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# Magic Properties of Special Class of Trees

M.Murugan

(School of Science, Tamil Nadu Open University, 577, Anna Salai, Chennai - 600 015, India)

E-mail: muruganganesan@yahoo.in

**Abstract**: In this paper, we consider special class of trees called uniform $k$-distant trees, which have many interesting properties. We show that they have an edge-magic total labeling, a super edge-magic total labeling, a $(a, d)$-edge-antimagic vertex labeling, an $(a, d)$-edge-antimagic total labeling, a super $(a, d)$- edge-antimagic total labeling. Also we introduce a new labeling called edge bi-magic vertex labeling and prove that every uniform $k$-distant tree has edge bi-magic vertex labeling.

**Key Words**: $k$-distant tree, magic labeling, anti-magic labeling, total labeling, Smarandache anti-magic labeling.

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## §1. Introduction

For graph theory terminology and notation, we follow either Bondy and Murty [3] or Murugan [8]. In this paper, we consider a graph to be finite and without loops or multiple edges. The vertex set of a graph $G$ is denoted by $V(G)$, whereas the edge set of $G$ is denoted by $E(G)$.

A labeling of a graph is a function that sends some set of graph element to a set of positive integers. If the domain is $V(G)$ or $E(G)$ or $V(G) \cup E(G)$, then the labeling is called vertex labeling or edge labeling or total labeling respectively. The edge-weight of an edge $uv$ under a vertex labeling is the sum of the vertex labels at its ends; under a total labeling, we also add the label of $uv$.

Trees are important family of graphs and posses many interesting properties. The famous Graceful Tree Conjecture, also known as Ringel-Kotzig or Rosa's or even Ringel-Kotzig-Rosa Conjecture, says that all trees have a graceful labeling was mentioned in [11]. Yao et al. [5] have conjectured that every tree is $(k, d)$-graceful for $k > 1$ and $d > 1$. Hedge [6] has conjectured that all trees are $(k, d)$-balanced for some values of $k$ and $d$. A caterpillar is a tree with the property that the removal of its endpoints leaves a path. A lobster is a tree with the property that the removal of the endpoints leaves a caterpillar. Bermond [2] conjectured that lobsters are graceful and this is still open.

The conjecture, *All Trees are Harmonious* is still open and is unsettled for many years. Gallian in his survey [5] of graph labeling, has mentioned that no attention has been given to

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analyze the harmonious property of lobsters. It is clear that uniform 2-distant trees are special lobsters. Also, Atif Abueida and Dan Roberts [1] have proved that uniform $k$-distant trees admit a harmonious labeling, when they have even number of vertices. Murugan [9] has proved that all uniform $k$-distant trees are harmonious. In this paper, we analyze some interesting properties of uniform $k$-distant trees.

## §2. $k$-Distant Trees

A $k$-distant tree consists of a main path called the *spine*, such that each vertex on the spine is joined by an edge to at most one path on $k$-vertices. Those paths are called *tails* (i.e. each tail must be incident with a vertex on the spine). When every vertex on the spine has exactly one incident tail of length $k$, we call the tree a uniform $k$-distant tree.

A uniform $k$-distant tree with odd number of vertices is called a uniform $k$-distant odd tree. A uniform $k$-distant tree with even number of vertices is called a uniform $k$-distant even tree.

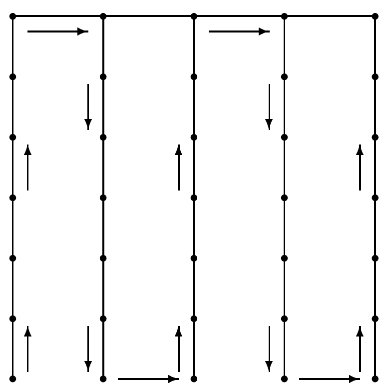

**Figure 1** Order to name the vertices

To prove our results, we name the vertices and edges of any uniform $k$-distant tree as in Figure 2 with the help of Figure 1. The arrows on the Figure 1 show the order of naming the vertices and edges.

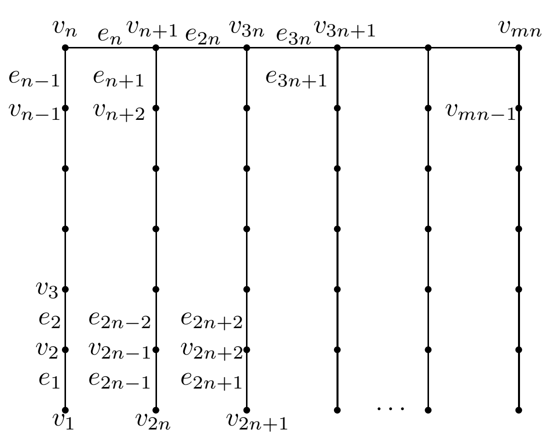


**Figure 2** Uniform $k$-distant tree

## §3. Variations of Magic Labelings

In this section, we list a few existing labelings which are useful for the development of this paper and we introduce a new labeling called *edge bi-magic vertex labeling.* Let $G$ be a graph with vertex set $V$ and edge set $E$.

**Definition** 3.1 (Edge-Magic Total Labeling) *An edge-magic total labeling of a graph* $G(V,E)$ *is a bijection* $f$ *from* $V \cup E$ *to* $\{1,2,\ldots,|V \cup E|\}$ *such that for all edges* $xy$, $f(x)+f(y)+f(xy)$ *is constant.*

This was introduced by Kotzig and Rosa [7] and rediscovered by Ringel and Llado [10].

**Definition** 3.2(Super Edge-Magic Total Labeling) *A super edge-magic total labeling of a graph* $G(V,E)$ *is an edge-magic total labeling with the additional property that the vertex labels are* 1 *to* $|V|$.

This was introduced by Enomoto et al. [4].

**Definition** 3.3($(a,d)$-Edge Antimagic Vertex Labeling) *An* $(a,d)$*-edge antimagic vertex labeling is a bijection from* $V(G)$ *onto* $\{1,2,\ldots,|V(G)|\}$ *such that the set of edge-weights of all edges in* $G$ *is*

$$\{a, a+d, \ldots, a+(|E(G)|-1)d\}$$

*where* $a > 0$ *and* $d \geqslant 0$ *are two fixed integers.*

This was introduced by Simanjuntak et al. [12].

**Definition** 3.4($(a,d)$-Edge Antimagic Total Labeling) *An* $(a,d)$*-edge antimagic total labeling is a bijection from* $V(G) \cup E(G)$ *onto the set* $\{1,2,\ldots,|V(G)|+|E(G)|\}$ *so that the set of edge-weights of all edges in* $G$ *is equal to* $\{a, a+d, \ldots, a+(|E(G)|-1)d\}$, *for two integers* $a > 0$ *and* $d \geqslant 0$.

This was introduced by Simanjuntak et al. [12].

**Definition** 3.5(Super $(a,d)$-Edge-Antimagic Total Labeling) *An* $(a,d)$*-edge-antimagic total labeling will be called super if it has the property that the vertex-labels are the integers* $1,2,\cdots,|V(G)|$.

**Definition** 3.6(Edge Bi-Magic Total Labeling) *An edge bi-magic total labeling of a graph* $G(V,E)$ *is a bijection* $f$ *from* $V \cup E$ *to* $\{1,2,\ldots,|V \cup E|\}$ *such that for all edges* $xy$, $f(x)+f(y)+f(xy)$ *is* $k_1$ *or* $k_2$ *where* $k_1$ *and* $k_2$ *are constants.*

This was introduced by Vishnupriya et al. [13]. Now we introduce *edge bi- magic vertex labeling.*

**Definition** 3.7(Edge Bi-Magic Vertex Labeling) *An edge bi-magic vertex labeling of a graph* $G(V,E)$ *is a bijection* $f$ *from* $V$ *to* $\{1,2,\ldots,|V(G)|\}$ *such that for all edges* $xy$, $f(x)+f(y)$ *is*

*$k_1$ or $k_2$ where $k_1$ and $k_2$ are constants.*

**Definition** 3.8(Smarandache anti-Magic Labeling) *Let $G$ be a graph and $H < G$. A Smarandache antimagic labeling on $H$ is a bijection from $V(H) \cup E(H)$ onto the set $\{1, 2, \ldots, |V(H)| + |E(H)|\}$ so that the set of edge-weights of all edges in $H$ is equal to $\{a, a+d, \cdots, a+(|E(H)|-1)d\}$ for two given integers $a > 0$ and $d \geqslant 0$, and $f(x) + f(y) + f(xy)$ is constant for all edges $xy$ in $E(G) \setminus E(H)$. Clearly, a Smarandache antimagic labeling on $G$ is nothing else but an $(a, d)$-edge antimagic total labeling.*

## §4. Results

**Theorem** 4.1 *Every uniform $k$-distant tree has an edge-magic total labeling.*

*Proof* Consider a uniform $k$-distant tree $T$ with $q$ edges. Since it is a tree, $q = p-1$, where $p$ is the number of vertices of $T$.

Define a labeling $f$ from $V(T) \cup E(T)$ into $\{1, 2, \ldots, p+q\}$ such that

$$\begin{aligned} f(v_i) &= \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd} \\ \left\lceil \dfrac{p}{2} \right\rceil + \dfrac{i}{2} & \text{if } i \text{ is even} \end{cases} \\ f(e_i) &= 2p - i \end{aligned}$$

We note that the sum of the labels of two consecutive vertices on the spine (that is, labels on the edges of the spine) is equal to the sum of the labels at the end vertices of the corresponding tail (for example, sum of the labels of $v_n$ and $v_{n+1}$ is equal to sum of the labels of $v_1$ and $v_{2n}$), by construction and labeling.

**Case 1** $i$ is odd.

Consider

$$\begin{aligned} f(v_i) + f(v_{i+1}) + f(e_i) &= \frac{i+1}{2} + \left\lceil \frac{p}{2} \right\rceil + \frac{i+1}{2} + 2p - i \\ &= \left\lceil \frac{p}{2} \right\rceil + 2p + i + 1 - i \\ &= 2p + \left\lceil \frac{p}{2} \right\rceil + 1 \end{aligned}$$

**Case 2** $i$ is even.

Consider

$$\begin{aligned} f(v_i) + f(v_{i+1}) + f(e_i) &= \left\lceil \frac{p}{2} \right\rceil + \frac{i}{2} + \frac{i+2}{2} + 2p - i \\ &= \left\lceil \frac{p}{2} \right\rceil + 2p + i + 1 - i \\ &= 2p + \left\lceil \frac{p}{2} \right\rceil + 1 \end{aligned}$$

Since $f(v_i) + f(v_{i+1}) + f(e_i) = 2p + \left\lceil \frac{p}{2} \right\rceil + 1$, $T$ has an edge-magic total labeling. □

**Theorem** 4.2 *Every uniform $k$-distant tree has a super edge-magic total labeling.*

*Proof* Consider the edge-magic total labeling of an uniform $k$-distant tree as in Theorem 4.1. Since the vertex labels are 1 to $|V|$, $T$ has a super edge-magic total labeling. □

**Theorem** 4.3 *Every uniform $k$-distant tree has a $(a, d)$-edge-antimagic vertex labeling.*

*Proof* Consider a uniform $k$-distant tree $T$ with $q$ edges. Since it is a tree, $q = p - 1$, where $p$ is the number of vertices of $T$. Define a labeling $f$ from $V(T)$ into $\{1, 2, \ldots, p\}$ such that

$$f(v_i) = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd} \\ \left\lceil \dfrac{p}{2} \right\rceil + \dfrac{i}{2} & \text{if } i \text{ is even} \end{cases}$$

We note that the sum of the labels of two consecutive vertices on the spine (that is, labels on the edges of the spine) is equal to the sum of the labels at the end vertices of the corresponding tail (for example, sum of the labels of $v_n$ and $v_{n+1}$ is equal to sum of the labels of $v_1$ and $v_{2n}$), by construction and labeling.

**Case 1** $i$ is odd.

Consider
$$f(v_i) + f(v_{i+1}) = \frac{i+1}{2} + \left\lceil \frac{p}{2} \right\rceil + \frac{i+1}{2} = \left\lceil \frac{p}{2} \right\rceil + i + 1$$
Now
$$f(v_{i+1}) + f(v_{i+2}) = \left\lceil \frac{p}{2} \right\rceil + \frac{i+1}{2} + \frac{i+3}{2} = \left\lceil \frac{p}{2} \right\rceil + i + 2.$$
Therefore, each $f(v_i) + f(v_{i+1})$ is distinct and differ by 1.

**Case 2** $i$ is even.

Consider
$$f(v_i) + f(v_{i+1}) = \left\lceil \frac{p}{2} \right\rceil + \frac{i}{2} + \frac{i+2}{2} = \left\lceil \frac{p}{2} \right\rceil + i + 1$$
Now
$$f(v_{i+1}) + f(v_{i+2}) = \frac{i+2}{2} + \left\lceil \frac{p}{2} \right\rceil + \frac{i+2}{2} = \left\lceil \frac{p}{2} \right\rceil + i + 2.$$

Therefore, each $f(v_i) + f(v_{i+1})$ is distinct and differ by 1. Hence, $T$ is $(a, d)$-edge antimagic vertex labeling, where $a = f(v_1) + f(v_2) = 1 + \left\lceil \frac{p}{2} \right\rceil + 1 = \left\lceil \frac{p}{2} \right\rceil + 2$ and $d = 1$. Hence, every uniform $k$-distant tree has a $(a, d)$ edge-antimagic vertex labeling. □

**Theorem** 4.4 *Every uniform $k$-distant tree has a $(a, d)$-edge-antimagic total labeling.*

*Proof* Consider a uniform $k$-distant tree $T$ with $q$ edges. Since it is a tree, $q = p - 1$, where $p$ is the number of vertices of $T$.

Define a labeling $f$ from $V(T)\cup E(T)$ into $\{1,2,\ldots,p+q\}$ such that

$$\begin{aligned} f(v_i) &= \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd} \\ \left\lceil \dfrac{p}{2} \right\rceil + \dfrac{i}{2} & \text{if } i \text{ is even} \end{cases} \\ f(e_i) &= p+i \end{aligned}$$

We note that the sum of the labels of two consecutive vertices on the spine (that is, labels on the edges of the spine) is equal to the sum of the labels at the end vertices of the corresponding tail (for example, sum of the labels of $v_n$ and $v_{n+1}$ is equal to sum of the labels of $v_1$ and $v_{2n}$), by construction and labeling.

**Case 1** $i$ is odd.

Consider

$$\begin{aligned} f(v_i)+f(v_{i+1})+f(e_i) &= \frac{i+1}{2} + \left\lceil \frac{p}{2} \right\rceil + \frac{i+1}{2} + p + i \\ &= \left\lceil \frac{p}{2} \right\rceil + p + i + 1 + i \\ &= p + \left\lceil \frac{p}{2} \right\rceil + 2i + 1. \end{aligned}$$

Now

$$\begin{aligned} f(v_{i+1})+f(v_{i+2})+f(e_{i+1}) &= \left\lceil \frac{p}{2} \right\rceil + \frac{i+1}{2} + \frac{i+3}{2} + p + i + 1 \\ &= \left\lceil \frac{p}{2} \right\rceil + p + i + 2 + i + 1 \\ &= p + \left\lceil \frac{p}{2} \right\rceil + 2i + 3. \end{aligned}$$

**Case 2** $i$ is even.

Consider

$$\begin{aligned} f(v_i)+f(v_{i+1})+f(e_i) &= \left\lceil \frac{p}{2} \right\rceil + \frac{i}{2} + \frac{i+2}{2} + p + i \\ &= \left\lceil \frac{p}{2} \right\rceil + p + i + 1 + i \\ &= p + \left\lceil \frac{p}{2} \right\rceil + 2i + 1 \end{aligned}$$

Now

$$\begin{aligned} f(v_{i+1})+f(v_{i+2})+f(e_{i+1}) &= \frac{i+2}{2} + \left\lceil \frac{p}{2} \right\rceil + \frac{i+2}{2} + p + i + 1 \\ &= \left\lceil \frac{p}{2} \right\rceil + p + i + 2 + i + 1 \\ &= p + \left\lceil \frac{p}{2} \right\rceil + 2i + 3. \end{aligned}$$

Therefore, each $f(v_i)+f(v_{i+1})+f(e_i)$ is distinct and the edge labels increase by 2. Hence,

$T$ is $(a, d)$-edge antimagic total labeling, where $a = f(v_1)+f(v_2)+f(e_1) = 1+\left\lceil \frac{p}{2} \right\rceil+1+p+1 = p + \left\lceil \frac{p}{2} \right\rceil + 3$ and $d = 2$. Hence, every uniform $k$-distant tree has a $(a, d)$ edge-antimagic total labeling. □

**Theorem** 4.5 *Every uniform $k$-distant tree has a super $(a, d)$-edge-antimagic total labeling.*

*Proof* Consider the edge-magic total labeling of an uniform $k$-distant tree as in Theorem 4.3. Since the vertex labels are 1 to $|V|$, $T$ has a super $(a, d)$-edge-antimagic total labeling. □

**Theorem** 4.6 *Every uniform $k$-distant tree has a edge bi-magic vertex labeling.*

*Proof* Consider a uniform $k$-distant tree $T$ with $q$ edges. Since it is a tree, $q = p-1$, where $p$ is the number of vertices of $T$. Define a labeling $f$ from $V(T)$ into $\{1, 2, \ldots, p\}$ such that

$$f(v_i) = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd} \\ p - \dfrac{i-2}{2} & \text{if } i \text{ is even} \end{cases}$$

We note that the sum of the labels of two consecutive vertices on the spine (that is, labels on the edges of the spine) is equal to the sum of the labels at the end vertices of the corresponding tail (for example, sum of the labels of $v_n$ and $v_{n+1}$ is equal to sum of the labels of $v_1$ and $v_{2n}$), by construction and labeling.

**Case 1** $i$ is odd.

Consider

$$f(v_i) + f(v_{i+1}) = \frac{i+1}{2} + p - \frac{i-1}{2} = p + 1$$

Now

$$f(v_{i+1}) + f(v_{i+2}) = p - \frac{i-1}{2} + \frac{i+3}{2} = p + 2$$

**Case 2** $i$ is even.

Consider

$$f(v_i) + f(v_{i+1}) = p - \frac{i-2}{2} + \frac{i+2}{2} = p + 2$$

Now

$$f(v_{i+1}) + f(v_{i+2}) = \frac{i+2}{2} + p - \frac{i}{2} = p + 1$$

Therefore, each edge has either $p+1$ or $p+2$ as edge weight. Hence every uniform $k$-distant tree has a edge bi-magic vertex labeling. □

## §5. Conclusion

Uniform $k$-distant trees are special class of trees which have many interesting properties. In this paper we have proved that every uniform $k$-distant tree has an edge-magic total labeling, a super edge-magic total labeling, a $(a, d)$-edge-antimagic vertex labeling, a $(a, d)$-edge-antimagic total labeling, a super $(a, d)$- edge-antimagic total labeling and a edge bi-magic vertex labeling.

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