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Multiple-attribute Decision-Making Method under a Single-Valued Neutrosophic Hesitant Fuzzy Environment

Abstract: On the basis of the combination of single-valued neutrosophic sets and hesitant fuzzy sets, this article proposes a single-valued neutrosophic hesitant fuzzy set (SVNHFS) as a further generalization of the concepts of fuzzy set, intuitionistic fuzzy set, single-valued neutrosophic set, hesitant fuzzy set, and dual hesitant fuzzy set. Then, we introduce the basic operational relations and cosine measure function of SVNHFSs. Also, we develop a single-valued neutrosophic hesitant fuzzy weighted averaging (SVNHFWA) operator and a single-valued neutrosophic hesitant fuzzy weighted geometric (SVNHFWG) operator and investigate their properties. Furthermore, a multiple-attribute decision-making method is established on the basis of the SVNHFWA and SVNHFWG operators and the cosine measure under a single-valued neutrosophic hesitant fuzzy environment. Finally, an illustrative example of investment alternatives is given to demonstrate the application and effectiveness of the developed approach.

Keywords: Single-valued neutrosophic hesitant fuzzy set, single-valued neutrosophic hesitant fuzzy weighted averaging (SVNHFWA) operator, single-valued neutrosophic hesitant fuzzy weighted geometric (SVNHFWG) operator, decision making.

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1 Introduction

The neutrosophic set [2] is a powerful general formal framework that generalizes the concept of the classic set, fuzzy set, interval-valued fuzzy set, intuitionistic fuzzy set, interval-valued intuitionistic fuzzy set, paraconsistent set, dialetheist set, paradoxist set, and tautological set. In the neutrosophic set, indeterminacy is quantified explicitly, and truth membership, indeterminacy membership, and falsity membership are independent. However, the neutrosophic set generalizes the above-mentioned sets from a philosophical point of view, and its functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$, i.e., $T_A(x): X \to]^-0, 1^+[$, $I_A(x): X \to]^-0, 1^+[$, and $F_A(x): X \to]^-0, 1^+[$. Thus, it will be difficult to apply in real scientific and engineering areas [5, 6]. Therefore, Wang et al. [5, 6] proposed the concepts of an interval neutrosophic set (INS) and a single-valued neutrosophic set (SVNS), which are the subclasses of a neutrosophic set, and provided the set-theoretic operators and various properties of SVNSs and INSs. SVNSs and INSs provide an additional possibility to represent uncertain, imprecise, incomplete, and inconsistent information that exists in the real world. They would be more suitable to handle indeterminate information and inconsistent information. Recently, Ye [13] presented the correlation coefficient of SVNSs based on the extension of the correlation of intuitionistic fuzzy sets and proved that the cosine similarity measure is a special case of the correlation coefficient of SVNSs, and then applied it to single-valued neutrosophic decision-making problems. Then, Ye [14] presented another form of correlation coefficient between SVNSs and applied it to multiple-attribute

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decision making under a single-valued neutrosophic environment. Ye [15] proposed a single-valued neutrosophic cross-entropy measure and applied it to multicriteria decision-making problems with single-valued neutrosophic information. Ye [16] also introduced the Hamming and Euclidean distances between INSs and the similarity measures, and then applied them to decision-making problems in an interval neutrosophic setting. Furthermore, Ye [17] presented the concept of a simplified neutrosophic set (SNS), which is a subclass of the neutrosophic set and includes an SVNS and an INS, and defined some operations of SNSs. Then, he developed a simplified neutrosophic weighted averaging (SNWA) operator, a simplified neutrosophic weighted geometric (SNWG) operator, and a multicriteria decision-making method using the SNWA and SNWG operators and the cosine measure of SNSs under a simplified neutrosophic environment.

In fuzzy decision-making problems, however, decision makers sometimes find it difficult to determine the membership of an element to a set, and in some circumstances they cause this difficult problem of giving a few different values due to doubt. To deal with such cases, Torra and Narukawa [3] and Torra [4] presented the concept of a hesitant fuzzy set (HFS) as another extension of the fuzzy set. Xu and Xia [9, 10] defined some similarity measures, distance and correlation measures of HFSs, and applied them to multicriteria decision making. Then, Xia and Xu [8] developed a series of aggregation operators for hesitant fuzzy information and applied them in solving decision-making problems. Xu and Xia [11] introduced hesitant fuzzy entropy and cross-entropy and their use in multiple-attribute decision making. Zhu et al. [19] introduced hesitant fuzzy geometric Bonferroni means and applied the proposed aggregation operators to multicriteria decision making. Wei [7] also introduced hesitant fuzzy prioritized operators and their application to multiple-attribute decision making. Chen et al. [1] generalized the concept of HFS to that of interval-valued HFS (IVHFS) in which the membership degrees of an element to a given set are not exactly defined but denoted by several possible interval values, and gave systematic aggregation operators to aggregate interval-valued hesitant fuzzy information. They then developed an approach to group decision making based on interval-valued hesitant preference relations to consider the differences of opinions between individual decision makers. Xu et al. [12] investigated the aggregation of hesitancy fuzzy information, proposed several series of aggregation operators, and discussed their connections. Then, they applied the Choquet integral to obtain the weights of criteria and established a group decision-making method under a hesitant fuzzy environment.

However, in some situations, decision makers sometimes cause this difficult problem of assigning a few different values for satisfied and unsatisfied degrees. Thus, the HFS and the IVHFS are difficult to use for such a decision-making problem. To handle such cases, Zhu et al. [20] originally introduced a dual HFS (DHFS) as a generalization of HFSs, which encompasses fuzzy sets, intuitionistic fuzzy sets, HFSs, and fuzzy multisets as special cases [20]. The DHFS consists of two parts – the membership hesitancy function and the non-membership hesitancy function – and can handle two kinds of hesitancy in this situation. Then, Ye [18] proposed a correlation coefficient of DHFSs and applied it to multiple-attribute decision-making problems under a dual hesitant fuzzy environment.

As mentioned above, hesitancy is the most common problem in decision making, for which HFS can be considered as a suitable means allowing several possible degrees for an element to a set. However, in an HFS, there is only one truth-membership hesitant function, and it cannot express this problem with a few different values assigned by truth-membership degrees, indeterminacy-membership degrees, and falsity-membership degrees, due to doubts of decision makers. Thus, in this situation, it can represent only one kind of hesitancy information and cannot express three kinds of hesitancy information. An SVNS is an instance of a neutrosophic set that provides an additional possibility to represent uncertain, imprecise, incomplete, and inconsistent information that exists in the real world. It would be more suitable to handle indeterminate information and inconsistent information. However, it can only express a truth-membership degree, an indeterminacy-membership degree, and a falsity-membership degree, and it cannot represent this problem with a few different values assigned by truth-membership degrees, indeterminacy-membership degrees, and falsity-membership degrees, respectively, due to doubts of decision makers. In such a situation, the aforementioned algorithms based on neutrosophic sets or HFSs and their decision-making methods are difficult to use for such a decision-making problem with three kinds of hesitancy information that exists in the real world. To handle this case, we need to introduce a concept of a single-valued neutrosophic HFS (SVNHFS), which is a combination of HFS and SVNS, and some algorithms for SVNHFSs. The SVNHFS consists of three parts – the truth-membership hesitancy function, indeterminacy-membership hesitancy function, and falsity-membership hesitancy function – and can express three kinds of hesitancy information in this situation. Therefore, the purposes of this article are (i) to propose an SVNHFS based on the combination of SVNS and HFS, (ii) to introduce the basic operational relations and cosine measure function of SVNHFSs, (iii) to develop a single-valued neutrosophic hesitant fuzzy weighted averaging (SVNHFWA) operator and a single-valued neutrosophic hesitant fuzzy weighted geometric (SVNHFWG) operator and investigate their properties, and (iv) to establish a multiple-attribute decision making method based on the SVNHFWA and SVNHFWG operators and the cosine measure under a single-valued neutrosophic hesitant fuzzy environment.

The rest of the article is organized as follows. Section 2 briefly describes some concepts of neutrosophic sets, SVNSs, HFSs, and DHFSs. Section 3 proposes the concept of SVNHFSs and defines the corresponding basic operations and cosine measure for SVNHFSs. In Section 4, we develop the SVNHFWA and SVNHFWG operators and investigate their properties. Section 5 establishes a decision-making approach based on the SVNHFWA and SVNHFWG operators and the cosine measure. An illustrative example validating our approach is presented and discussed in Section 6. Section 7 contains the conclusion and future research direction.

2 Preliminaries

2.1 Some Concepts of Neutrosophic Sets and SVNSs

A neutrosophic set is a part of neutrosophy, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra [2], and is a powerful general formal framework that generalizes the above-mentioned sets from a philosophical point of view. Smarandache [2] originally gave the following definition of a neutrosophic set.

Definition 1 [2]**.** Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or nonstandard subsets of $]^-0, 1^+[$, i.e., $T_A(x): X \to]^-0, 1^+[$, $I_A(x): X \to]^-0, 1^+[$, and $F_A(x): X \to]^-0, 1^+[$.

There is no restriction on the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$; thus, $^-0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$ [2].

Definition 2 [2]**.** The complement of a neutrosophic set A is denoted by A^c and is defined as $T_A^c(x) = \{1^+\} \ominus T_A(x)$, $I_A^c(x) = \{1^+\} \ominus I_A(x)$, and $F_A^c(x) = \{1^+\} \ominus F_A(x)$ for every x in X.

Definition 3 [2]**.** A neutrosophic set A is contained in the other neutrosophic set B, $A \subseteq B$, if and only if $\inf T_A(x) \le \inf T_B(x)$, $\sup T_A(x) \le \sup T_B(x)$, $\inf I_A(x) \ge \inf I_B(x)$, $\sup I_A(x) \ge \sup I_B(x)$, $\inf F_A(x) \ge \inf F_B(x)$, and $\sup F_A(x) \ge \sup F_B(x)$ for every x in X.

For application in real scientific and engineering areas, Wang et al. [6] proposed the concept of an SVNS, which is an instance of a neutrosophic set. In the following, we introduce the definition of SVNS [6].

Definition 4 [6]**.** Let X be a space of points (objects) with generic elements in X denoted by x. An SVNS A in X is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$, where $T_A(x)$, $I_A(x)$, and $F_A(x) \in [0, 1]$ for each point x in X. Then, an SVNS A can be expressed as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}.$$

Thus, the SVNS satisfies the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

The following relations for SVNSs, A, B, are defined in Ref. [6]:

1. $A \subseteq B$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, and $F_A(x) \geq F_B(x)$ for any x in X.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $A^c = \{\langle x, F_A(x), 1 - I_A(x), T_A(x)\rangle \mid x \in X\}$.

For convenience, an SVNS A is denoted by $A = \langle T_A(x), I_A(x), F_A(x)\rangle$ for any x in X. For two SVNSs A and B, the operational relations [6] are defined as follows:
1. $A \cup B = \langle \max(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \min(F_A(x), F_B(x))\rangle$ for any x in X.
2. $A \cap B = \langle \min(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \max(F_A(x), F_B(x))\rangle$ for any x in X.
3. $A \times B = \langle T_A(x) + T_B(x) - T_A(x)T_B(x), I_A(x)I_B(x), F_A(x)F_B(x)\rangle$ for any x in X.

2.2 Hesitant Fuzzy Sets

As a generalization of fuzzy set, HFS [3, 4] is a very useful tool in some situations where there are some difficulties in determining the membership of an element to a set caused by a doubt between a few different values. As there are several possible values in determining the membership of an element to a set, Torra and Narukawa [3] and Torra [4] first proposed the concept of HFS, which is defined as follows.

Definition 5 [3, 4]**.** Let X be a fixed set; an HFS A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of [0, 1], which can be represented as the following mathematical symbol:

$$A = \{\langle x, h_A(x)\rangle \mid x \in X\},$$

where $h_A(x)$ is a set of some different values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to A. For convenience, we call $h_A(x)$ a hesitant fuzzy element [9, 10].

Definition 6 [3, 4]**.** Given a hesitant fuzzy element h, its lower and upper bounds are defined as $h^-(x) = \min h(x)$ and $h^+(x) = \max h(x)$, respectively.

Definition 7 [3, 4]**.** Given a hesitant fuzzy element h, $A_{\text{env}}(h)$ is called the envelope of h, which is represented by $(h^-, 1 - h^+)$, with the lower bound h^- and upper bound h^+.

From this definition, Torra and Narukawa [3] and Torra [4] gave the relationship between an HFS and an intuitionistic fuzzy set, i.e., $A_{\text{env}}(h)$ is defined as $\{<x, \mu(x), \nu(x)>\}$ with μ and ν defined by $\mu(x) = h^-(x)$, $\nu(x) = 1 - h^+(x)$, $x \in X$.

Given three hesitant fuzzy elements h, h_1, and h_2, Torra [4] defined some operations in them as follows:
1. $h^c = \bigcup_{\gamma \in h} \{1 - \gamma\}$.
2. $h_1 \cup h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \max\{\gamma_1, \gamma_2\}$.
3. $h_1 \cap h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \min\{\gamma_1, \gamma_2\}$.

Definition 8 [8]**.** For a hesitant element h, $s(h) = \frac{1}{\#h}\sum_{\gamma \in h} \gamma$ is called the score function of h, where $\#h$ is the number of the elements in h. For two hesitant elements h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

According to the relationship between a hesitant fuzzy element and an intuitionistic fuzzy value, Xia and Xu [8] defined some operations on the hesitant fuzzy elements h, h_1, and h_2 and a positive scale λ:

1. $h^\lambda = \bigcup_{\gamma \in h} \{\gamma^\lambda\}$.
2. $\lambda h = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^\lambda\}$.
3. $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$.
4. $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}$.

2.3 Dual Hesitant Fuzzy Sets

Zhu et al. [20] defined a DHFS as an extension of an HFS, in terms of two functions that return two sets of membership values and non-membership values, respectively, for each element in the domain as follows.

Definition 9 [20]**.** Let X be a fixed set; then, a DHFS D on X is defined as

$$D=\{\langle x, h(x), g(x)\rangle | x\in X\},$$

in which $h(x)$ and $g(x)$ are two sets of some values in [0, 1], denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set D, respectively, with the conditions $0 \le \gamma, \eta \le 1$, and $0 \le \gamma^+ + \eta^+ \le 1$, where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma \in h^+(x) = \cup_{\gamma \in h(x)} \max\{\gamma\}$, and $\eta^+ \in g^+(x) = \cup_{\eta \in g(x)} \max\{\eta\}$ for $x \in X$.

For convenience, the pair $d(x) = \{h(x), g(x)\}$ is called a dual hesitant fuzzy element (DHFE) denoted by $d = \{h, g\}$.

Definition 10 [20]**.** Let d_1 and d_2 be two DHFEs in a fixed set X; then, their union and intersection are defined, respectively, by

1. $d_1 \cup d_2 = \{h \in (h_1 \cup h_2) \,|\, h \ge \max(h_1^-, h_2^-),\, g \in (g_1 \cap g_2) \,|\, g \le \min(g_1^+, g_2^+)\}.$
2. $d_1 \cap d_2 = \{h \in (h_1 \cap h_2) \,|\, h \le \min(h_1^+, h_2^+),\, g \in (g_1 \cup g_2) \,|\, g \ge \max(g_1^-, g_2^-)\}.$

Then, Zhu et al. [19] gave the following operations:

1. $d_1 \oplus d_2 = \{h_1 \oplus h_2, g_1 \otimes g_2\} = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2} \{\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}, \{\eta_1\eta_2\}\}.$
2. $d_1 \otimes d_2 = \{h_1 \otimes h_2, g_1 \oplus g_2\} = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1, \gamma_2 \in h_2, \eta_2 \in g_2} \{\{\gamma_1\gamma_2\}, \{\eta_1 + \eta_2 - \eta_1\eta_2\}\}.$
3. $\lambda d_1 = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \{\{1-(1-\gamma_1)^\lambda\}, \{\eta_1^\lambda\}\}, \lambda > 0.$
4. $d_1^\lambda = \bigcup_{\gamma_1 \in h_1, \eta_1 \in g_1} \{\{\gamma_1^\lambda\}, \{1-(1-\eta_1)^\lambda\}\}, \lambda > 0.$

To compare the DHFEs, Zhu et al. [20] gave the following comparison laws.

Definition 11 [20]**.** Let $d_1 = \{h_1, g_1\}$ and $d_2 = \{h_2, g_2\}$ be any two DHFEs; then, the score function of d_i ($i = 1, 2$) is $S(d_i) = (1/\#h_i)\sum_{\gamma_i \in h_i} \gamma_i - (1/\#g_i)\sum_{\eta_i \in g_i} \eta_i$ ($i = 1, 2$) and the accuracy function of d_i ($i = 1, 2$) is $P(d_i) = (1/\#h_i)\sum_{\gamma_i \in h_i} \gamma_i + (1/\#g_i)\sum_{\eta_i \in g_i} \eta_i$ ($i = 1, 2$), where $\#h_i$ and $\#g_i$ are the numbers of the elements in h_i and g_i, respectively; then

i. If $S(d_1) > S(d_2)$, then d_1 is superior to d_2, denoted by $d_1 \succ d_2$.

ii. If $S(d_1) = S(d_2)$, then
   1. If $P(d_1) = P(d_2)$, then d_1 is equivalent to d_2, denoted by $d_1 \sim d_2$.
   2. If $P(d_1) > P(d_2)$, then d_1 is superior to d_2, denoted by $d_1 \succ d_2$.

3 Single-Valued Neutrosophic Hesitant Fuzzy Set

HFSs can reflect the original information given by the decision makers as much as possible, and more values are obtained from the decision makers or experts, which belonging to [0, 1] may be assigned to easily express hesitant judgments. Therefore, in this section, the concept of an SVNHFS is presented on the basis of the combination of SVNSs and HFSs as a further generalization of the concept of SVNSs and HFSs, which is defined as follows.

Definition 12. Let X be a fixed set; an SVNHFS on X is defined as

$$N=\{\langle x, \tilde{t}(x), \tilde{i}(x), \tilde{f}(x)\rangle \mid x\in X\},$$

in which $\tilde{t}(x)$, $\tilde{i}(x)$, and $\tilde{f}(x)$ are three sets of some values in [0, 1], denoting the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees of the element $x \in X$ to the set N, respectively, with the conditions $0 \le \delta, \gamma, \eta \le 1$ and $0 \le \gamma^+ + \delta^+ + \eta^+ \le 3$, where $\gamma\in\tilde{t}(x)$, $\delta\in\tilde{i}(x)$, $\eta\in\tilde{f}(x)$, $\gamma^+\in\tilde{t}^+(x)=\bigcup_{\gamma\in\tilde{t}(x)}\max\{\gamma\}$, $\delta^+\in\tilde{i}^+(x)=\bigcup_{\delta\in\tilde{i}(x)}\max\{\delta\}$, and $\eta^+\in\tilde{f}^+(x)=\bigcup_{\eta\in\tilde{f}(x)}\max\{\eta\}$ for $x\in X$.

For convenience, the three tuple $n(x)=\{\tilde{t}(x), \tilde{i}(x), \tilde{f}(x)\}$ is called a single-valued neutrosophic hesitant fuzzy element (SVNHFE) or a triple hesitant fuzzy element, which is denoted by the simplified symbol $n=\{\tilde{t}, \tilde{i}, \tilde{f}\}$.

From Definition 12, we can see that the SVNHFS consists of three parts, i.e., the truth-membership hesitancy function, the indeterminacy-membership hesitancy function, and the falsity-membership hesitancy function, supporting a more exemplary and flexible access to assign values for each element in the domain, and can handle three kinds of hesitancy in this situation. Thus, the existing sets, including fuzzy sets, intuitionistic fuzzy sets, SVNSs, HFSs, and DHFSs, can be regarded as special cases of SVNHFSs.

Then, we can define the union and intersection of SVNHFEs as follows.

Definition 13. Let n_1 and n_2 be two SVNHFEs in a fixed set X; then, their union and intersection are defined, respectively, by

1. $$n_1\cup n_2=\{\tilde{t}\in(\tilde{t}_1\cup\tilde{t}_2)\mid\tilde{t}\ge\max(\tilde{t}_1^-, \tilde{t}_2^-), \tilde{i}\in(\tilde{i}_1\cap\tilde{i}_2)\mid\tilde{i}\le\min(\tilde{i}_1^+, \tilde{i}_2^+), \tilde{f}\in(\tilde{f}_1\cap\tilde{f}_2)\mid\tilde{f}\le\min(\tilde{f}_1^+, \tilde{f}_2^+)\}.$$

2. $$n_1\cap n_2=\{\tilde{t}\in(\tilde{t}_1\cap\tilde{h}_2)\mid\tilde{t}\le\min(\tilde{t}_1^+, \tilde{t}_2^+), \tilde{i}\in(\tilde{i}_1\cup\tilde{i}_2)\mid\tilde{i}\ge\max(\tilde{i}_1^-, \tilde{i}_2^-), \tilde{f}\in(\tilde{f}_1\cup\tilde{f}_2)\mid\tilde{f}\ge\max(\tilde{f}_1^-, \tilde{f}_2^-)\}.$$

Therefore, for two SVNHFEs n_1 and n_2, we can give the following basic operations:

1. $$n_1\oplus n_2=\{\tilde{t}_1\oplus\tilde{t}_2, \tilde{i}_1\otimes\tilde{i}_2, \tilde{f}_1\otimes\tilde{f}_2\}=\bigcup_{\gamma_1\in\tilde{t}_1,\delta_1\in\tilde{i}_1,\eta_1\in\tilde{f}_1,\gamma_2\in\tilde{t}_2,\delta_2\in\tilde{i}_2,\eta_2\in\tilde{f}_2}\{\{\gamma_1+\gamma_2-\gamma_1\gamma_2\}, \{\delta_1\delta_2\}, \{\eta_1\eta_2\}\}.$$

2. $$n_1\otimes n_2=\{\tilde{t}_1\otimes\tilde{t}_2, \tilde{i}_1\oplus\tilde{i}_2, \tilde{f}_1\oplus\tilde{f}_2\}=\bigcup_{\gamma_1\in\tilde{t}_1,\delta_1\in\tilde{i}_1,\eta_1\in\tilde{f}_1,\gamma_2\in\tilde{t}_2,\delta_2\in\tilde{i}_2,\eta_2\in\tilde{f}_2}\{\{\gamma_1\gamma_2\}, \{\delta_1+\delta_2-\delta_1\delta_2\}, \{\eta_1+\eta_2-\eta_1\eta_2\}\}.$$

3. $$\lambda n_1=\bigcup_{\gamma_1\in\tilde{t}_1,\delta_1\in\tilde{i}_1,\eta_1\in\tilde{f}_1}\{\{1-(1-\gamma_1)^\lambda\}, \{\delta_1^\lambda\}, \{\eta_1^\lambda\}\}, \lambda>0.$$

4. $$n_1^\lambda=\bigcup_{\gamma_1\in\tilde{t}_1,\delta_1\in\tilde{i}_1,\eta_1\in\tilde{f}_1}\{\{\gamma_1^\lambda\}, \{1-(1-\delta_1)^\lambda\}, \{1-(1-\eta_1)^\lambda\}\}, \lambda>0.$$

On the basis of the cosine measure of SVNSs [13, 17], we give the following cosine measure between SVNHFEs.

Definition 14. Let $n_1=\{\tilde{t}_1, \tilde{i}_1, \tilde{f}_1\}$ and $n_2=\{\tilde{t}_2, \tilde{i}_2, \tilde{f}_2\}$ be any two SVNHFEs; thus, the cosine measure between n_1 and n_2 is defined by

$$\cos(n_1, n_2)=\frac{\left(\frac{1}{l_1}\sum_{\gamma_1\in\tilde{t}_1}\gamma_1\right)\cdot\left(\frac{1}{l_2}\sum_{\gamma_2\in\tilde{t}_2}\gamma_2\right)+\left(\frac{1}{p_1}\sum_{\delta_1\in\tilde{i}_1}\delta_1\right)\cdot\left(\frac{1}{p_2}\sum_{\delta_2\in\tilde{i}_2}\delta_2\right)+\left(\frac{1}{q_1}\sum_{\eta_1\in\tilde{f}_1}\eta_1\right)\cdot\left(\frac{1}{q_2}\sum_{\eta_2\in\tilde{f}_2}\eta_2\right)}{\sqrt{\left(\frac{1}{l_1}\sum_{\gamma_1\in\tilde{t}_1}\gamma_1\right)^2+\left(\frac{1}{p_1}\sum_{\delta_1\in\tilde{i}_1}\delta_1\right)^2+\left(\frac{1}{q_1}\sum_{\eta_1\in\tilde{f}_1}\eta_1\right)^2}\sqrt{\left(\frac{1}{l_2}\sum_{\gamma_2\in\tilde{t}_2}\gamma_2\right)^2+\left(\frac{1}{p_2}\sum_{\delta_2\in\tilde{i}_2}\delta_2\right)^2+\left(\frac{1}{q_2}\sum_{\eta_2\in\tilde{f}_2}\eta_2\right)^2}}, \tag{1}$$

where l_i, p_i, and q_i for $i = 1, 2$ are the numbers of the elements in $\tilde{t}_i, \tilde{i}_i, \tilde{f}_i$ for $i = 1, 2$, respectively, and $\cos(n_1, n_2) \in [0, 1]$.

To compare the SVNHFEs, we give the following comparative laws based on the cosine measure.

Let $n_1 = \{\tilde{t}_1, \tilde{i}_1, \tilde{f}_1\}$ and $n_2 = \{\tilde{t}_2, \tilde{i}_2, \tilde{f}_2\}$ be any two SVNHFEs; thus, the cosine measure between n_i ($i = 1, 2$) and the ideal element $n^* = <1, 0, 0>$ is obtained by applying Eq. (1) as follows:

$$\cos(n_i, n^*) = \frac{\frac{1}{l_i}\sum_{\gamma_i \in \tilde{t}_i}\gamma_i}{\sqrt{\left(\frac{1}{l_i}\sum_{\gamma_i \in \tilde{t}_i}\gamma_i\right)^2 + \left(\frac{1}{p_i}\sum_{\delta_i \in \tilde{i}_i}\delta_i\right)^2 + \left(\frac{1}{q_i}\sum_{\eta_i \in \tilde{f}_i}\eta_i\right)^2}}, \tag{2}$$

where l_i, p_i, and q_i for $i = 1, 2$ are the numbers of the elements in $\tilde{t}_i, \tilde{i}_i, \tilde{f}_i$ for $i = 1, 2$, respectively, and $\cos(n_i, n^*) \in [0, 1]$ for $i = 1, 2$. Then, there are the following comparative laws based on the cosine measure:

1. If $\cos(n_1, n^*) > \cos(n_2, n^*)$, then n_1 is superior to n_2, denoted by $n_1 \succ n_2$.
2. If $\cos(n_1, n^*) = \cos(n_2, n^*)$, then n_1 is equivalent to n_2, denoted by $n_1 \sim n_2$.

4 Weighted Aggregation Operators for SVNHFEs

On the basis of the basic operations of SVNHFEs in Section 3, this section proposes the following two weighted aggregation operators for SVNHFEs as a further generalization of the weighted aggregation operators of HFSs and SNSs [8, 17], and investigates their properties as the weighted aggregation operators are more basic operators in information aggregation and decision making.

4.1 SVNHFWA Operator

Definition 15. Let n_j ($j = 1, 2, \ldots, k$) be a collection of SVNHFEs; then, we define the SVNHFWA operator as follows:

$$\mathrm{SVNHFWA}(n_1, n_2, \ldots, n_k) = \sum_{j=1}^{k} w_j n_j, \tag{3}$$

where $\boldsymbol{w} = (w_1, w_2, \ldots, w_k)^T$ is the weight vector of n_j ($j = 1, 2, \ldots, k$), and $w_j > 0$, $\sum_{j=1}^{k} w_j = 1$.

On the basis of the basic operations of SVNHFEs described in Section 3 and Definition 15, we can derive Theorem 1.

Theorem 1. *Let n_j ($j = 1, 2, \ldots, k$) be a collection of SVNHFEs; then, the aggregated result of the SVNHFWA operator is also an SVNHFE, and*

$$\begin{aligned}\mathrm{SVNHFWA}(n_1, n_2, \ldots, n_k) &= \sum_{j=1}^{k} w_j n_j \\ &= \bigcup_{\gamma_1 \in \tilde{t}_1, \gamma_2 \in \tilde{t}_2, \ldots, \gamma_k \in \tilde{t}_k, \delta_1 \in \tilde{i}_1, \delta_2 \in \tilde{i}_2, \ldots, \delta_k \in \tilde{i}_k, \eta_1 \in \tilde{f}_1, \eta_2 \in \tilde{f}_2, \ldots, \eta_k \in \tilde{f}_k} \left\{\left\{1 - \prod_{j=1}^{k}(1-\gamma_j)^{w_j}\right\}, \left\{\prod_{j=1}^{k}\delta_j^{w_j}\right\}, \left\{\prod_{j=1}^{k}\eta_j^{w_j}\right\}\right\},\end{aligned} \tag{4}$$

where $\boldsymbol{w} = (w_1, w_2, \ldots, w_k)^T$ is the weight vector of n_j ($j = 1, 2, \ldots, k$), and $w_j > 0$, $\sum_{j=1}^{k} w_j = 1$.

Proof. The proof of Eq. (4) can be done by means of mathematical induction.

1. When $k=2$, then,

$$w_1 n_1 = \bigcup_{\gamma_1 \in \tilde{t}_1, \delta_1 \in \tilde{i}_1, \eta_1 \in \tilde{f}_1} \{\{1-(1-\gamma_1)^{w_1}\}, \{\delta_1^{w_1}\}, \{\eta_1^{w_1}\}\},$$

$$w_2 n_2 = \bigcup_{\gamma_2 \in \tilde{t}_2, \delta_2 \in \tilde{i}_2, \eta_2 \in \tilde{f}_2} \{\{1-(1-\gamma_2)^{w_2}\}, \{\delta_2^{w_2}\}, \{\eta_2^{w_2}\}\}.$$

Thus,

$$\begin{aligned} w_1 n_1 \oplus w_2 n_2 = & \bigcup_{\gamma_1 \in \tilde{t}_1, \gamma_2 \in \tilde{t}_2, \delta_1 \in \tilde{i}_1, \delta_2 \in \tilde{i}_2, \eta_1 \in \tilde{f}_1, \eta_2 \in \tilde{f}_2} \\ & \{\{1-(1-\gamma_1)^{w_1} + (1-(1-\gamma_2)^{w_2}) - (1-(1-\gamma_1)^{w_1})(1-(1-\gamma_2)^{w_2}\}, \{\delta_1^{w_1}\delta_2^{w_2}\}, \{\eta_1^{w_1}\eta_2^{w_2}\}\}. \\ = & \bigcup_{\gamma_1 \in \tilde{t}_1, \gamma_2 \in \tilde{t}_2, \delta_1 \in \tilde{i}_1, \delta_2 \in \tilde{i}_2, \eta_1 \in \tilde{f}_1, \eta_2 \in \tilde{f}_2} \{\{1-(1-\gamma_1)^{w_1}(1-\gamma_2)^{w_2}\}, \{\delta_1^{w_1}\delta_2^{w_2}\}, \{\eta_1^{w_1}\eta_2^{w_2}\}\}. \end{aligned} \quad (5)$$

2. When $k=m$, by applying Eq. (4), we get

$$\begin{aligned} \text{SVNHFWA}\{n_1, n_2, \ldots, n_m\} = \sum_{j=1}^{m} w_j n_j = & \bigcup_{\gamma_1 \in \tilde{t}_1, \gamma_2 \in \tilde{t}_2, \ldots, \gamma_m \in \tilde{t}_m, \delta_1 \in \tilde{i}_1, \delta_2 \in \tilde{i}_2, \ldots, \delta_m \in \tilde{i}_m, \eta_1 \in \tilde{f}_1, \eta_2 \in \tilde{f}_2, \ldots, \eta_m \in \tilde{f}_m} . \\ & \left\{\left\{1-\prod_{j=1}^{m}(1-\gamma_j)^{w_j}\right\}, \left\{\prod_{j=1}^{m}\delta_j^{w_j}\right\}, \left\{\prod_{j=1}^{m}\eta_j^{w_j}\right\}\right\} \end{aligned} \quad (6)$$

3. When $k=m+1$, by applying Eqs. (5) and (6), we can get

$$\begin{aligned} \text{SVNHFWA}(n_1, n_2, \ldots, n_{m+1}) = \sum_{j=1}^{m+1} w_j n_j = & \bigcup_{\gamma_1 \in \tilde{t}_1, \gamma_2 \in \tilde{t}_2, \ldots, \gamma_{m+1} \in \tilde{t}_{m+1}, \delta_1 \in \tilde{i}_1, \delta_2 \in \tilde{i}_2, \ldots, \delta_{m+1} \in \tilde{i}_{m+1}, \eta_1 \in \tilde{f}_1, \eta_2 \in \tilde{f}_2, \ldots, \eta_{m+1} \in \tilde{f}_{m+1}} \\ & \left\{\left\{1-\prod_{j=1}^{m}(1-\gamma_j)^{w_j} + (1-(1-\gamma_{m+1})^{w_{m+1}}) - \left(1-\prod_{j=1}^{m}(1-\gamma_j)^{w_j}\right)\left(1-(1-\gamma_{m+1})^{w_{m+1}}\right)\right\},\right. \\ & \left.\left\{\prod_{j=1}^{m}\delta_j^{w_j}\delta_{m+1}^{w_{m+1}}\right\}, \left\{\prod_{j=1}^{m}\eta_j^{w_j}\eta_{m+1}^{w_{m+1}}\right\}\right\} \\ = & \bigcup_{\gamma_1 \in \tilde{t}_1, \gamma_2 \in \tilde{t}_2, \ldots, \gamma_{m+1} \in \tilde{t}_{m+1}, \delta_1 \in \tilde{i}_1, \delta_2 \in \tilde{i}_2, \ldots, \delta_{m+1} \in \tilde{i}_{m+1}, \eta_1 \in \tilde{f}_1, \eta_2 \in \tilde{f}_2, \ldots, \eta_{m+1} \in \tilde{f}_{m+1}} \left\{\left\{1-\prod_{j=1}^{m+1}(1-\gamma_j)^{w_j}\right\}, \left\{\prod_{j=1}^{m+1}\delta_j^{w_j}\right\}, \left\{\prod_{j=1}^{m+1}\eta_j^{w_j}\right\}\right\}. \end{aligned} \quad (7)$$

Therefore, considering the above results, we have Eq. (4) for any k. This completes the proof. □

It is obvious that the SVNHFWA operator has the following properties:

1. Idempotency: Let n_j ($j = 1, 2, \ldots, k$) be a collection of SVNHFEs. If n_i ($j = 1, 2, \ldots, k$) is equal, i.e., $n_j = n$ for $j = 1, 2, \ldots, k$, then SVNHFWA $(n_1, n_2, \ldots, n_k) = n$.
2. Boundedness: Let n_j ($j = 1, 2, \ldots, k$) be a collection of SVNHFEs and let $n^- = \{\{r^-\}, \{\delta^+\}, \{\eta^+\}\}$ and $n^+ = \{\{r^+\}, \{\delta^-\}, \{\eta^-\}\}$, where $\gamma^- = \bigcup_{\gamma_j \in \tilde{t}_j} \min_j\{\gamma_j\}$, $\delta^- = \bigcup_{\delta_j \in \tilde{i}_j} \min_j\{\delta_j\}$, $\eta^- = \bigcup_{\eta_j \in \tilde{f}_j} \min_j\{\eta_j\}$, $\gamma^+ = \bigcup_{\gamma_j \in \tilde{t}_j} \max_j\{\gamma_j\}$, $\delta^+ = \bigcup_{\delta_j \in \tilde{i}_j} \max_j\{\delta_j\}$, and $\eta^+ = \bigcup_{\eta_j \in \tilde{f}_j} \max_j\{\eta_j\}$ for $j = 1, 2, \ldots, k$; then, $n^- \le$ SVNHFWA $(n_1, n_2, \ldots, n_k) \le n^+$.
3. Monotonity: Let n_j ($j = 1, 2, \ldots, k$) be a collection of SVNHFEs. If $n_j \le n_j'$ for $j = 1, 2, \ldots, k$, then $\text{SVNHFWA}(n_1, n_2, \ldots, n_k) \le \text{SVNHFWA}(n_1', n_2', \ldots, n_k')$.

4.2 SVNHFWG Operator

Definition 16. Let n_j ($j = 1, 2, \ldots, k$) be a collection of SVNHFEs; then, we define the SVNHFWG operator as follows:

$$\mathrm{SVNHFWG}(n_1, n_2, \ldots, n_k) = \prod_{j=1}^{k} n_j^{w_j}, \tag{8}$$

where $\boldsymbol{w} = (w_1, w_2, \ldots, w_k)^{\mathrm{T}}$ is the weight vector of n_j ($j = 1, 2, \ldots, k$), and $w_j > 0$, $\sum_{j=1}^{k} w_j = 1$.

On the basis of the basic operations of SVNHFEs described in Section 3 and Definition 16, we can derive Theorem 2.

Theorem 2. *Let n_j ($j = 1, 2, \ldots, k$) be a collection of SVNHFEs; then, the aggregated result of the SVNHFWG operator is also an SVNHFE, and*

$$\mathrm{SVNHFWG}(n_1, n_2, \ldots, n_k) = \prod_{j=1}^{k} n_j^{w_j}$$
$$= \bigcup_{\gamma_1 \in \tilde{t}_1, \gamma_2 \in \tilde{t}_2, \ldots, \gamma_k \in \tilde{t}_k, \delta_1 \in \tilde{i}_1, \delta_2 \in \tilde{i}_2, \ldots, \delta_k \in \tilde{i}_k, \eta_1 \in \tilde{f}_1, \eta_2 \in \tilde{f}_2, \ldots, \eta_k \in \tilde{f}_k} \left\{ \left\{ \prod_{j=1}^{k} \gamma_j^{w_j} \right\}, \left\{ 1 - \prod_{j=1}^{k} (1-\delta_j)^{w_j} \right\}, \left\{ 1 - \prod_{j=1}^{k} (1-\eta_j)^{w_j} \right\} \right\}, \tag{9}$$

where $\boldsymbol{w} = (w_1, w_2, \ldots, w_k)^{T}$ is the weight vector of n_j ($j = 1, 2, \ldots, k$), and $w_j > 0$, $\sum_{j=1}^{k} w_j = 1$.

Similar to the manner of the above proof, we can give the proof of Theorem 2 (omitted).

It is obvious that the SVNHFWG operator also has the following properties:

1. Idempotency: Let n_j ($j = 1, 2, \ldots, k$) be a collection of SVNHFEs. If n_i ($j = 1, 2, \ldots, k$) is equal, i.e., $n_j = n$ for $j = 1, 2, \ldots, k$, then SVNHFWG $(n_1, n_2, \ldots, n_k) = n$.
2. Boundedness: Let n_j ($j = 1, 2, \ldots, k$) be a collection of SVNHFEs and let $n^- = \{\{r^-\}, \{\delta^+\}, \{\eta^+\}\}$ and $n^+ = \{\{r^+\}, \{\delta^-\}, \{\eta^-\}\}$, where $\gamma^- = \bigcup_{\gamma_j \in \tilde{t}_j} \min_j \{\gamma_j\}$, $\delta^- = \bigcup_{\delta_j \in \tilde{i}_j} \min_j \{\delta_j\}$, $\eta^- = \bigcup_{\eta_j \in \tilde{f}_j} \min_j \{\eta_j\}$, $\gamma^+ = \bigcup_{\gamma_j \in \tilde{t}_j} \max_j \{\gamma_j\}$, $\delta^+ = \bigcup_{\delta_j \in \tilde{i}_j} \max_j \{\delta_j\}$, and $\eta^+ = \bigcup_{\eta_j \in \tilde{f}_j} \max_j \{\eta_j\}$ for $j = 1, 2, \ldots, k$; then, $n^- \le$ SVNHFWG $(n_1, n_2, \ldots, n_k) \le n^+$.
3. Monotonity: Let n_j ($j = 1, 2, \ldots, k$) be a collection of SVNHFEs. If $n_j \le n_j'$ for $j = 1, 2, \ldots, k$; then, $\mathrm{SVNHFWG}(n_1, n_2, \ldots, n_k) \le \mathrm{SVNHFWG}(n_1', n_2', \ldots, n_k')$.

5 Decision-Making Method Based on the SVNHFWA and SVNHFWG Operators

In this section, we apply the SVNHFWA and SVNHFWG operators to multiple-attribute decision-making problems with single-valued neutrosophic hesitant fuzzy information.

For a multiple-attribute decision-making problem under a single-valued neutrosophic hesitant fuzzy environment, let $A = \{A_1, A_2, \ldots, A_m\}$ be a discrete set of alternatives and $C = \{C_1, C_2, \ldots, C_k\}$ be a discrete set of attributes. When the decision makers are required to evaluate the alternative A_i ($i = 1, 2, \ldots, m$) under the attribute C_j ($j = 1, 2, \ldots, k$), they may assign a set of several possible values to each of truth-membership degrees, indeterminacy-membership degrees, and falsity-membership degrees to which an alternative A_i ($i = 1, 2, \ldots, m$) satisfies and/or dissatisfies an attribute C_j ($j = 1, 2,\ldots, k$), and then these evaluated values can be expressed as an SVNHFE $n_{ij} = (\tilde{t}_{ij}, \tilde{i}_{ij}, \tilde{f}_{ij})$ ($i = 1, 2,\ldots, m$; $j = 1, 2,\ldots, k$). Thus, we can elicit a single-valued neutrosophic hesitant fuzzy decision matrix $D = (n_{ij})_{m\times k}$, where n_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, k$) is in the form of SVNHFEs.

The weight vector of attributes for the different importance of each attribute is given as $\boldsymbol{w} = (w_1, w_2,\ldots, w_k)^{\mathrm{T}}$, where $w_j \ge 0$, $j = 1, 2, \ldots, k$, and $\sum_{j=1}^{k} w_j = 1$. Then, we use the SVNHFWA or SVNHFWG operator and the cosine measure to develop an approach for multiple-attribute decision-making problems with single-valued neutrosophic hesitant fuzzy information, which can be described as follows:

Step 1. Aggregate all SVNHFEs of n_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, k$) by using the SVNHFWA operator to derive the collective SVNHFE n_i ($i = 1, 2, \ldots, m$) for an alternative A_i ($i = 1, 2, \ldots, m$):

$$\begin{aligned} n_i &= \mathrm{SVNHFWA}(n_{i1}, n_{i2}, \ldots, n_{ik}) = \sum_{j=1}^{k} w_j n_{ij} \\ &= \bigcup_{\gamma_{i1}\in\tilde{t}_{i1},\gamma_{i2}\in\tilde{t}_{i2},\ldots,\gamma_{ik}\in\tilde{t}_{ik},\delta_{i1}\in\tilde{i}_{i1},\delta_{i2}\in\tilde{i}_{i2},\ldots,\delta_{ik}\in\tilde{i}_{ik},\eta_{i1}\in\tilde{f}_{i1},\eta_{i2}\in\tilde{f}_{i2},\ldots,\eta_{ik}\in\tilde{f}_{ik}} \left\{\left\{1-\prod_{j=1}^{k}(1-\gamma_{ij})^{w_j}\right\}, \left\{\prod_{j=1}^{k}\delta_{ij}^{w_j}\right\}, \left\{\prod_{j=1}^{k}\eta_{ij}^{w_j}\right\}\right\}, \end{aligned} \quad (10)$$

or by using the SVNHFWG operator:

$$\begin{aligned} n_i &= \mathrm{SVNHFWG}(n_{i1}, n_{i2}, \ldots, n_{ik}) = \prod_{j=1}^{k} n_{ij}^{w_j} \\ &= \bigcup_{\gamma_{i1}\in\tilde{t}_{i1},\gamma_{i2}\in\tilde{t}_{i2},\ldots,\gamma_{ik}\in\tilde{t}_{ik},\delta_{i1}\in\tilde{i}_{i1},\delta_{i2}\in\tilde{i}_{i2},\ldots,\delta_{ik}\in\tilde{i}_{ik},\eta_{i1}\in\tilde{f}_{i1},\eta_{i2}\in\tilde{f}_{i2},\ldots,\eta_{ik}\in\tilde{f}_{ik}} \left\{\left\{\prod_{j=1}^{k}\gamma_{ij}^{w_j}\right\}, \left\{1-\prod_{j=1}^{k}(1-\delta_{ij})^{w_j}\right\}, \left\{1-\prod_{j=1}^{k}(1-\eta_{ij})^{w_j}\right\}\right\}. \end{aligned} \quad (11)$$

Step 2. Calculate the measure values of the collective SVNHFE n_i ($i = 1, 2, \ldots, m$) and the ideal element $n^* = <1, 0, 0>$ by Eq. (2).

Step 3. Rank the alternatives and select the best one(s) in accordance with the measure values.

Step 4. End.

6 Illustrative Example

An illustrative example about investment alternatives for a multiple-attribute decision-making problem adapted from Ye [13, 18] is used as the demonstration of the applications of the proposed decision-making method under a single-valued neutrosophic hesitant fuzzy environment. There is an investment company that wants to invest a sum of money in the best option available. There is a panel with four possible alternatives in which to invest the money: (i) A_1 is a car company; (ii) A_2 is a food company; (iii) A_3 is a computer company; and (iv) A_4 is an arms company. The investment company must make a decision according to the following three attributes: (i) C_1 is the risk; (ii) C_2 is the growth; and (iii) C_3 is the environmental impact. The attribute weight vector is given as $\boldsymbol{w} = (0.35, 0.25, 0.4)^{\mathrm{T}}$. The four possible alternatives, A_i ($i = 1, 2, 3, 4$), are to be evaluated using the single-valued neutrosophic hesitant fuzzy information by some decision makers or experts under three attributes, C_j ($j = 1, 2, 3$).

For the evaluation of an alternative A_i ($i = 1, 2, 3, 4$) with respect to an attribute C_j ($j = 1, 2, 3$), it is obtained from the evaluation of some decision makers. For example, three decision makers discuss the degrees that an alternative A_1 should satisfy a criterion C_1 can be assigned with 0.5, 0.4, and 0.3; the degrees that an alternative A_1 with respect to an attribute C_1 may be unsure can be assigned with 0.1; and the degrees that an alternative dissatisfies an attribute C_1 can be assigned with 0.4 by two of three decision makers and 0.3 by one of three decision makers. For the single-valued neutrosophic hesitant fuzzy notation, they can be expressed as {{0.3, 0.4, 0.5}, {0.1}, {0.3, 0.4}}. Thus, when the four possible alternatives with respect to the above three attributes are evaluated by the three decision makers, we can obtain the single-valued neutrosophic hesitant fuzzy decision matrix D shown in Table 1.

Then, we use the developed approach to obtain the ranking order of the alternatives and the most desirable one(s), which can be described as follows:

Step 1. Aggregate all SVNHFEs of n_{ij} ($i = 1, 2, 3, 4$; $j = 1, 2, 3$) by using the SVNHFWA operator to derive the collective SVNHFE n_i ($i = 1, 2, 3, 4$) for an alternative A_i ($i = 1, 2, 3, 4$). Take an alternative A_1; for example, we have

Table 1. Single-Valued Neutrosophic Hesitant Fuzzy Decision Matrix *D*.

	C_1	C_2	C_3
A_1	{{0.3, 0.4, 0.5}, {0.1}, {0.3, 0.4}}	{{0.5, 0.6}, {0.2, 0.3}, {0.3,0.4}}	{{0.2, 0.3}, {0.1, 0.2}, {0.5, 0.6}}
A_2	{{0.6, 0.7}, {0.1, 0.2}, {0.2, 0.3}}	{{0.6, 0.7}, {0.1}, {0.3}}	{{0.6, 0.7}, {0.1, 0.2}, {0.1, 0.2}}
A_3	{{0.5, 0.6}, {0.4}, {0.2, 0.3}}	{{0.6}, {0.3}, {0.4}}	{{0.5, 0.6}, {0.1}, {0.3}}
A_4	{{0.7, 0.8}, {0.1}, {0.1, 0.2}}	{{0.6, 0.7}, {0.1}, {0.2}}	{{0.3, 0.5}, {0.2}, {0.1, 0.2, 0.3}}

$$\begin{aligned}
n_1 &= \text{SVNHFWA}(n_{11}, n_{12}, n_{13}) = \sum_{j=1}^{3} w_j n_{1j} \\
&= \bigcup_{\gamma_{11}\in\tilde{t}_{11},\gamma_{12}\in\tilde{t}_{12},\gamma_{13}\in\tilde{t}_{13},\delta_{11}\in\tilde{i}_{11},\delta_{12}\in\tilde{i}_{12},\delta_{13}\in\tilde{i}_{13},\eta_{11}\in\tilde{f}_{11},\eta_{12}\in\tilde{f}_{12},\eta_{13}\in\tilde{f}_{13}} \left\{\left\{1-\prod_{j=1}^{3}(1-\gamma_{1j})^{w_j}\right\}, \left\{\prod_{j=1}^{3}\delta_{1j}^{w_j}\right\}, \left\{\prod_{j=1}^{3}\eta_{1j}^{w_j}\right\}\right\} \\
&= \{\{1-(1-0.3)^{0.35}(1-0.5)^{0.25}(1-0.2)^{0.4},\ 1-(1-0.4)^{0.35}(1-0.5)^{0.25}(1-0.2)^{0.4}, \\
&\quad 1-(1-0.5)^{0.35}(1-0.5)^{0.25}(1-0.2)^{0.4},\ 1-(1-0.3)^{0.35}(1-0.6)^{0.25}(1-0.2)^{0.4}, \\
&\quad 1-(1-0.4)^{0.35}(1-0.6)^{0.25}(1-0.2)^{0.4},\ 1-(1-0.5)^{0.35}(1-0.6)^{0.25}(1-0.2)^{0.4}, \\
&\quad 1-(1-0.3)^{0.35}(1-0.5)^{0.25}(1-0.3)^{0.4},\ 1-(1-0.4)^{0.35}(1-0.5)^{0.25}(1-0.3)^{0.4}, \\
&\quad 1-(1-0.5)^{0.35}(1-0.5)^{0.25}(1-0.3)^{0.4},\ 1-(1-0.3)^{0.35}(1-0.6)^{0.25}(1-0.3)^{0.4}, \\
&\quad 1-(1-0.4)^{0.35}(1-0.6)^{0.25}(1-0.3)^{0.4},\ 1-(1-0.5)^{0.35}(1-0.6)^{0.25}(1-0.3)^{0.4}\}, \\
&\quad \{0.1^{0.35}0.2^{0.25}0.1^{0.4},\ 0.1^{0.35}0.2^{0.25}0.2^{0.4},\ 0.1^{0.35}0.3^{0.25}0.1^{0.4},\ 0.1^{0.35}0.3^{0.25}0.2^{0.4}\}, \\
&\quad \{0.3^{0.35}0.3^{0.25}0.5^{0.4},\ 0.4^{0.35}0.3^{0.25}0.5^{0.4},\ 0.3^{0.35}0.4^{0.25}0.5^{0.4},\ 0.4^{0.35}0.4^{0.25}0.5^{0.4}, \\
&\quad 0.3^{0.35}0.3^{0.25}0.6^{0.4},\ 0.4^{0.35}0.3^{0.25}0.6^{0.4},\ 0.3^{0.35}0.4^{0.25}0.6^{0.4},\ 0.4^{0.35}0.4^{0.25}0.6^{0.4}\}
\end{aligned}$$

and obtain the following collective SVNHFE n_1:

$$n_1 = \{\{0.3212, 0.3565, 0.3568, 0.358, 0.3903, 0.3914, 0.3917, 0.3966, 0.4234, 0.428, 0.4293, 0.459\}, \{0.1189, 0.1316, 0.1569, 0.1737\}, \{0.3680, 0.3955, 0.3959, 0.407, 0.4254, 0.4373, 0.4378, 0.4704\}\}.$$

Similar to the above calculation, we can derive the following collective SVNHFE n_i (i = 2, 3, 4):

$$n_2 = \{\{0.6, 0.6278, 0.6383, 0.6435, 0.6634, 0.6682, 0.6776, 0.7\}, \{0.1, 0.1275, 0.132, 0.1682\}, \{0.1677, 0.1933, 0.2213, 0.2551\}\};$$
$$n_3 = \{\{0.5233, 0.5578, 0.5629, 0.6\}, \{0.2138\}, \{0.2797, 0.3224\}\};$$
$$n_4 = \{\{0.5476, 0.579, 0.6045, 0.6074, 0.632, 0.6347, 0.6569, 0.6807\}, \{0.1320\}, \{0.1189, 0.1516, 0.1569, 0.1845, 0.2, 0.2352\}\}.$$

Step 2. Calculate the measure values of the collective SVNHFE n_i (i = 1, 2, 3, 4) and the ideal element $n^* = \langle 1, 0, 0\rangle$ by Eq. (2):

$$\cos(n_1, n^*) = 0.6636,\ \cos(n_2, n^*) = 0.9350,\ \cos(n_3, n^*) = 0.8353, \text{ and } \cos(n_4, n^*) = 0.9426.$$

Step 3. Rank the alternatives in accordance with the measure values $A_4 \succ A_2 \succ A_3 \succ A_1$. Therefore, we can see that the alternative A_4 is the best choice.

If we utilize the SVNHFWG operator for the multiple-attribute decision-making problem, the decision-making procedure can be described as follows:

Step 1′. Aggregate all SVNHFEs of n_{ij} (i = 1, 2, 3, 4; j = 1, 2, 3) by using the SVNHFWG operator to derive the collective SVNHFE n_i (i = 1, 2, 3, 4) for the alternative A_i (i = 1, 2, 3, 4). Take an alternative A_1; for example, we have

$$n_1 = \text{SVNHFWG}(n_{11}, n_{12}, n_{13}) = \prod_{j=1}^{3} n_{ij}^{w_j}$$
$$= \bigcup_{\gamma_{11}\in\tilde{t}_{11}, \gamma_{12}\in\tilde{t}_{12}, \gamma_{13}\in\tilde{t}_{13}, \delta_{11}\in\tilde{i}_{11}, \delta_{12}\in\tilde{i}_{12}, \delta_{13}\in\tilde{i}_{13}, \eta_{11}\in\tilde{f}_{11}, \eta_{12}\in\tilde{f}_{12}, \eta_{13}\in\tilde{f}_{13}} \left\{\left\{\prod_{j=1}^{3}\gamma_{1j}^{w_j}\right\}, \left\{1-\prod_{j=1}^{3}(1-\delta_{1j})^{w_j}\right\}, \left\{1-\prod_{j=1}^{3}(1-\eta_{1j})^{w_j}\right\}\right\}$$
$$=\{\{0.3^{0.35}0.5^{0.25}0.2^{0.4},\ 0.4^{0.35}0.5^{0.25}0.2^{0.4},\ 0.5^{0.35}0.5^{0.25}0.2^{0.4},\ 0.3^{0.35}0.6^{0.25}0.2^{0.4},\ 0.4^{0.35}0.6^{0.25}0.2^{0.4},$$
$$0.5^{0.35}0.6^{0.25}0.2^{0.4},\ 0.3^{0.35}0.5^{0.25}0.3^{0.4},\ 0.4^{0.35}0.5^{0.25}0.3^{0.4},\ 0.5^{0.35}0.5^{0.25}0.3^{0.4},\ 0.3^{0.35}0.6^{0.25}0.3^{0.4},$$
$$0.4^{0.35}0.6^{0.25}0.3^{0.4},\ 0.5^{0.35}0.6^{0.25}0.3^{0.4}\},\ \{1-(1-0.1)^{0.35}(1-0.2)^{0.25}(1-0.1)^{0.4},$$
$$1-(1-0.1)^{0.35}(1-0.2)^{0.25}(1-0.2)^{0.4},\ 1-(1-0.1)^{0.35}(1-0.3)^{0.25}(1-0.1)^{0.4},$$
$$1-(1-0.1)^{0.35}(1-0.3)^{0.25}(1-0.2)^{0.4}\},\ \{1-(1-0.3)^{0.35}(1-0.3)^{0.25}(1-0.5)^{0.4},$$
$$1-(1-0.4)^{0.35}(1-0.3)^{0.25}(1-0.5)^{0.4},\ 1-(1-0.3)^{0.35}(1-0.4)^{0.25}(1-0.5)^{0.4},$$
$$1-(1-0.4)^{0.35}(1-0.4)^{0.25}(1-0.5)^{0.4},\ 1-(1-0.3)^{0.35}(1-0.3)^{0.25}(1-0.6)^{0.4},$$
$$1-(1-0.4)^{0.35}(1-0.3)^{0.25}(1-0.6)^{0.4},\ 1-(1-0.3)^{0.35}(1-0.4)^{0.25}(1-0.6)^{0.4},$$
$$1-(1-0.4)^{0.35}(1-0.4)^{0.25}(1-0.6)^{0.4}\}\}$$

and obtain the following collective SVNHFE n_1:

$$n_1 = \{\{0.2898, 0.3033, 0.3205, 0.3355, 0.3409, 0.3466, 0.3568, 0.3627, 0.377, 0.3946, 0.4076, 0.4266\}, \{0.1261, 0.1548, 0.1663, 0.1937\}, \{0.3881, 0.4113, 0.4203, 0.4404, 0.4422, 0.4615, 0.4698, 0.4898\}\}.$$

Similar to the above calculation, we can derive the following collective SVNHFE n_i ($i = 2, 3, 4$):

$$n_2 = \{\{0.6, 0.6236, 0.6333, 0.6382, 0.6581, 0.6632, 0.6735, 0.7\}, \{0.1, 0.1363, 0.1414, 0.1761\}, \{0.1889, 0.226, 0.2263, 0.2616\}\};$$
$$n_3 = \{\{0.5233, 0.5578, 0.5629, 0.6\}, \{0.2666\}, \{0.2942, 0.3265\}\};$$
$$n_4 = \{\{0.4799, 0.4988, 0.5029, 0.5226, 0.5887, 0.6119, 0.6169, 0.6411\}, \{0.1414\}, \{0.1261, 0.1614, 0.1663, 0.2, 0.2097, 0.2416\}\}.$$

Step 2′. Calculate the measure $\cos(n_i, n^*)$ ($i = 1, 2, 3, 4$) of the collective SVNHFE n_i ($i = 1, 2, 3, 4$) for the alternative A_i ($i = 1, 2, 3, 4$) and the ideal element $n^* = <1, 0, 0>$ by Eq. (2):

$$\cos(n_1, n^*) = 0.604,\ \cos(n_2, n^*) = 0.9259,\ \cos(n_3, n^*) = 0.808, \text{ and } \cos(n_4, n^*) = 0.9232.$$

Step 3′. Rank the alternatives in accordance with the measure values $A_2 \succ A_4 \succ A_3 \succ A_1$. Therefore, we can see that the alternative A_2 is the best choice.

Obviously, the above two kinds of ranking orders are the same as the ones in Refs. [13, 18]. However, we can see that the different ranking orders are obtained from the SVNHFWA and SVNHFWG operators, as there are different focal points between the SVNHFWA operator and the SVNHFWG operator. The SVNHFWA operator emphasizes the group's major points, whereas the SVNHFWG operator emphasizes individual major points.

In this section, we have proposed the approach to solve a single-valued neutrosophic hesitant fuzzy multiple-attribute decision-making problem. The above example clearly indicates that the proposed decision-making method is applicable and effective under a single-valued neutrosophic hesitant fuzzy environment. Comparing the SVNHFS with the HFS and the DHFS, the SVNHFS contains more information because it takes into account the information of its truth-membership hesitant degrees, indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees, whereas the HFS only contains the information of its membership hesitant degrees. Then, the DHFS is a further generalization of the HFS, which includes the information of its membership hesitant degrees and non-membership hesitant degrees. Therefore, the decision-making method proposed in this article can deal with not only single-valued neutrosophic hesitant fuzzy decision-making problems but also indeterminate and inconsistent decision-making problems. To some extent, the decision-making method in single-valued neutrosophic hesitant fuzzy setting is more general and more practical than existing decision-making methods in fuzzy setting, intuitionistic fuzzy setting, hesitant fuzzy setting, dual hesitant fuzzy setting, and single-valued neutrosophic setting because SVNHFSs include the aforementioned fuzzy sets.

7 Conclusion

This article introduced the concept of SVNHFSs based on the combination of both HFSs and SVNSs as a further generalization of these fuzzy concepts, and defined some basic operations of SVNHFEs and the cosine measure of SVNHFEs. Then, we proposed the SVNHFWA and SVNHFWG operators and investigated their properties. Furthermore, the two aggregation operators were applied to multiple-attribute decision-making problems under a single-valued neutrosophic hesitant fuzzy environment, in which attribute values with respect to alternatives are evaluated by the form of SVNHFEs and the attribute weights are known information. We used the SVNHFWA and SVNHFWG operators and the cosine measure to rank the alternatives and determine the best one(s) according to the measure values. Finally, an illustrative example was provided to illustrate the application of the developed approach. The proposed multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment is more suitable for real scientific and engineering applications because the proposed decision-making method include more much information and can deal with indeterminate and inconsistent decision-making problems. Therefore, when we encounter some situations that are represented by indeterminate information and inconsistent information, the proposed decision-making method demonstrates great superiority in dealing with the single-valued neutrosophic hesitant fuzzy information. In the future, we shall further develop more aggregation operators for SVNHFEs and apply them to solve practical applications in these areas, such as group decision making, expert system, information fusion system, fault diagnoses, and medical diagnoses.

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