

As we can see from Table 4, the score functions of the aggregation results using the different aggregation parameters λ and γ are different, but the rankings of the alternatives are the same in this example. The decision makes can choose the appropriate values in accordance with their preferences. In general, we can take the values of the parameter $\gamma = 1,2$, they are Algebraic aggregation operators and Einstein aggregation operators, and $\lambda \to 0, \lambda = 1,2$, they are geometric operator, arithmetic operator and quadratic operator.

In order to verify the effective of the proposed method, we can compare with the method shown in literature [38]. Firstly, there are the same ranking results of these methods. Secondly, the aggregation operators proposed in this paper are more general and more flexible according to the different parameter values λ and γ.

6. Conclusions

This paper puts forward a new method to solve MAGDM problems with single valued neutrosophic information. We have defined Hamacher operation rules of single valued neutrosophic numbers by using Hamacher t-conorm and t-norm, and discussed some properties of them. Further, we have developed some new aggregation operators based on Hamacher operations and generalized aggregation operators for single valued neutrosophic information, including the generalized neutrosophic number Hamacher weighted averaging (GNNHWA) operator, generalized neutrosophic number Hamacher ordered weighted averaging (GNNHOWA) operator, and generalized neutrosophic number Hamacher hybrid averaging (GNNHHA) operator, and discussed various properties of these operators and analyzed some special cases of them. Moreover, we applied the developed operators to deal with the MAGDM problems with single valued neutrosophic information, and proposed a new method. This research has showed the proposed operators extended the Algebraic and Einstein aggregation operators, also extended the arithmetic aggregation operators, geometric aggregation operators and quadratic aggregation operators, and the proposed method is more general and more flexible according to the different parameter values λ and γ. In further research, it is necessary and meaningful to apply the proposed operators to real decision making problems, or extend them to the other domains such as pattern recognition, fuzzy cluster analysis and uncertain programming, etc.

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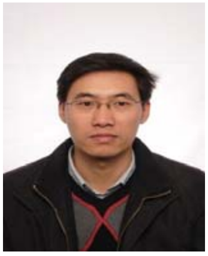

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