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Some arithmetical properties of the Smarandache series

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Abstract The Smarandache function $S(n)$ is defined as the minimal positive integer m such that $n|m!$. The main purpose of this paper is to study the analyze converges questions for some series of the form $\sum\limits_{n=1}^{\infty}\frac{1}{S(n)^{\delta}}$, i.e., we proved the series $\sum\limits_{n=1}^{\infty}\frac{1}{S(n)^{\delta}}$ diverges for any $\delta \leq 1$, and $\sum\limits_{n=1}^{\infty}\frac{1}{S(n)^{\epsilon S(n)}}$ converges for any $\epsilon > 0$.

Keywords Smarandache function, smarandache series, converges.

§1. Introduction and results

For every positive integer n, let $S(n)$ be the minimal positive integer m such that $n|m!$, i.e.,

$$S(n) = \min\{m : m \in \mathbb{N}, n|m!\}.$$

This function is known as Smarandache function [1]. Easily, one has $S(1) = 1, S(2) = 2, S(3) = 3, S(4) = 4, S(5) = 5, S(6) = 3, S(7) = 7, S(8) = 4, S(9) = 6, S(10) = 5, \cdots$.

Use the standard factorization of $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $p_1 < p_2 < \cdots < p_k$, it's trivial to have

$$S(n) = \max_{1\leq i\leq k}\{S(p_i^{\alpha_i})\}.$$

Many scholars have studied the properties of $S(n)$, for example, M. Farris and P. Mitchell [2] show the boundary of $S(p^{\alpha})$ as

$$(p-1)\alpha + 1 \leq S(p^{\alpha}) \leq (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Z. Xu [3] noticed the following interesting relationship formula

$$\pi(x) = -1 + \sum_{n=2}^{[x]} \left[\frac{S(n)}{n}\right],$$

by the fact that $S(p) = p$ for p prime and $S(n) < n$ except for the case $n = 4$ and $n = p$, where $\pi(x)$ denotes the number of prime up to x, and $[x]$ the greatest integer less or equal to x. Those and many other interesting results on Smarandache function $S(n)$, readers may refer to [2]-[6].

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Let p be a fixed prime and $n \in \mathbb{N}$, the primitive numbers of power p, denoted by $S_p(n)$, is defined by

$$S_p(n) = \min\{m : m \in \mathbb{N}, p^n | m!\} = S(p^n).$$

Z. Xu [3] obtained the identity between Riemann zeta function $\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}, \sigma > 1$ and an infinite series involving $S_p(n)$ as

$$\sum_{n=1}^{\infty} \frac{1}{S_p^s(n)} = \frac{\zeta(s)}{p^s - 1},$$

and he also obtained some other asymptotic formulae for $S_p(n)$. F. Luca [4] proved the series $\sum\limits_{n=1}^{\infty} \frac{1}{S(n)^{S(n)^\delta}}$ converges for all $\delta \geq 1$ and diverges for all $\delta < 1$, and the series $\sum\limits_{n=1}^{\infty} \frac{1}{S(n)^{\epsilon \log n}}$ converges for any $\varepsilon > 0$.

In this note, we studied the analyze converges problems for the infinite series involving $S(n)$. That is, we shall prove the following conclusions:

Theorem 1.1. For any $\delta \leq 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^\delta}$$

diverges.

Theorem 1.2. For any $\varepsilon > 0$, the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^{\varepsilon S(n)}}$$

converges.

§2. Some lemmas

To complete the proof of theorems, we need two Lemmas.

Lemma 2.1. Let p be any fixed prime. Then for any real number $x \geq 1$, we have the asymptotic formula:

$$\sum_{\substack{n=1 \\ S_p(n) \leq x}}^{\infty} \frac{1}{S_p(n)} = \frac{1}{p-1}\left(\ln x + \gamma + \frac{p \ln p}{p-1}\right) + O(x^{-\frac{1}{2}} + \epsilon),$$

where γ is the Euler constant, ϵ denotes any fixed positive numbers.

Proof. See Theorem 2 of [3].

Lemma 2.2.[7] Let $\epsilon > 0$ and $d(n)$ denotes the divisor function of positive integer n. Then

$$d(n) = O(n^\epsilon) \leq C_\epsilon n^\epsilon,$$

where the o-constant C_ϵ depends on ϵ.

Proof. The proof follows [7] by writing $n = \prod\limits_{p|n} p^\alpha$, the standard factorization of n. Then

$$p^{\alpha\epsilon} \geq 2^{\alpha\epsilon} = e^{\alpha\epsilon \ln 2} \geq \alpha\epsilon \ln 2 \geq \frac{1}{2}(a+1)\epsilon \ln 2.$$

If $p^\epsilon \geq 2$, then $p^{\alpha\epsilon} \geq 2^\alpha \geq \alpha + 1$. Therefore,

$$\frac{d(n)}{n^\epsilon} = \prod_{p|n} \frac{\alpha+1}{p^{\alpha\epsilon}} = \prod_{\substack{p|n \\ p^\epsilon<2}} \frac{\alpha+1}{p^{\alpha\epsilon}} \prod_{\substack{p|n \\ p^\epsilon\geq 2}} \frac{\alpha+1}{p^{\alpha\epsilon}} \geq \prod_{\substack{p|n \\ p^\epsilon<2}} \frac{\alpha+1}{\frac{1}{2}(a+1)\epsilon\ln 2} \prod_{\substack{p|n \\ p^\epsilon\geq 2}} \frac{\alpha+1}{\alpha+1}.$$

The last item in above inequality is $\prod\limits_{\substack{p|n \\ p^\epsilon<2}} \frac{2}{\epsilon\ln 2}$, which is less than $\prod\limits_{p^\epsilon<2} \frac{2}{\epsilon\ln 2} = C_\epsilon$, say, the o-constant C_ϵ depends on ϵ.

§3. Proof of theorems

Proof of Theorem 1.

We may treat the case $\delta = 1$ first. By Lemma 1 and the notation $S_p(n) = S(p^n)$, we have

$$\sum_{n=1}^{\infty} \frac{1}{S(p^n)} = \lim_{x\to+\infty} \sum_{\substack{n=1 \\ S_p(n)\leq x}}^{\infty} \frac{1}{S_p(n)} = \infty.$$

Obviously, for $\delta \leq 1$, $\sum\limits_{n=1}^{\infty} \frac{1}{S(n)^\delta}$ diverges follows easily by the trivial inequality:

$$\sum_{n=1}^{\infty} \frac{1}{S(n)^\delta} \geq \sum_{n=1}^{\infty} \frac{1}{S(n)} \geq \sum_{n=1}^{\infty} \frac{1}{S(p^n)},$$

complete the proof.

Proof of Theorem 2.

It certainly suffices to assume that $\epsilon \leq 1$. We rewrite series $\sum\limits_{n=1}^{\infty} \frac{1}{S(n)^{\varepsilon S(n)}}$ as

$$\sum_{k=1}^{\infty} \frac{u(k)}{k^{\varepsilon k}},$$

where $u(k) = \sharp\{n : S(n) = k\}$. For every positive integer n such that $S(n) = k$ is a divisor of $k!$, i.e. $u(k) \leq d(k!)$. By Lemma 2 and the inequality bellow

$$(k!)^2 = \prod_{j=1}^{k} j(k+1-j) < \prod_{j=1}^{k} \left(\frac{k+1}{2}\right)^2 = \left(\frac{k+1}{2}\right)^{2k}.$$

we have

$$u(k) \leq d(k!) \leq C_\epsilon (k!)^\epsilon < C_\epsilon \left(\frac{k+1}{2}\right)^{\epsilon k}.$$

where C_ϵ means that the constant depending on ϵ.

Therefore, recalling that the properties of the sequence $\left(1+\frac{1}{k}\right)^k$, we have

$$\sum_{k=1}^{\infty} \frac{u(k)}{k^{\varepsilon k}} \leq C_\epsilon \sum_{k=1}^{\infty} \frac{1}{k^{\varepsilon k}} \left(\frac{k+1}{2}\right)^{\epsilon k} = C_\epsilon \sum_{k=1}^{\infty} \frac{1}{2^{\varepsilon k}} \left(\frac{k+1}{k}\right)^{\epsilon k} < C_1 \sum_{k=1}^{\infty} \frac{1}{2^{\varepsilon k}},$$

for some constant C_1, it follows that series $\sum_{k=1}^{\infty} \frac{u(k)}{k^{\varepsilon k}}$ is bounded above by

$$C_1 \sum_{k=1}^{\infty} \frac{1}{2^{\varepsilon k}} = \frac{C_1}{2^{\varepsilon} - 1},$$

completing the proof.

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