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Mean Labelings on Product Graphs

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Abstract: Let G be a (p,q) graph and let $f : V(G) \to \{0,1,\cdots,q\}$ be an injection. Then G is said to have a mean labeling if for each edge uv, there exists an induced injective map $f^* : E(G) \to \{1,2,\cdots,q\}$ defined by

$$f^*(uv) = \frac{f(u)+f(v)}{2} \text{ if } f(u)+f(v) \text{ is even, and}$$
$$= \frac{f(u)+f(v)+1}{2} \text{ if } f(u)+f(v) \text{ is odd}$$

We extend this notion to *Smarandachely near m-mean labeling* if for each edge $e = uv$ and an integer $m \geq 2$, the induced Smarandachely m-labeling f^* is defined by

$$f^*(e) = \left\lceil \frac{f(u)+f(v)}{m} \right\rceil.$$

A graph that admits a Smarandachely near mean m-labeling is called *Smarandachely near m-mean graph*. The graph G is said to be a near mean graph if the injective map $f : V(G) \to \{1,2,\cdots,q-1,q+1\}$ induces $f^* : E(G) \to \{1,2,\cdots,q\}$ which is also injective, defined as above. In this paper we investigate the direct product of paths for their meanness and the Cartesian product of P_n and K_4 for its near-meanness.

Key Words: Smarandachely near m-labeling, Smarandachely near m-mean graph, mean graph, near-mean graph, direct product, Cartesian product.

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§1. Introduction

By a graph we mean a finite, undirected graph without loops or multiple edges. For all the terminology and notations in graph theory we follow [2] and [5] and for the terminology regarding labeling we follow [1]. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The direct product of G and H is denoted by $G \times H$ and is defined as a graph with vertex set $V(G) \times V(H)$ and edge set

$$\{(g,h),(g',h')/gg' \in E(G) \text{ and } hh' \in E(H)\}.$$

The Cartesian product of G and H is denoted by $G \Box H$ and is defined as a graph with

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vertex set $V(G) \times V(H)$ and edge set $\{(g,h),(g',h')/\text{either } (g=g' \text{ and } h \text{ adj } h') \text{ or } (g \text{ adj } g' \text{ and } h=h')\}$. The concept of mean labeling was introduced in [6] and the notion of near-mean labeling was introduced in [3].

In [4], various product graphs are proved as near-mean graphs.

§2. Direct Product of Graphs

Definition 2.1 *The direct product of G and H is the graph denoted by $G \times H$, whose vertex set is $V(G) \times V(H)$ and for which vertices (g,h) and (g',h') are adjacent precisely if $gg' \in E(G)$ and $hh' \in E(H)$. Thus*

$$\begin{aligned} V(G \times H) &= \{(g,h)/g \in V(G) \text{ and } h \in V(H)\} \\ E(G \times H) &= \{(g,h)(g',h')/gg' \in E(G) \text{ and } hh' \in E(H)\} \end{aligned}$$

Remark 2.1 $P_m \times P_n$ is a disconnected graph with two components. Direct product is both commutative and associative. The maps $(x_1,x_2) \mapsto (x_2,x_1)$ and $((x_1,x_2),x_3) \mapsto (x_1(x_2,x_3))$ give rise to the following isomorphisms

$$G_1 \times G_2 \cong G_2 \times G_1, \quad (G_1 \times G_2) \times G_3 \cong G_1 \times (G_2 \times G_3)$$

Theorem 2.1 *$P_3 \times P_m$ is a mean graph when $m \geqslant 3$ and is odd.*

Proof Let $u_{ij};\ i=1,2,3; j=1,2,\cdots,m$ be the vertices of $P_3 \times P_m$. Note that this graph has $3m$ vertices and $4(m-1)$ edges. Define $f: V(P_3 \times P_m) \to \{0,1,\cdots,q\}$ such that

$$\begin{aligned} f(u_{11}) &= 0 \\ f(u_{1j}) &= \begin{cases} 2j-3 & ; j=3,5,\cdots,m \\ 2m & ; j=2 \\ f(u_{1,j-2})+j-k & ; j=4,6,\ldots,m-1; k=1,2,3;1,2,3;1,2,3\cdots \end{cases} \\ f(u_{2j}) &= \begin{cases} 2(j-1) & ; j=2,4,\cdots,m-1 \\ 2(m-1) & ; j=1 \\ f(u_{2,j-2})+4 & ; j=3,5,\cdots,m \end{cases} \\ f(u_{3j}) &= \begin{cases} 2j-1 & ; j=1,3,\cdots,m \\ 2m+1 & ; j=2 \\ f(u_{3,j-2})+4 & ; j=4,6\cdots,m-1 \end{cases} \end{aligned}$$

It can be easily verified that f is one one which induces the edge labels $f^*(E(P_3 \times P_m))$. Hence the theorem. □

Example 2.1 The Fig.1 following shows the mean labeling of $P_3 \times P_7$.

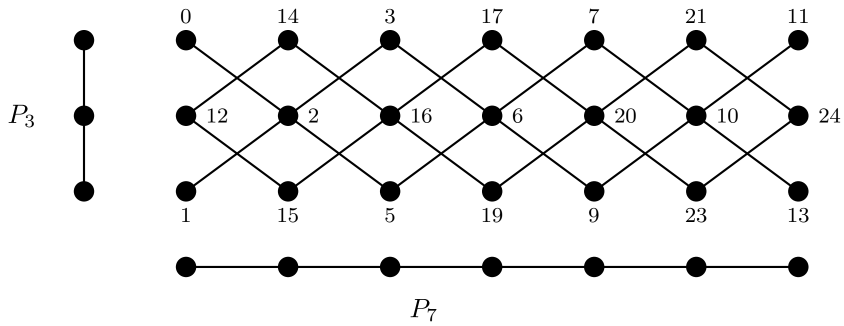

Fig 1

Theorem 2.2 *$P_5 \times P_m$ admits mean labeling when $m \geqslant 7$ and is odd.*

Proof Let u_{ij}, $i = 1, 2, \cdots, 5$ and $j = 1, 2, \cdots, m$ be the vertices of $P_5 \times P_m$. Consider $f : V(P_5 \times P_m) \to \{0, 1, \cdots, q\}$ which is defined as

$$\begin{aligned} f(u_{11}) &= 0 \\ f(u_{i1}) &= i - 2, \quad i = 3, 5 \\ f(u_{ij}) &= f(u_{i,j-2}) + 8, \quad i = 1, 3, 5; \;\; j = 3, 5, \cdots, m \\ f(u_{i2}) &= i, \quad i = 2, 4 \\ f(u_{ik}) &= f(u_{i,k-2}) + 8, \quad i = 2, 4; k = 4, 6, \cdots, m-1 \end{aligned}$$

And when $i = 1, 3, 5$, $f(u_{i2}) = f(u_{5,m}) + i$; when $i = 2, 4$, $f(u_{i1}) = f(u_{4,m-1}) + i - 1$; when $i = 1, 2, \cdots, 5; l = 3, 4, \cdots, m$, $f(u_{il}) = f(u_{i,l-2}) + 8$.

From the definition of labelings on $V(P_5 \times P_m)$, we can infer that the vertex labels are in an increasing sequence. That is the sequence such as:

For $j = 1, 3, \cdots, m, \langle u_{1j} \rangle, \langle u_{3j} \rangle$ and $\langle u_{5j} \rangle$; for $j = 2, 4, \cdots, m-1$, $\langle u_{2j} \rangle, \langle u_{4j} \rangle$ and for $k = 2, 4, \cdots, m-1$, $\langle u_{1k} \rangle, \langle u_{3k} \rangle, \langle u_{5k} \rangle$; for $k = 1, 3, \cdots, m$, $\langle u_{2k} \rangle$ and $\langle u_{4k} \rangle$, occur as an arithmetic progression.

Also we have

$$\begin{aligned} f(u_{11}) &= 0, \quad f(u_{31}) = 1 \\ f(u_{51}) &= 3, \quad f(u_{22}) = 2 \\ f(u_{42}) &= 4 \end{aligned}$$

Hence f_p is one- one with $f_p^* = \{1, 2, \cdots, q\}$. □

Remark 2.2 $P_n \times P_m$ are not mean graphs for all m. Since $P_2 \times P_m$ being a disjoint union of two P_m paths, it has $2(m-1)$ edges on $2m$ vertices. This implies that the number of edges is less than the number of vertices by 2. Hence we cannot label them with $\{0, 1, \cdots, q\}$.

Conjecture 2.1 *For m even $P_3 \times P_m$ and $P_5 \times P_m$ are not mean graphs.*

§3. Cartesian Product of Graphs

Definition 3.1 *Let G and H be graphs with $V(G) = V_1$ and $V(H) = V_2$. The cartesian product of G and H is the graph $G\Box H$ whose vertex set is $V_1 \times V_2$ such that two vertices $u = (x, y)$ and $v = (x', y')$ are adjacent if and only if either $x = x'$ and y is adjacent to y' in H or $y = y'$ and x is adjacent to x' in G. That is u adj v in $G\Box H$ whenever $[x = x'$ and y adj $y']$ or $[y = y'$ and x adj $x']$.*

Definition 3.2 *Let P_n be a path on n vertices and K_4 be the complete graph on 4 vertices. The cartesian product of P_n and K_4 is $P_n\Box K_4$ with $4n$ vertices and $10n - 4$ edges.*

Theorem 3.1 *$P_n\Box K_4$ is a near mean graph.*

Proof Let $G = P_n\Box K_4$ with $V(G) = \{u_{i1}, u_{i2}, u_{i3}, u_{i4}/i = 1, 2, \cdots, n\}$. Define $f : V(G) \to \{0, 1, \cdots, q - 1, q + 1\}$ such that

$$\begin{aligned}
f(u_{11}) &= 0, \quad f(u_{i1}) = 5(2i-1),\ i = 2, 4, \ldots, n \\
&= 5(2i-2),\ i \neq 1,\ \text{odd} \\
f(u_{i2}) &= 10(i-1) + 2 \\
f(u_{i3}) &= 5(2i-1) + (-1)^i 2 \\
f(u_{i4}) &= \begin{cases} 5(2i-1) + 3, & i \text{ odd} \\ 5(2i-3) + 4 & i \text{ even} \end{cases}
\end{aligned}$$

The edge labels induced by f are as follows:

When i is even,

$$\begin{aligned}
f^*(u_{i1}u_{i2}) &= \frac{1}{2}\Bigg[f(u_{i1}) + f(u_{i2}) + 1\Bigg] \\
&= \frac{1}{2}\Bigg[5(2i-1) + 5(2i-2) + 2 + 1\Bigg] \\
&= 10i - 6, \quad i = 2, 4, \ldots, n
\end{aligned}$$

When i is odd,

$$\begin{aligned}
f^*(u_{i1}u_{i2}) &= \frac{f(u_{i1}) + f(u_{i2})}{2} \\
&= \frac{5(2i-2) + 5(2i-2) + 2}{2} \\
&= 5(2i-2) + 1, \quad i = 1, 3, 5, \ldots
\end{aligned}$$

Hence the edges $u_{i1}u_{i2}$ carry labels $1, 14, 21, \cdots, 10(n-1)+1$ if n is odd or $1, 14, 21, \cdots, 10n-6$

if n is even.

$$\begin{aligned} f^*(u_{i1}, u_{i+1,1}) &= \frac{f(u_{i1}) + f(u_{i+1,1}) + 1}{2}, \quad i = 1, 2, \cdots, n-1 \\ &\quad \text{(since } f(u_{i1}) \text{ and } f(u_{i+1,1}) \text{ are of opposite parity)} \\ &= \frac{1}{2}[5(2i-1) + 5(2(i+1)-2) + 1] \\ &= 10i - 2 \end{aligned}$$

Hence the edges $u_{i1}, u_{i+1,1}$ have labels as $8, 18, 28, \cdots, 10n-12$.

$$\begin{aligned} f^*(u_{i2}, u_{i+1,2}) &= \frac{f(u_{i2}) + f(u_{i+1,2})}{2} \\ &\quad \text{(since } f(u_{i2}), f(u_{i+1,2}) \text{ are of same parity)} \\ &= 10i - 3, \; i = 1, 2, \cdots, (n-1) \end{aligned}$$

The edges $u_{i2}, u_{i+1,2}$ have $7, 17, 27, \cdots, 10n-13$ as labels.

$$\begin{aligned} f^*(u_{i3}, u_{i+1,3}) &= \frac{f(u_{i3}) + f(u_{i+1,3})}{2} \\ &= 10i, \; i = 1, 2, \cdots, (n-1) \end{aligned}$$

Therefore, $u_{i3}u_{i+1,3}$ assume labels $10, 20, 30, \cdots, 10(n-1)$,

$$\begin{aligned} f^*(u_{i4}, u_{i+1,4}) &= \frac{f(u_{i4}) + f(u_{i+1,4}) + 1}{2} \\ &\quad \text{(since both vertex labels are of opposite parity)} \\ &= \frac{1}{2}[5(2i-1) + 3 + 5(2i-1) + 4 + 1] \\ &= 10i - 1 \\ \text{or} \quad &= \frac{1}{2}[5(2i-3) + 4 + 5(2i+1) + 3 + 1] = 10i - 1 \end{aligned}$$

Therefore $u_{i4}u_{i+1,4}$ have labels as $9, 19, \cdots, 10n-11$.

When i is odd,

$$\begin{aligned} f^*(u_{i2}, u_{i4}) &= \frac{f(u_{i2}) + f(u_{i4})}{2} \\ &= \frac{5(2i-2) + 2 + 5(2i-1) + 3}{2} \\ &= 10i - 5 \end{aligned}$$

When i is even,

$$\begin{aligned} f^*(u_{i2}u_{i4}) &= \frac{f(u_{i2}) + f(u_{i4}) + 1}{2} \\ &= \frac{10(i-1) + 2 + 5(2i-3) + 4 + 1}{2} \\ &= 10i - 9 \end{aligned}$$

Hence $5, 11, 25, \cdots 10n-9$ if n is even or $5, 11, 25, \cdots, 10n-5$ if n is odd, correspond to the edges $u_{i2}u_{i4}$

$$f^*(u_{i2}, u_{i3}) = \frac{f(u_{i2}) + f(u_{i3}) + 1}{2} = 10i - 6 + (-1)^i$$

So the edges $u_{i2}u_{i3}$ have labels $3, 15, 23, \cdots, 10n-6+(-1)^n$.

$$\begin{aligned} f^*(u_{i3}, u_{i4}) &= \frac{f(u_{i3})+f(u_{i4})}{2} = 10i-7 \text{ if } i \text{ is even, or} \\ &= \frac{f(u_{i3})+f(u_{i4})+1}{2} = 10i-4 \text{ if } i \text{ is odd} \end{aligned}$$

So the values taken by $u_{i3}u_{i4}$ are $6, 13, 26, \cdots 10n-7$ if n is even or $6, 13, \cdots, 10n-4$ if n is odd.

If i is odd,

$$f^*(u_{i1}, u_{i3}) = \frac{f(u_{i1})+f(u_{i3})+1}{2} = 10i-8$$

If i is even,

$$f^*(u_{i1}, u_{i3}) = \frac{f(u_{i1})+f(u_{i3})}{2} = 10i-4$$

If i is odd,

$$f^*(u_{i1}, u_{i4}) = \frac{f(u_{i1})+f(u_{i4})}{2} = 10i-6$$

If i is even,

$$f^*(u_{i1}, u_{i4}) = \frac{f(u_{i1})+f(u_{i4})}{2} = 10i-8$$

Hence the edge values of $u_{i1}u_{ij}$ are $1, 2, 4, \cdots, 10n-8, 10n-6, 10n-4$ if n is even, or $1, 2, \cdots, 10n-9, 10n-8, 10n-6$ if n is odd. Hence the theorem. □

Example 3.1 The Fig.2 following shows the near mean labeling of $P_4\square K_4$.

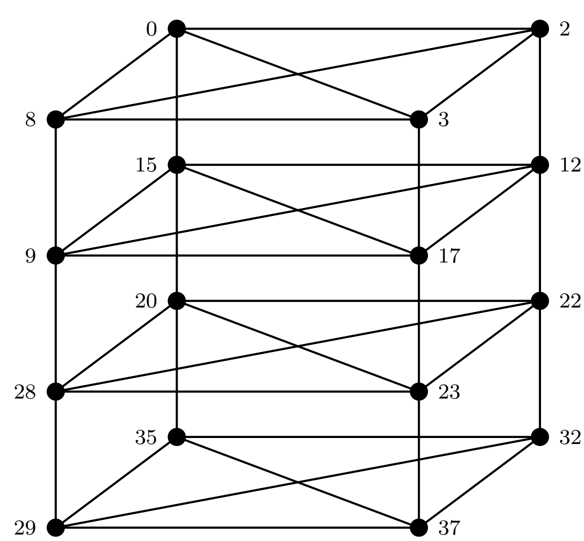

Fig 2

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