SMARANDACHE CURVES ACCORDING TO CURVES ON A SPACELIKE SURFACE IN MINKOWSKI 3-SPACE $\mathbb{R}^3_1$

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Abstract. In this paper, we introduce Smarandache curves according to the Lorentzian Darboux frame of a curve on spacelike surface in Minkowski 3-space $\mathbb{R}^3_1$. Also, we obtain the Sabban frame and the geodesic curvature of the Smarandache curves and give some characterizations on the curves when the curve α is an asymptotic curve or a principal curve. And, we give an example to illustrate these curves.

1. Introduction

In the theory of curves in the Euclidean and Minkowski spaces, one of the interesting problem is the characterization of a regular curve. In the solution of the problem, the curvature functions κ and τ of a regular curve have an effective role. It is known that the shape and size of a regular curve can be determined by using its curvatures κ and τ. Another approach to the solution of the problem is to consider the relationship between the corresponding Frenet vectors of two curves. For instance, Bertrand curves and Mannheim curves arise from this relationship. Another example is the Smarandache curves. They are the objects of Smarandache geometry, i.e. a geometry which has at least one Smarandachely denied axiom ([3]). The axiom is said Smarandachely denied, if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the Theory of relativity and the Parallel Universes.

By definition, if the position vector of a curve β is composed by the Frenet frame's vectors of another curve α, then the curve β is called a Smarandache curve ([7]). Special Smarandache curves in the Euclidean and Minkowski spaces are studied by some authors ([1, 4, 5, 6, 9, 10]). For instance the special Smarandache curves according to Darboux frame in $\mathbb{E}^3$ are characterized in [8].

In this paper, we define Smarandache curves according to the Lorentzian Darboux frame of a curve on spacelike surface in Minkowski 3-space $\mathbb{R}^3_1$. Inspired by above papers we investigate the geodesic curvature and the Sabban Frame's vectors of Smarandache curves. In section 2, we explain the basic concepts of Minkowski 3-space and give Lorentzian Darboux frame that will be use throughout the paper. Section 3 is devoted to the study of four Smarandache curves, $\mathbf{T}\eta$-Smarandache curve, $\mathbf{T}\xi$-Smarandache curve, $\eta\xi$-Smarandache curve and $\mathbf{T}\eta\xi$-Smarandache curve by considering the relationship with invariants k_n, $k_g(s)$ and $\tau_g(s)$ of curve on spacelike surface in Minkowski 3-space $\mathbb{R}^3_1$. Also, we give some characterizations on the curves when the curve α is an asymptotic curve or a principal curve. Finally, we illustrate these curves with an example.

2. Basic Concepts

The Minkowski 3-space $\mathbb{R}^3_1$ is the Euclidean 3-space $\mathbb{R}^3$ provided with the standard flat metric given by

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2, \tag{2.1}$$

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where (x_1, x_2, x_3) is a rectangular Cartesian coordinate system of $\mathbb{R}^3_1$. Since $\langle \cdot, \cdot \rangle$ is an indefinite metric, recall that a non-zero vector $\mathbf{x} \in \mathbb{R}^3_1$ can have one of three Lorentzian causal characters: it can be spacelike if $\langle \mathbf{x},\ \mathbf{x} \rangle > 0$, timelike if $\langle \mathbf{x},\ \mathbf{x} \rangle < 0$ and null (lightlike) if $\langle \mathbf{x},\ \mathbf{x} \rangle = 0$. In particular, the norm (length) of a vector $\mathbf{x} \in \mathbb{R}^3_1$ is given by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x},\ \mathbf{x} \rangle|}$ and two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be orthogonal, if $\langle \mathbf{x},\ \mathbf{y} \rangle = 0$. For any $\mathbf{x} = (x_1,\ x_2,\ x_3)$ and $\mathbf{y} = (y_1,\ y_2,\ y_3)$ in the space $\mathbb{R}^3_1$, the *pseudo vector product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathbf{x} \times \mathbf{y} = (-x_2 y_3 + x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1). \tag{2.2}$$

Next, recall that an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}^3_1$, can locally be *spacelike, timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ are respectively *spacelike, timelike* or *null (lightlike)* for every $s \in I$.([2]). If $\|\alpha'(s)\| \neq 0$ for every $s \in I$, then α is a *regular curve* in $\mathbb{R}^3_1$. A spacelike (timelike) regular curve α is parameterized by pseudo-arclength parameter s which is given by $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3_1$, then the tangent vector $\alpha'(s)$ along α has unit length, that is, $\langle \alpha'(s),\ \alpha'(s) \rangle = 1$ ($\langle \alpha'(s),\ \alpha'(s) \rangle = -1$) for all $s \in I$, respectively.

Remark 1. *Let* $\mathbf{x} = (x_1,\ x_2,\ x_3)$, $\mathbf{y} = (y_1,\ y_2,\ y_3)$ *and* $\mathbf{z} = (z_1,\ z_2,\ z_3)$ *be vectors in* $\mathbb{R}^3_1$. *Then:*

$$\begin{array}{lrcl} (i) & \langle \mathbf{x} \times \mathbf{y},\ \mathbf{z} \rangle & = & \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \\ (ii) & \mathbf{x} \times (\mathbf{y} \times \mathbf{z}) & = & -\langle \mathbf{x},\ \mathbf{z} \rangle \mathbf{y} + \langle \mathbf{x},\ \mathbf{y} \rangle \mathbf{z}, \\ (iii) & \langle \mathbf{x} \times \mathbf{y},\ \mathbf{x} \times \mathbf{y} \rangle & = & -\langle \mathbf{x},\ \mathbf{x} \rangle \langle \mathbf{y},\ \mathbf{y} \rangle + \langle \mathbf{x},\ \mathbf{y} \rangle^2, \end{array}$$

where $\times$ *is the pseudo vector product in the space* $\mathbb{R}^3_1$.

Lemma 1. *In the Minkowski* 3*-space* $\mathbb{R}^3_1$, *following properties are satisfied (*[2]*):*
(i) two timelike vectors are never orthogonal;
(ii) two null vectors are orthogonal if and only if they are linearly dependent;
(iii) timelike vector is never orthogonal to a null vector.

Let $\phi : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3_1,\ \phi(U) = M$ and $\gamma : I \subset \mathbb{R} \longrightarrow U$ be a spacelike embedding and a regular curve, respectively. Then we have a curve α on the surface M is defined by $\alpha(s) = \phi(\gamma(s))$ and since ϕ is a spacelike embedding, we have a unit timelike normal vector field η along the surface M is defined by

$$\eta \equiv \frac{\phi_x \times \phi_y}{\|\phi_x \times \phi_y\|}. \tag{2.3}$$

Since M is a spacelike surface, we can choose a future directed unit timelike normal vector field η along the surface M. Hence we have a pseudo-orthonormal frame $\{\mathbf{T}, \eta, \xi\}$ which is called the *Lorentzian Darboux frame* along the curve α where $\xi(s) = \mathbf{T}(s) \times \eta(s)$ is an unit spacelike vector. The corresponding Frenet formulae of α read

$$\begin{bmatrix} \mathbf{T}' \\ \eta' \\ \xi' \end{bmatrix} = \begin{bmatrix} 0 & k_n & k_g \\ k_n & 0 & \tau_g \\ -k_g & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \eta \\ \xi \end{bmatrix}, \tag{2.4}$$

where $k_n(s) = -\langle \mathbf{T}'(s), \eta(s) \rangle$, $k_g(s) = \langle \mathbf{T}'(s), \xi(s) \rangle$ and $\tau_g(s) = -\langle \xi'(s), \eta(s) \rangle$ are the *asymptotic curvature*, the *geodesic curvature* and the *principal curvature* of α on the surface M in $\mathbb{R}^3_1$, respectively, and s is arclength parameter of α. In particular, the following relations hold

$$\mathbf{T} \times \eta = \xi, \quad \eta \times \xi = -\mathbf{T}, \quad \xi \times \mathbf{T} = \eta. \tag{2.5}$$

Both k_n and k_g may be positive or negative. Specifically, k_n is positive if α curves towards the normal vector η, and k_g is positive if α curves towards the tangent normal vector ξ.

Also, the curve α is characterized by k_n, k_g and τ_g as the follows:

$$\alpha \text{ is } \begin{cases} \text{an asymptotic curve iff } k_n \equiv 0, \\ \text{a geodesic curve iff } k_g \equiv 0, \\ \text{a principal curve iff } \tau_g \equiv 0. \end{cases} \tag{2.6}$$

Since α is a unit-speed curve, $\ddot{\alpha}$ is perpendicular to $\mathbf{T}$, but $\ddot{\alpha}$ may have components in the normal and tangent normal directions:

$$\ddot{\alpha} = k_n \eta + k_g \xi.$$

These are related to the *total curvature* κ of α by the formula

$$\kappa^2 = \|\ddot{\alpha}\|^2 = k_g^2 - k_n^2. \tag{2.7}$$

From (2.7) we can give the following Proposition.

Proposition 1. *Let M be a spacelike surface in $\mathbb{R}_1^3$. Let $\alpha = \alpha(s)$ be a regular unit speed curves lying fully with the Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$ on the surface M in $\mathbb{R}_1^3$. There is not a geodesic curve on M.*

The *pseudosphere* with center at the origin and of radius $r = 1$ in the Minkowski 3-space $\mathbb{R}_1^3$ is a quadric defined by

$$S_1^2 = \left\{ \vec{x} \in \mathbb{R}_1^3 \middle| -x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

Let $\beta : I \subset \mathbb{R} \longrightarrow S_1^2$ be a curve lying fully in pseudosphere S_1^2 in $\mathbb{R}_1^3$. Then its position vector β is a spacelike, which means that the tangent vector $T_\beta = \beta'$ can be a spacelike, a timelike or a null. Depending on the causal character of T_β, we distinguish the following three cases, [5].

Case 1. T_β ***is a unit spacelike vector***

Then we have orthonormal Sabban frame $\{\beta(s), T_\beta(s), \xi_\beta(s)\}$ along the curve α, where $\xi_\beta(s) = -\beta(s) \times T_\beta(s)$ is the unit timelike vector. The corresponding Frenet formulae of β, according to the Sabban frame read

$$\begin{bmatrix} \beta' \\ T_\beta' \\ \xi_\beta' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -\bar{k}_g(s) \\ 0 & -\bar{k}_g(s) & 0 \end{bmatrix} \begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix}. \tag{2.8}$$

where $\bar{k}_g(s) = \det(\beta(s), T_\beta(s), T_\beta'(s))$ is the geodesic curvature of β and s is the arclength parameter of β. In particular, the following relations hold

$$\beta \times T_\beta = -\xi_\beta, \quad T_\beta \times \xi_\beta = \beta, \quad \xi_\beta \times \beta = T_\beta. \tag{2.9}$$

Case 2. T_β ***is a unit timelike vector***

Hence we have orthonormal Sabban frame $\{\beta(s), T_\beta(s), \xi_\beta(s)\}$ along the curve β, where $\xi_\beta(s) = \beta(s) \times T_\beta(s)$ is the unit spacelike vector. The corresponding Frenet formulae of β, according to the Sabban frame read

$$\begin{bmatrix} \beta' \\ T_\beta' \\ \xi_\beta' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \bar{k}_g(s) \\ 0 & \bar{k}_g(s) & 0 \end{bmatrix} \begin{bmatrix} \beta \\ T_\beta \\ \xi_\beta \end{bmatrix}. \tag{2.10}$$

where $\bar{k}_g(s) = \det(\beta(s), T_\beta(s), T_\beta'(s))$ is the geodesic curvature of β and s is the arclength parameter of β. In particular, the following relations hold

$$\beta \times T_\beta = \xi_\beta, \quad T_\beta \times \xi_\beta = \beta, \quad \xi_\beta \times \beta = -T_\beta. \tag{2.11}$$

Case 3. T_β ***is a null vector***

It is known that the only null curves lying on pseudosphere S_1^2 are the null straight lines, which are the null geodesics.

3. Smarandache curves according to curves on a spacelike surface in Minkowski 3-space $\mathbb{R}^3_1$

In the following section, we define the Smarandache curves according to the Lorentzian Darboux frame in Minkowski 3-space. Also, we obtain the Sabban frame and the geodesic curvature of the Smarandache curves lying on pseudosphere S^2_1 and give some characterizations on the curves when the curve α is an asymptotic curve or a principal curve.

Definition 1. *Let $\alpha = \alpha(s)$ be a spacelike curve lying fully on the spacelike surface M in $\mathbb{R}^3_1$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then $\mathbf{T}\eta$-Smarandache curve of α is defined by*

$$\beta(s^\star(s)) = \frac{1}{\sqrt{2}}(a\mathbf{T}(s) + b\eta(s)), \tag{3.1}$$

where $a, b \in \mathbb{R}_0$ and $a^2 - b^2 = 2$.

Definition 2. *Let $\alpha = \alpha(s)$ be a spacelike curve lying fully on the spacelike surface M in $\mathbb{R}^3_1$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then $\mathbf{T}\xi$-Smarandache curve of α is defined by*

$$\beta(s^\star(s)) = \frac{1}{\sqrt{2}}(a\mathbf{T}(s) + b\xi(s)), \tag{3.2}$$

where $a, b \in \mathbb{R}_0$ and $a^2 + b^2 = 2$.

Definition 3. *Let $\alpha = \alpha(s)$ be a spacelike curve lying fully on the spacelike surface M in $\mathbb{R}^3_1$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then $\eta\xi$-Smarandache curve of α is defined by*

$$\beta(s^\star(s)) = \frac{1}{\sqrt{2}}(a\eta(s) + b\xi(s)), \tag{3.3}$$

where $a, b \in \mathbb{R}_0$ and $b^2 - a^2 = 2$.

Definition 4. *Let $\alpha = \alpha(s)$ be a spacelike curve lying fully on the spacelike surface M in $\mathbb{R}^3_1$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then $\mathbf{T}\eta\xi$-Smarandache curve of α is defined by*

$$\beta(s^\star(s)) = \frac{1}{\sqrt{3}}(a\mathbf{T}(s) + b\eta(s) + c\xi(s)), \tag{3.4}$$

where $a, b \in \mathbb{R}_0$ and $a^2 - b^2 + c^2 = 3$.

Thus, there are two following cases:

Case 4. *(α is an asymptotic curve). Then, we have the following theorems.*

Theorem 1. *Let $\alpha = \alpha(s)$ be an asymptotic spacelike curve lying fully on the spacelike surface M in $\mathbb{R}^3_1$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then*
(i) if $ak_g + b\tau_g \neq 0$ for all s, then the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of the $\mathbf{T}\eta$-Smarandache curve β is given by

$$\begin{bmatrix} \beta \\ \mathbf{T}_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} & 0 \\ 0 & 0 & \epsilon \\ \epsilon\frac{b}{\sqrt{2}} & \epsilon\frac{a}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \eta \\ \xi \end{bmatrix},$$

and the geodesic curvature $\bar{k}_g$ of the curve β reads

$$\bar{k}_g = \frac{a\tau_g - bk_g}{\sqrt{2}},$$

where $\epsilon = sign\,(ak_g + b\tau_g)$ for all s and

$$\frac{ds^*}{ds} = \epsilon\frac{ak_g + b\tau_g}{\sqrt{2}}.$$

(ii) if $ak_g + b\tau_g = 0$ for all s, then the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of the $\mathbf{T}\eta$-Smarandache curve β is a null geodesic.

Proof. We assume that the curve α is an asymptotic curve. Differentiating the equation (3.1) with respect to s and using (2.4) we obtain

$$\begin{aligned}\beta' &= \frac{d\beta}{ds} \\ &= \frac{1}{\sqrt{2}}\left(ak_g + b\tau_g\right)\xi,\end{aligned}$$

and

$$\left\langle \beta', \beta' \right\rangle = \frac{\left(ak_g + b\tau_g\right)^2}{2}.$$

Then, there are two following cases:

(i). If $ak_g + b\tau_g \neq 0$ for all s, since $\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds}$, then the tangent vector $\mathbf{T}_\beta$ of the curve β is a spacelike vector such that

$$\mathbf{T}_\beta = \epsilon\xi, \tag{3.5}$$

where

$$\frac{ds^*}{ds} = \epsilon\frac{ak_g + b\tau_g}{\sqrt{2}}.$$

On the other hand, from the equations (3.1) and (3.5) it can be easily seen that

$$\begin{aligned}\xi_\beta &= -\beta \times T_\beta \\ &= \epsilon\frac{b}{\sqrt{2}}\mathbf{T} + \epsilon\frac{a}{\sqrt{2}}\eta,\end{aligned} \tag{3.6}$$

is a unit timelike vector.
Consequently, the geodesic curvature $\bar{k}_g$ of the curve $\beta = \beta(s^*)$ is given by

$$\begin{aligned}\bar{k}_g &= \det\left(\beta,\ T_\beta,\ \mathbf{T}'_\beta\right) \\ &= \frac{a\tau_g - bk_g}{\sqrt{2}}\end{aligned}$$

From (3.1), (3.5) and (3.6) we obtain the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of β.

(ii). If $ak_g + b\tau_g = 0$ for all s, then β' is null. So, the tangent vector $\mathbf{T}_\beta$ of the curve β is a null vector. It is known that the only null curves lying on pseudosphere S_1^2 are the null straight lines, which are the null geodesics. $\square$

In the theorems which follow, in a similar way as in Theorem 1 we obtain the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ and the geodesic curvature $\bar{k}_g$ of a spacelike Smarandache curve. We omit the proofs of Theorems 2, 3 and 4, since they are analogous to the proof of Theorem 1

Theorem 2. *Let $\alpha = \alpha(s)$ be an asymptotic spacelike curve lying fully on the spacelike surface M in $\mathbb{R}_1^3$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then*
(i) if $2k_g^2 - (b\tau_g)^2 \neq 0$ for all s, then the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of the $\mathbf{T}\xi$-Smarandache curve β is given by

$$\begin{bmatrix}\beta \\ \mathbf{T}_\beta \\ \xi_\beta\end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{2}} & 0 & \frac{b}{\sqrt{2}} \\ \frac{-bk_g}{\sqrt{\epsilon\left(2k_g^2 - \left(b\tau_g\right)^2\right)}} & \frac{b\tau_g}{\sqrt{\epsilon\left(2k_g^2 - \left(b\tau_g\right)^2\right)}} & \frac{ak_g}{\sqrt{\epsilon\left(2k_g^2 - \left(b\tau_g\right)^2\right)}} \\ \frac{b\tau_g}{\sqrt{\epsilon\left(2k_g^2 - \left(b\tau_g\right)^2\right)}} & \frac{-2k_g}{\sqrt{\epsilon\left(2k_g^2 - \left(b\tau_g\right)^2\right)}} & \frac{ab\tau_g}{\sqrt{\epsilon\left(2k_g^2 - \left(b\tau_g\right)^2\right)}} \end{bmatrix} \begin{bmatrix}\mathbf{T} \\ \eta \\ \xi\end{bmatrix},$$

and the geodesic curvature $\bar{k}_g$ of the curve β reads

$$\bar{k}_g = \frac{\left(\begin{array}{c} \left(a^2b^3\tau_g^3 - b^4\tau_g^3 + 4b\tau_g k_g^2\right) k_g' + \left(b^4\tau_g^2 k_g - a^2b^3\tau_g^2 k_g - 4bk_g^3\right)\tau_g' \\ + \left(2ab^2 + 2ab - 4a\right)\tau_g k_g^4 - \left(ab^4 + ab^3\right)\tau_g^3 k_g^2 + ab^4\tau_g^5 \end{array}\right)}{\left(4k_g^2 - 2\left(b\tau_g\right)^2\right)^2},$$

where $\epsilon = sign\left(2k_g^2 - \left(b\tau_g\right)^2\right)$ for all s and

$$\frac{ds^*}{ds} = \sqrt{\epsilon\frac{2k_g^2 - \left(b\tau_g\right)^2}{2}}.$$

(ii) if $ak_g + b\tau_g = 0$ for all s, then the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of the $\mathbf{T}\eta$-Smarandache curve β is a null geodesic.

Theorem 3. *Let $\alpha = \alpha(s)$ be an asymptotic spacelike curve lying fully on the spacelike surface M in $\mathbb{R}_1^3$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then*
(i) if $\left(bk_g\right)^2 - 2\tau_g^2 \neq 0$ for all s, then the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of the $\eta\xi$-Smarandache curve β is given by

$$\begin{bmatrix} \beta \\ \mathbf{T}_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} 0 & \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} \\ \frac{-bk_g}{\sqrt{\epsilon\left(\left(bk_g\right)^2 - 2\tau_g^2\right)}} & \frac{b\tau_g}{\sqrt{\epsilon\left(\left(bk_g\right)^2 - 2\tau_g^2\right)}} & \frac{a\tau_g}{\sqrt{\epsilon\left(\left(bk_g\right)^2 - 2\tau_g^2\right)}} \\ \frac{-2\tau_g}{\sqrt{2\epsilon\left(\left(bk_g\right)^2 - 2\tau_g^2\right)}} & \frac{b^2k_g}{\sqrt{2\epsilon\left(\left(bk_g\right)^2 - 2\tau_g^2\right)}} & \frac{-abk_g}{\sqrt{2\epsilon\left(\left(bk_g\right)^2 - 2\tau_g^2\right)}} \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \eta \\ \xi \end{bmatrix},$$

and the geodesic curvature $\bar{k}_g$ of the curve β reads

$$\bar{k}_g = \frac{\left(\begin{array}{c} \left(4\sqrt{2}b\tau_g^3 - \left(\sqrt{2}a^2b^3 + \sqrt{2}b^5\right)\tau_g k_g^2\right) k_g' + \left(\left(\sqrt{2}a^2b^3 + \sqrt{2}b^5\right)k_g^3 - 4\sqrt{2}b\tau_g^2 k_g\right)\tau_g' \\ -\sqrt{2}ab^4k_g^5 + 2\sqrt{2}ab^4\tau_g^2k_g^3 + \left(4\sqrt{2}a - 4\sqrt{2}ab^2\right)\tau_g^4 k_g \end{array}\right)}{2\left(b^2k_g^2 - 2\tau_g^2\right)^2},$$

where $\epsilon = sign\left(\left(bk_g\right)^2 - 2\tau_g^2\right)$ for all s and

$$\frac{ds^*}{ds} = \sqrt{\epsilon\frac{\left(bk_g\right)^2 - 2\tau_g^2}{2}}.$$

(ii) if $\left(bk_g\right)^2 - 2\tau_g^2 = 0$ for all s, then the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of the $\mathbf{T}\eta$-Smarandache curve β is a null geodesic.

Theorem 4. *Let $\alpha = \alpha(s)$ be an asymptotic spacelike curve lying fully on the spacelike surface M in $\mathbb{R}_1^3$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then*
(i) if $3\left(k_g^2 - \tau_g^2\right) + \left(bk_g + a\tau_g\right)^2 \neq 0$ for all s, then the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of the $\mathbf{T}\eta\xi$-Smarandache curve β is given by

$$\begin{aligned} \beta &= \frac{a\mathbf{T} + b\eta + c\xi}{\sqrt{3}}, \\ \mathbf{T}_\beta &= \frac{-ck_g\mathbf{T} + c\tau_g\eta + \left(ak_g + b\tau_g\right)\xi}{\sqrt{\epsilon\left(3\left(k_g^2 - \tau_g^2\right) + \left(bk_g + a\tau_g\right)^2\right)}}, \\ \xi_\beta &= \frac{\left(b\left(ak_g + b\tau_g\right) - c^2\tau_g\right)\mathbf{T} + \left(c^2k_g + a\left(ak_g + b\tau_g\right)\right)\eta - \left(ac\tau_g + bck_g\right)\xi}{\sqrt{\epsilon\left(9\left(k_g^2 - \tau_g^2\right) + 3\left(bk_g + a\tau_g\right)^2\right)}}, \end{aligned}$$

and the geodesic curvature $\bar{k}_g$ of the curve β reads

$$\bar{k}_g = \frac{\left(b\left(ak_g + b\tau_g\right) - c^2\tau_g\right)f_1 - \left(c^2k_g + a\left(ak_g + b\tau_g\right)\right)f_2 - \left(ac\tau_g + bck_g\right)f_3}{\sqrt{3}\left(3\left(k_g^2 - \tau_g^2\right) + \left(bk_g + a\tau_g\right)^2\right)^2},$$

where $\epsilon = sign\left(3\left(k_g^2-\tau_g^2\right)+\left(bk_g+a\tau_g\right)^2\right)$ *for all* s *and*

$$\frac{ds^*}{ds}=\sqrt{\epsilon\frac{\left(bk_g\right)^2-2\tau_g^2}{2}},$$

and

$$\begin{aligned}
f_1 &= \left(\left(a^2c-3c\right)\tau_g^2+abc\tau_g k_g\right)k_g'+\left(\left(3c-a^2c\right)\tau_g k_g-abck_g^2\right)\tau_g' \\
&\quad+\left(3a+ab^2\right)k_g^4+\left(a^3+2ab^2-3a\right)\tau_g^2k_g^2+\left(b^3+2a^2b+3b\right)\tau_g k_g^3 \\
&\quad+\left(a^2b-3b\right)\tau_g^3k_g \\
f_2 &= \left(abc\tau_g^2+\left(b^2c+3c\right)\tau_g k_g\right)k_g'-\left(abc\tau_g k_g+\left(3c+b^2c\right)k_g^2\right)\tau_g' \\
&\quad+\left(3b-a^2b\right)\tau_g^4-\left(b^3+2a^2b+3b\right)\tau_g^2k_g^2-\left(ab^2+3a\right)\tau_g k_g^3 \\
&\quad+\left(3a-2ab^2-a^3\right)\tau_g^3k_g \\
f_3 &= \left(\left(-a^3+3a+ab^2\right)\tau_g^2+\left(b^3+3b-a^2b\right)\tau_g k_g\right)k_g' \\
&\quad+\left(\left(-b^3-3b+a^2b\right)k_g^2+\left(a^3-ab^2-3a\right)\tau_g k_g\right)\tau_g' \\
&\quad+\left(3c+b^2c\right)k_g^4+\left(3c-a^2c\right)\tau_g^4+2abc\tau_g k_g^3-2abc\tau_g^3k_g \\
&\quad+\left(a^2c-6c-b^2c\right)\tau_g^2k_g^2
\end{aligned}$$

(ii) if $3\left(k_g^2-\tau_g^2\right)+\left(bk_g+a\tau_g\right)^2=0$ *for all* s*, then the Sabban frame* $\{\beta,\ \mathbf{T}_\beta,\xi_\beta\}$ *of the* $\mathbf{T}\eta$*-Smarandache curve* β *is a null geodesic.*

Case 5. *(α is a principal curve). Then, we have the following theorems.*

Theorem 5. *Let* $\alpha=\alpha(s)$ *be a principal spacelike curve lying fully on the spacelike surface* M *in* $\mathbb{R}_1^3$ *with the moving Lorentzian Darboux frame* $\{\mathbf{T},\eta,\xi\}$*. Then the* $\mathbf{T}\eta$*-Smarandache curve* β *is spacelike and the Sabban frame* $\{\beta,\ \mathbf{T}_\beta,\xi_\beta\}$ *is given by*

$$\begin{bmatrix}\beta\\ \mathbf{T}_\beta\\ \xi_\beta\end{bmatrix}=\begin{bmatrix}\frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} & 0\\ \frac{bk_n}{\sqrt{\left(ak_g\right)^2-2k_n^2}} & \frac{ak_n}{\sqrt{\left(ak_g\right)^2-2k_n^2}} & \frac{ak_g}{\sqrt{\left(ak_g\right)^2-2k_n^2}}\\ \frac{abk_g}{\sqrt{2\left(ak_g\right)^2-4k_n^2}} & \frac{a^2k_g}{\sqrt{2\left(ak_g\right)^2-4k_n^2}} & \frac{-2k_n}{\sqrt{2\left(ak_g\right)^2-4k_n^2}}\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \eta\\ \xi\end{bmatrix},$$

and the geodesic curvature $\bar{k}_g$ *of the curve* β *reads*

$$\bar{k}_g=\frac{abk_g g_1-a^2k_g g_2-2k_n g_3}{\sqrt{2}\left(a^2k_g^2-2k_n^2\right)^2},$$

where

$$\frac{ds^*}{ds}=\sqrt{\frac{\left(ak_g\right)^2-2k_n^2}{2}},$$

and

$$\begin{aligned}
g_1 &= ba^2k_gk_nk_g'-ba^2k_g^2k_n'+a^3k_g^4-\left(a^3+2a\right)k_g^2k_n^2+2ak_n^4, \\
g_2 &= -a^3k_g^2k_n'+a^3k_gk_nk_g'-ba^2k_g^2k_n^2+2bk_n^4, \\
g_3 &= 2ak_n^2k_g'-2ak_gk_nk_n'-ba^2k_g^3k_n+2bk_gk_n^3.
\end{aligned}$$

Proof. We assume that the curve α is a principal curve. Differentiating the equation (3.1) with respect to s and using (2.4) we obtain

$$\begin{aligned}
\beta' &= \frac{d\beta}{ds} \\
&= \frac{1}{\sqrt{2}}(bk_n\mathbf{T}+ak_n\eta+ak_g\xi),
\end{aligned}$$

and

$$\langle \beta', \beta' \rangle = \frac{(ak_g)^2 - 2k_n^2}{2},$$

where from (2.7) $(ak_g)^2 - 2k_n^2 > 0$ for all s. Since $\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds}$, the tangent vector $\mathbf{T}_\beta$ of the curve β is a spacelike vector such that

$$\mathbf{T}_\beta = \frac{1}{\sqrt{(ak_g)^2 - 2k_n^2}}(bk_n\mathbf{T} + ak_n\eta + ak_g\xi), \tag{3.7}$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{(ak_g)^2 - 2k_n^2}{2}}.$$

On the other hand, from the equations (3.1) and (3.7) it can be easily seen that

$$\begin{aligned} \xi_\beta &= -\beta \times T_\beta \\ &= \frac{1}{\sqrt{2(ak_g)^2 - 4k_n^2}}(abk_g\mathbf{T} + a^2k_g\eta - 2k_n\xi), \end{aligned} \tag{3.8}$$

is a unit timelike vector.
Consequently, the geodesic curvature $\bar{k}_g$ of the curve $\beta = \beta(s^*)$ is given by

$$\begin{aligned} \bar{k}_g &= \det\left(\beta,\ T_\beta,\ \mathbf{T}'_\beta\right) \\ &= \frac{abk_g g_1 - a^2k_g g_2 - 2k_n g_3}{\sqrt{2}\left(a^2k_g^2 - 2k_n^2\right)^2} \end{aligned}$$

where

$$\begin{aligned} g_1 &= ba^2k_gk_nk_g' - ba^2k_g^2k_n' + a^3k_g^4 - \left(a^3 + 2a\right)k_g^2k_n^2 + 2ak_n^4 \\ g_2 &= -a^3k_g^2k_n' + a^3k_gk_nk_g' - ba^2k_g^2k_n^2 + 2bk_n^4 \\ g_3 &= 2ak_n^2k_g' - 2ak_gk_nk_n' - ba^2k_g^3k_n + 2bk_gk_n^3 \end{aligned}$$

From (3.1), (3.7) and (3.8) we obtain the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of β. $\square$

In the theorems which follow, in a similar way as in Theorem 5 we obtain the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ and the geodesic curvature $\bar{k}_g$ of a spacelike Smarandache curve. We omit the proofs of Theorems 6 and 8, since they are analogous to the proof of Theorem 5

Theorem 6. *Let $\alpha = \alpha(s)$ be a principal spacelike curve lying fully on the spacelike surface M in $\mathbb{R}_1^3$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then the $\mathbf{T}\xi$-Smarandache curve β is spacelike and the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ is given by*

$$\begin{bmatrix} \beta \\ \mathbf{T}_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{2}} & 0 & \frac{b}{\sqrt{2}} \\ \frac{-bk_g}{\sqrt{2k_g^2 - (ak_n)^2}} & \frac{ak_n}{\sqrt{2k_g^2 - (ak_n)^2}} & \frac{ak_g}{\sqrt{2k_g^2 - (ak_n)^2}} \\ \frac{-abk_n}{\sqrt{4k_g^2 - 2(ak_n)^2}} & \frac{2k_g}{\sqrt{4k_g^2 - 2(ak_n)^2}} & \frac{-a^2k_n}{\sqrt{4k_g^2 - 2(ak_n)^2}} \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \eta \\ \xi \end{bmatrix},$$

and the geodesic curvature $\bar{k}_g$ of the curve β reads

$$\bar{k}_g = -\frac{abk_nh_1 + 2k_gh_2 + a^2k_nh_3}{\sqrt{2}\left(2k_g^2 - (ak_n)^2\right)^2},$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2k_g^2 - (ak_n)^2}{2}},$$

and

$$\begin{aligned} h_1 &= -ba^2k_n^2k_g' + ba^2k_gk_nk_n' + 2ak_g^4 - a^3k_g^2k_n^2 - 2ak_g^2k_n^2 + a^3k_n^4, \\ h_2 &= 2ak_gk_nk_g' - 2ak_g^2k_n' - 2bk_g^3k_n + ba^2k_gk_n^3, \\ h_3 &= a^3k_n^2k_g' - a^3k_gk_nk_n' - 2bk_g^4 + ba^2k_g^2k_n^2. \end{aligned}$$

Theorem 7. *Let $\alpha = \alpha(s)$ be a principal spacelike curve lying fully on the spacelike surface M in $\mathbb{R}_1^3$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then*
(i) if $ak_n - bk_g \neq 0$ for all s, the $\eta\xi$-Smarandache curve β is spacelike and the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ is given by

$$\begin{bmatrix} \beta \\ \mathbf{T}_\beta \\ \xi_\beta \end{bmatrix} = \begin{bmatrix} 0 & \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} \\ \epsilon & 0 & 0 \\ 0 & -\epsilon\frac{b}{\sqrt{2}} & \epsilon\frac{a}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \eta \\ \xi \end{bmatrix},$$

and the geodesic curvature $\bar{k}_g$ of the curve β reads

$$\bar{k}_g = -\frac{ak_g + bk_n}{\sqrt{2}},$$

where $\epsilon = sign\,(ak_n - bk_g)$ for all s and

$$\frac{ds^*}{ds} = \epsilon\frac{ak_n - bk_g}{\sqrt{2}}.$$

(ii) if $ak_n - bk_g = 0$ for all s, then the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of the $\eta\xi$-Smarandache curve β is a null geodesic.

Proof. We assume that the curve α is a principal curve. Differentiating the equation (3.3) with respect to s and using (2.4) we obtain

$$\begin{aligned} \beta' &= \frac{d\beta}{ds^*}\frac{ds^*}{ds} \\ &= \frac{1}{\sqrt{2}}\left(ak_n - bk_g\right)\mathbf{T}, \end{aligned}$$

and

$$\left\langle \beta', \beta' \right\rangle = \frac{\left(ak_n - bk_g\right)^2}{2}.$$

Then, there are two following cases:
(i). If $ak_n - bk_g \neq 0$ for all s, since $\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds}$, then we obtain the unit tangent vector $\mathbf{T}_\beta$ of the curve β is a spacelike vector such that

$$\mathbf{T}_\beta = \epsilon\mathbf{T}, \tag{3.9}$$

where

$$\frac{ds^*}{ds} = \epsilon\frac{ak_n - bk_g}{\sqrt{2}},$$

and $\epsilon = sign\,(ak_n - bk_g)$.

On the other hand, from the equations (3.3) and (3.9) it can be easily seen that

$$\begin{aligned} \xi_\beta &= -\beta \times T_\beta \\ &= -\epsilon\frac{b}{\sqrt{2}}\eta + \epsilon\frac{a}{\sqrt{2}}\xi, \end{aligned} \tag{3.10}$$

is a unit timelike vector.
Consequently, the geodesic curvature $\bar{k}_g$ of the curve $\beta = \beta(s^*)$ is given by

$$\begin{aligned} \bar{k}_g &= \det\left(\beta,\ T_\beta,\ \mathbf{T}_\beta'\right) \\ &= -\frac{ak_g + bk_n}{\sqrt{2}} \end{aligned}$$

From (3.3), (3.9) and (3.10) we obtain the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ of β.
(ii). If $ak_n - bk_g = 0$ for all s, then β' is null. So, the tangent vector $\mathbf{T}_\beta$ of the curve β is a null vector. It is known that the only null curves lying on pseudosphere S_1^2 are the null straight lines, which are the null geodesics. □

Theorem 8. *Let $\alpha = \alpha(s)$ be a principal spacelike curve lying fully on the spacelike surface M in $\mathbb{R}_1^3$ with the moving Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$. Then the $\mathbf{T}\eta\xi$-Smarandache curve β is spacelike and the Sabban frame $\{\beta,\ \mathbf{T}_\beta, \xi_\beta\}$ is given by*

$$\begin{aligned}
\beta &= \frac{1}{\sqrt{3}}(a\mathbf{T} + b\eta + c\xi), \\
\mathbf{T}_\beta &= \frac{(bk_n - ck_g)\,\mathbf{T} + ak_n\eta + ak_g\xi}{\sqrt{3\left(k_n^2 - k_g^2\right) + (bk_g - ck_n)^2}}, \\
\xi_\beta &= \frac{(abk_g - ack_n)\,\mathbf{T} + \left(\left(3 + b^2\right)k_g - bck_n\right)\eta - \left(\left(3 - c^2\right)k_n + bck_g\right)\xi}{\sqrt{9\left(k_n^2 - k_g^2\right) + 3\,(bk_g - ck_n)^2}},
\end{aligned}$$

and the geodesic curvature $\bar{k}_g$ of the curve β reads

$$\bar{k}_g = -\frac{(abk_g - ack_n)\,l_1 - \left(\left(3 + b^2\right)k_g - bck_n\right)l_2 - \left(\left(3 - c^2\right)k_n + bck_g\right)l_3}{\sqrt{2}\left(a^2k_g^2 - 2k_n^2\right)^2},$$

where

$$\frac{ds^*}{ds} = \frac{\sqrt{3\left(k_n^2 - k_g^2\right) + (bk_g - ck_n)^2}}{\sqrt{3}},$$

and

$$\begin{aligned}
l_1 &= \left(3abk_n^2 - 3ack_gk_n\right)k_g' + \left(3ack_g^2 - 3abk_gk_n\right)k_n' + (bk_n - ck_g)\,(bk_g - ck_n)^3 \\
&\quad + 3bck_g^4 - 3bck_n^4 + 3\left(b^2 + c^2\right)k_gk_n^3 - 3\left(b^2 + c^2\right)k_g^3k_n \\
l_2 &= \left(\left(3b^2 + 3c^2 + 9\right)k_n^2 - 6bck_gk_n\right)k_g' - \left(\left(3b^2 + 3c^2 + 9\right)k_gk_n - 6bck_g^2\right)k_n' \\
&\quad - \left(ac^3 + 3ac\right)k_n^4 + \left(3ab + 3abc^2\right)k_gk_n^3 + \left(ab^3 - 3ab\right)k_g^3k_n + \left(3ac - 3ab^2c\right)k_g^2k_n^2 \\
l_3 &= \left(\left(3b^2 + 3c^2 - 9\right)k_gk_n - 6bck_n^2\right)k_g' + \left(\left(9 - 3b^2 - 3c^2\right)k_g^2 + 6bck_gk_n\right)k_n' \\
&\quad + \left(ab^3 - 3ab\right)k_g^4 - \left(3ac + ac^3\right)k_gk_n^3 + \left(3ac - 3ab^2c\right)k_g^3k_n + \left(3ab + 3abc^2\right)k_g^2k_n^2
\end{aligned}$$

Example 1. *Let us define a spacelike ruled surface (see Figure 1) in the Minkowski* 3*-space such as*

$$\begin{aligned}
\phi : U \subset \mathbb{R}^2 &\longrightarrow \mathbb{R}_1^3 \\
(s,u) &\longrightarrow \phi(s,u) = \alpha\,(s) + u\mathbf{e}\,(s)
\end{aligned}$$

and

$$\phi(s,u) = (-u\sinh s,\ s,\ -u\cosh s)$$

where $u \in (-1,1)$.
Then we get the Lorentzian Darboux frame $\{\mathbf{T}, \eta, \xi\}$ *along the curve α as follows*

$$\begin{aligned}
\mathbf{T}(s) &= (0,\ 1,\ 0)\,, \\
\eta(s) &\equiv \frac{1}{\sqrt{1-u^2}}\left(\cosh s,\ -u,\ \sinh s\right), \\
\xi(s) &= \frac{1}{\sqrt{1-u^2}}\left(-\sinh s,\ 0,\ -\cosh s\right)
\end{aligned}$$

where $\xi(s)$ is a spacelike vectors and $\eta(s)$ is a unit timelike vector.

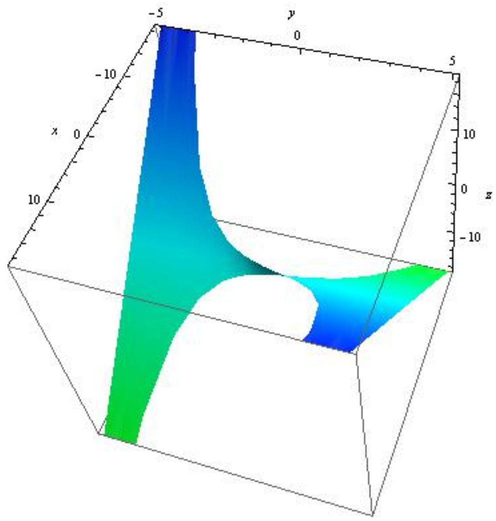


FIGURE 1. The spacelike surface $\phi(s,u)$

Moreover, the geodesic curvature $k_g(s)$*, the* asymptotic curvature $k_n(s)$ *and the* principal curvature $\tau_g(s)$ *of the curve* α *have the form*

$$\begin{aligned} k_g(s) &= \langle \mathbf{T}'(s), \xi(s) \rangle = 0, \\ k_n(s) &= -\langle \mathbf{T}'(s), \eta(s) \rangle = 0, \\ \tau_g(s) &= -\langle \xi'(s), \eta(s) \rangle = -\frac{1}{1-u^2}. \end{aligned}$$

Taking $a = \sqrt{3},\ b = 1$ *and using (3.1), we obtain that the* $\mathbf{T}\eta$*-Smarandache curve* β *of the curve* α *is given by (see Figure 2a)*

$$\beta(s^{\star}(s)) = \left(\frac{\cosh s}{\sqrt{1-u^2}},\ \sqrt{3} - \frac{u}{\sqrt{1-u^2}},\ \frac{\sinh s}{\sqrt{1-u^2}} \right)$$

Taking $a = b = 1$ *and using (3.2), we obtain that the* $\mathbf{T}\xi$*-Smarandache curve* β *of the curve* α *is given by (see Figure 2b)*

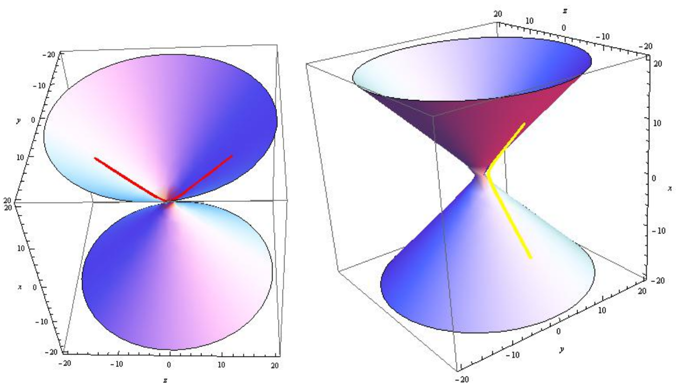
$$\beta(s^{\star}(s)) = \left(-\frac{\sinh s}{\sqrt{1-u^2}},\ 1,\ -\frac{\cosh s}{\sqrt{1-u^2}} \right).$$

Taking $a = \sqrt{3}$*,* $b = 1$ *and using (3.3), we obtain that the* $\eta\xi$*-Smarandache curve* β *of the curve* α *is given by (see Figure 3a)*

$$\beta(s^{\star}(s)) = -\frac{1}{\sqrt{1-u^2}} \left(\sinh s - \sqrt{3}\cosh s,\ \sqrt{3}u,\ \cosh s - \sqrt{3}\sinh s \right)$$

Taking $a = \sqrt{3},\ b = 1,\ c = 1$ *and using (3.4), we obtain that the* $\mathbf{T}\eta\xi$*-Smarandache curve* β *of the curve* α *is given by (see Figure 3b)*

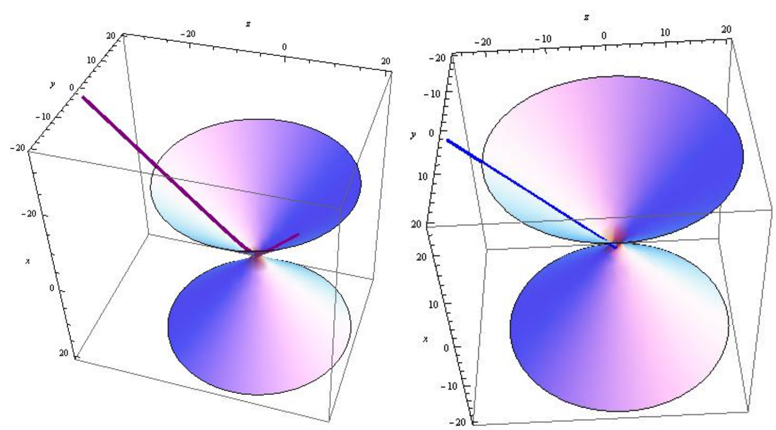
$$\beta(s^{\star}(s)) = \frac{1}{\sqrt{1-u^2}} \left(\cosh s - \sinh s,\ \sqrt{3-3u^2} - u,\ \sinh s - \cosh s \right).$$



(A) The $\mathbf{T}\eta$-Smarandache curve β on S_1^2 for $u = \frac{1}{\sqrt{3}}$

(B) The $\mathbf{T}\xi$-Smarandache curve β on S_1^2 for $u = \frac{1}{\sqrt{3}}$

FIGURE 2



(A) The $\eta\xi$-Smarandache curve β on S_1^2 for $u = \frac{1}{\sqrt{3}}$

(B) The $\mathbf{T}\eta\xi$-Smarandache curve β on S_1^2 for $u = \frac{1}{\sqrt{3}}$

FIGURE 3

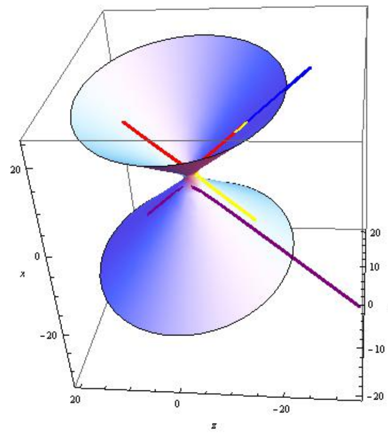


FIGURE 4. The Smarandache curves on S^2_1 for $u = \frac{1}{\sqrt{3}}$

4. Conflict of Interests

The author(s) declare(s) that there is no conflict of interests regarding the publication of this article.

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