International J.Math. Combin. Vol.3(2014), 89-96

Odd Sequential Labeling of Some New Families of Graphs

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Abstract: A graph $G = (V(G), E(G))$ with p vertices and q edges is said to be an *odd sequential graph* if there is an injection $f : V(G) \to \{0, 1, \cdots, q\}$ or if G is a tree then f is an injection $f : V(G) \to \{0, 1, \cdots, 2q-1\}$ such that when each edge xy is assigned the label $f(x) + f(y)$, the resulting edge labels are $\{1, 3, \cdots, 2q-1\}$. In this paper we initiate a study on some new families of odd sequential graphs generated by some graph operations on some standard graphs.

Key Words: Odd sequential labeling, Smarandachely odd sequential labeling, super subdivisions of a graph, shadow graph.

AMS(2010): 05C78

§1. Introduction

By a graph $G = (V(G), E(G))$ with p vertices and q edges we mean a simple, connected and undirected graph in this paper. A brief summary of definitions and other information is given in order to maintain compactness. The terms not defined here are used in the sense of Harary [3].

Definition 1.1 *The super subdivisions of a graph G produces a new graph by replacing each edge of G by a complete bipartite graph $K_{2,m}$ (where m is any positive integer) in such a way that the ends of each e_i are merged with two vertices of 2-vertices part of $K_{2,m}$ after removing the edge e_i from the graph G. It is denoted by $SS(G)$.*

Definition 1.2 *A comb is a caterpillar in which each vertex in the path is joined to exactly one pendant vertex.*

Definition 1.3 *For a graph G, its split graph is obtained by adding to each vertex v, a new vertex v' so that v' is adjacent to every vertex that is adjacent to v in G.*

Definition 1.4 *The shadow graph $D_2(G)$ of a connected graph G is obtained by taking two copies of G say G' and G'', then join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''.*

[1]Received December 9, 2013, Accepted August 31, 2014.

Definition 1.5 *A bistar is the graph obtained by joining the apex vertices of two copies of star* $K_{1,n}$ *by an edge.*

Definition 1.6([6]) *Let* $G = (V(G), E(G))$ *be a graph and* $G_1, G_2, \cdots, G_n$ *be* n *copies of graph* G. *Then the graph obtained by adding an edge between* G_i *and* G_{i+1}, *for* $i = 1, 2, \cdots, n-1$ *is called a path union of* G.

Definition 1.7 *If the vertices are assigned values subject to certain conditions then it is known as graph labeling.*

Graph labeling introduced by Rosa in [5] is now one of the fascinating areas of research with applications ranging from social sciences to computer science and from neural network to bio-technology to mention a few. A systematic study on various applications of graph labeling is carried out by Bloom and Golomb [1]. The famous Ringel-Kotzig [4] graceful tree conjecture and many illustrious works on it brought a tide of different labeling techniques like harmonious labeling, odd graceful labeling, edge graceful labeling etc. For detailed survey on graph labeling and related results we refer to Gallian [2]. The concept of odd sequential labeling was introduced by Singh and Varkey [7] which is defined as follows.

Definition 1.8 *A graph* $G = (V(G), E(G))$ *with* p *vertices and* q *edges is said to be an odd sequential graph if there is an injection* $f : V(G) \to \{0, 1, \cdots, q\}$ *or if* G *is a tree then* f *is an injection* $f : V(G) \to \{0, 1, \cdots, 2q-1\}$ *such that when each edge* xy *is assigned the label* $f(x) + f(y)$, *the resulting edge labels are* $\{1, 3, \cdots, 2q-1\}$.

The graph which admits odd sequential labeling is known as an *odd sequential graph*. Generally, a graph G is called *Smarandachely odd sequential* if there is a subset $V' \subset V(G)$ such that the resulting edge labels of $G \setminus \langle V' \rangle$ are $\{1, 3, \cdots, 2q'-1\}$, where $q' \leq q$. Clearly, if $V' = \emptyset$, such a Smarandachely odd sequential graph is nothing else but an odd sequential graph. In[7] it has been also proved that the graphs such as combs, grids, stars and rooted trees of level 2 are odd sequential while odd cycles are not.

Here we investigate odd sequential labeling of some new families of graphs generated by some graph operations on some standard graphs.

§2. Results on Odd Sequential Labeling

Theorem 2.1 *The graph* $C_n \odot nK_{1,m}$, *where* n *is even admits odd sequential labeling.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of C_n, where n is even. Let u_{ij} be the newly added vertices in C_n to form $C_n \odot nK_{1,m}$, where $1 \leq i \leq n$ and $1 \leq j \leq m$. To define $f : V(C_n \odot nK_{1,m}) \to \{0, 1, \cdots, q\}$ two cases are to be considered.

Case 1. $n \equiv 0(mod 4)$

Consider the following 4 subcases:

Subcase 1.1 $1 \leq i \leq \dfrac{n}{2}$

In this caes, let $f(v_i) = (m+1)(i-1)$ if i is odd and $i(m+1)-1$ if i is even.

Subcase 1.2 $\dfrac{n}{2}+1 \le i \le n$

In this case, let $f(v_i) = (m+1)(i-1)+2$ if i is odd and $i(m+1)-1$ if i is even.

Subcase 1.3 $1 \le i \le \dfrac{n}{2}$ and $1 \le j \le m$

In this case, let $f(u_{ij}) = i(m+1)-m+2(j-1)$ if i is odd and $(m+1)(i-2)+2j$ if i is even.

Subcase 1.4 $\dfrac{n}{2}+1 \le i \le n$ and $1 \le j \le m$

In this case, let $f(u_{ij}) = i(m+1)-m+2(j-1)$ if i is odd and $(m+1)(i-2)+2(j+1)$ if i is even.

Case 2. $n \equiv 2(mod4)$

Consider the following 5 subcases.

Subcase 2.1 $1 \le i \le \dfrac{n}{2}$

In this case, let $f(v_i) = (m+1)(i-1)$ if i is odd, $i(m+1)-1$ if i is even and $f(v_i) = i(m+1)+1$ if $i = \dfrac{n}{2}+1$.

Subcase 2.2 $\dfrac{n}{2}+2 \le i \le n$

In this case, let $f(v_i) = (m+1)(i-1)+2$ if i is odd and $i(m+1)-1$ if i is even.

Subcase 2.3 $1 \le i \le \dfrac{n}{2}+1$ and $1 \le j \le m$

In this case, let $f(u_{ij}) = i(m+1)-m+2(j-1)$ if i is odd, $(m+1)(i-2)+2j$ if i is even and $f(u_{ij}) = (m+1)(i-1)-1$ if $i = \dfrac{n}{2}+2$ and $j = 1$.

Subcase 2.4 $i = \dfrac{n}{2}+2$ and $2 \le j \le m$

In this case, let $f(u_{ij}) = i(m+1)-m+2(j-1)$ if i is odd and $(m+1)(i-2)+2(j+1)$ if i is even.

Subcase 2.5 $\dfrac{n}{2}+3 \le i \le n$ and $1 \le j \le m$

In this case, let $f(u_{ij}) = i(m+1)-m+2(j-1)$ if i is odd and $(m+1)(i-2)+2(j+1)$ if i is even.

In view of the above defined labeling pattern f satisfies the conditions for odd sequential labeling. That is $C_n \odot nK_{1,m}$ is an odd sequential graph. □

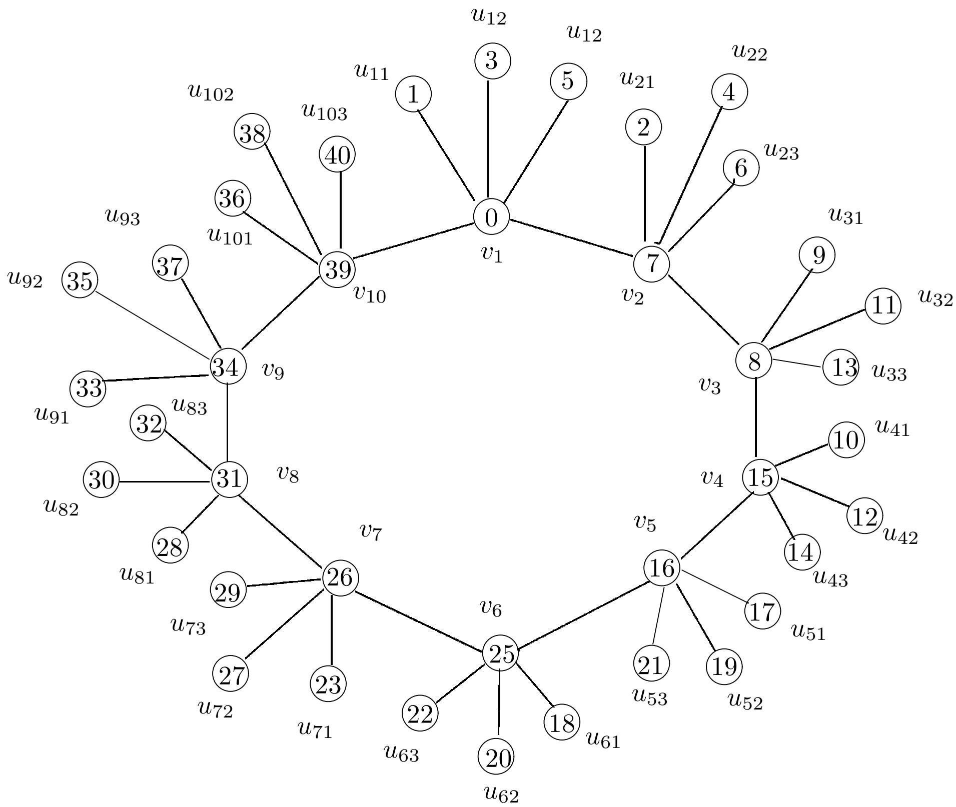

Figure 1

Illustration 2.2 The Figure 1 shows an odd sequential labeling of $C_{10} \odot 10K_{1,3}$.

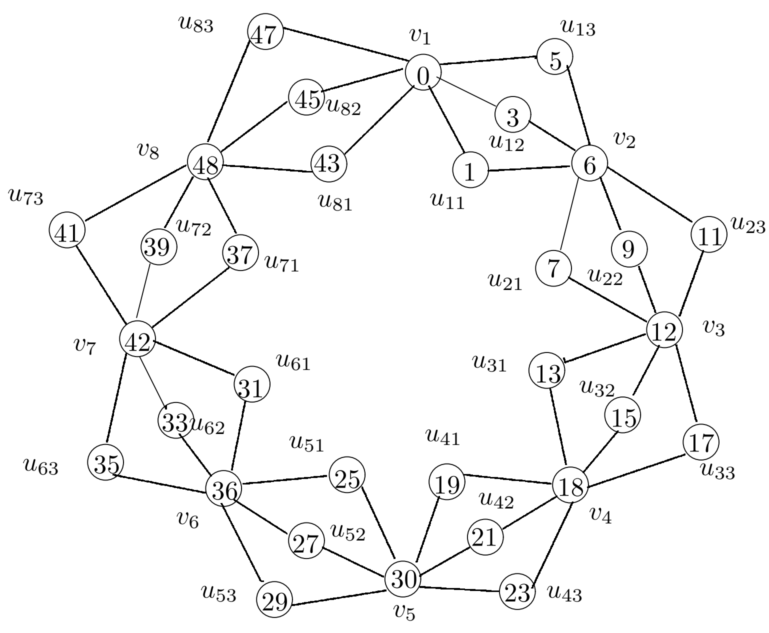

Figure 2

Theorem 2.3 *The graph $SS(C_n)$ where n is even admits odd sequential labeling.*

Proof Let C_n be the cycle containing n vertices $v_1, v_2, \cdots, v_n$, where n is even. Let e_i denotes the edge $v_i v_{i+1}$ in C_n. For $1 \leq i \leq n$ each edge e_i of cycle C_n is replaced by a complete bipartite graph $K_{2,m}$ where m is any positive integer. Let u_{ij} be the vertices of the m vertices part of $K_{2,m}$ where $1 \leq i \leq n$, $1 \leq j \leq m$. Define $f : V(SS(C_n)) \to \{0, 1, \cdots, q\}$ as follows.

Let $f(v_1) = 0$ and $f(v_i) = mi + (i-2)m$ if $2 \leq i \leq \frac{n}{2}$. For $\frac{n}{2} + 1 \leq i \leq n$, let $f(v_i) = 2mi$, $f(u_{1j}) = 2j - 1$ for $1 \leq j \leq m$. For $2 \leq i \leq n$, $1 \leq j \leq m$, let $f(u_{ij}) = mi + (i-2)m + 2j - 1$. Then the above defined labeling pattern f provides odd sequential labeling for $SS(C_n)$ where n is even. That is for even n, $SS(C_n)$ admits odd sequential labeling. □

Illustration 2.4 The Figure 2 shows an odd sequential labeling of $SS(C_8)$ with $K_{2,3}$.

Theorem 2.5 *The split graph of comb is an odd sequential graph.*

Proof Let $\{v_i, 1 \leq i \leq n\}$ and $\{v'_i, 1 \leq i \leq n\}$ be the vertices of comb in which $\{v'_i, 1 \leq i \leq n\}$ are the pendant vertices. Let $\{u_i, 1 \leq i \leq n\}$ and $\{u'_i, 1 \leq i \leq n\}$ be the newly added vertices and let G be the split graph of comb. Define $f : V(G) \to \{0, 1, \cdots, q\}$ as follows.

Let $f(v_i) = 6i - 4$ if i is odd and $6i - 3$ if i is even, and $f(v'_1) = 1$, $f(v'_3) = 15$. Let $f(v'_i) = 6i - 3$ if i is odd, $i \neq 1, 3$, and $6i - 4$ if i is even. Let $f(u_i) = 6i - 6$ if i is odd, and $6i - 7$ if i is even, and $f(u'_1) = 3$, $f(u'_3) = 11$. Let $f(u'_i) = 6i - 7$ if i is odd, $i \neq 1, 3$ and $6i - 6$ if i is even.

Then the above defined function provides an odd sequential labeling for the split graph of comb. That is, split graph of comb is an odd sequential graph. □

Illustration 2.6 The Figure 3 shows an odd sequential labeling of split graph of a comb.

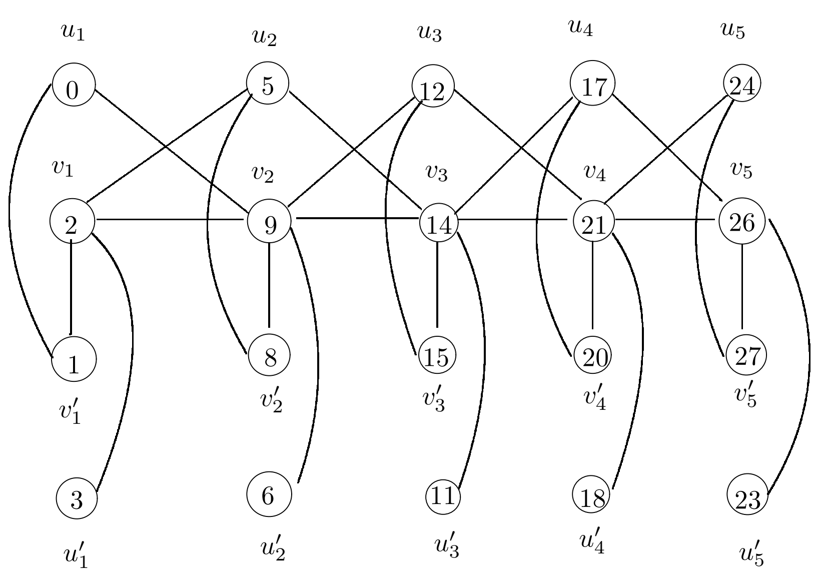

Figure 3

Theorem 2.7 *$D_2(Comb)$ admits sequential labeling.*

Proof Consider two copies of comb G_1 and G_2. Let $\{v_i, 1 \leq i \leq n\}$ and $\{v'_i, 1 \leq i \leq n\}$ be the vertices of comb G_1 and $\{u_i, 1 \leq i \leq n\}$ and $\{u'_i, 1 \leq i \leq n\}$ be the vertices of G_2. Let G be the shadow graph of the comb. Define $f : V(G) \to \{0, 1, \ldots, q\}$ as follows.

For $1 \leq i \leq n$, let $f(v_i) = 8i - 8$ if i is odd and $8i - 7$ if i is even. For $1 \leq i \leq n$, let $f(v'_i) = 8i - 7$ if i is odd, and $8i - 8$ if i is even. For $1 \leq i \leq n$, let $f(u_i) = 8i - 4$ if i is odd and $8i - 5$ if i is even. For $1 \leq i \leq n$, let $f(u'_i) = 8i - 5$ if i is odd and $8i - 4$ if i is even.

In view of the above defined labeling pattern f satisfies the conditions of odd sequential labeling. That is the $D_2(Comb)$ admits odd sequential labeling.

Illustration 2.8 The following Figure 4 shows an odd sequential labeling of $D_2(Comb)$.

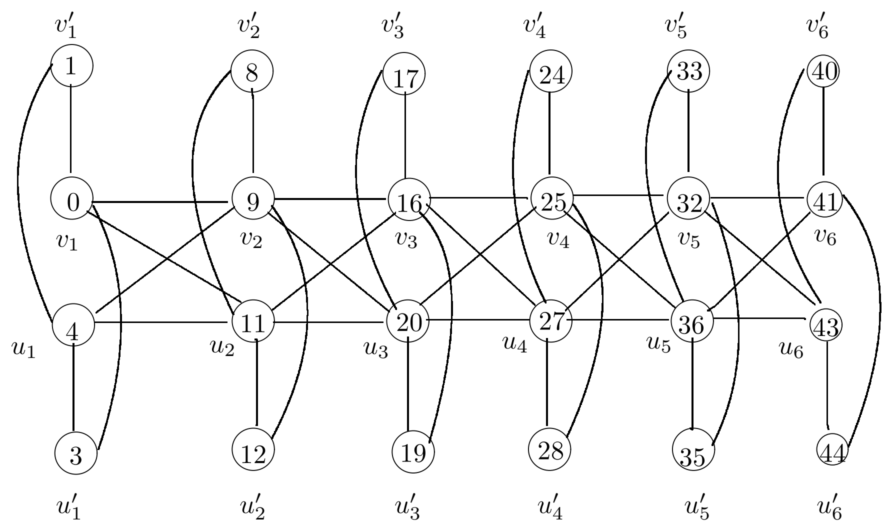

Figure 4

Theorem 2.9 *The graph $D_2(B_{n,n})$ is an odd sequential graph.*

Proof Consider two copies of $B(n, n)$ say $B_1(n, n)$ and $B_2(n, n)$. Let $\{v_1, v_2, v_{1j}, v_{2j}, 1 \leq j \leq n\}$ and $\{u_1, u_2, u_{1j}, u_{2j}, 1 \leq j \leq n\}$ be the vertices of $B_1(n, n)$ and $B_2(n, n)$ where v_1, v_2 and u_1, u_2 are the respective apex vertices. Let $D_2(B_{n,n})$ be the shadow graph of $B_1(n, n)$ and $B_2(n, n)$. Define $f : V(D_2(B_{n,n})) \to \{0, 1, \ldots, q\}$ as follows.

Let $f(v_1) = 0$, $f(v_2) = 8n + 1$, $f(u_1) = 4n$, $f(u_2) = 8n + 3$, $f(v_{1j}) = 4(j - 1) + 1$ if $1 \leq j \leq n$, $f(v_{2j}) = 4j$ if $1 \leq j \leq n - 1$, $f(v_{2n}) = 4(n + 1)$, $f(u_{1j}) = 4j - 1$ if $1 \leq j \leq n$ and $f(u_{2j}) = 4(n + j + 1)$ if $1 \leq j \leq n$.

The above defined function f provides an odd sequential labeling for $D_2(B_{n,n})$. That is $D_2(B_{n,n})$ is an odd sequential graph.

Illustration 2.10 The following *Figure 5* shows an odd sequential labeling of $D_2(B_{4,4})$.

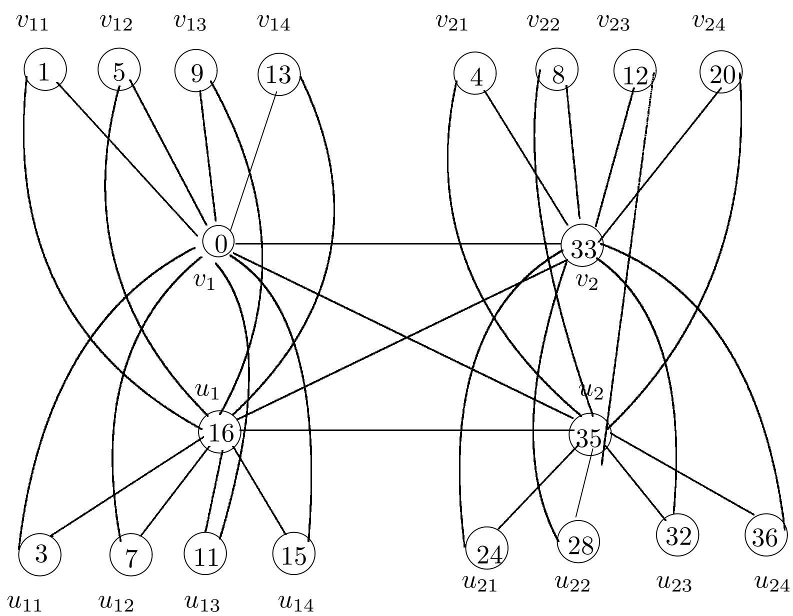

Figure 5

Theorem 2.11 *Path union graph of even cycle C_n is an odd sequential graph.*

Proof Consider k copies of even cycle C_n. Let $v_{i,j}$ be the vertices of C_n where $1 \leq i \leq k$ and $1 \leq j \leq n$. Without loss of generality let each copy of C_n is joined to its succeeding one by the edge $v_{i,1}v_{i+1,1}$; $1 \leq i \leq k-1$. Let G be the path union graph of k copies of even cycle C_n. Define $f : V(G) \to \{0, 1, \cdots, q\}$ as follows.

Case 1. k is odd.

In this case, let $f(v_{i,1}) = (n+1)(i-1)$, $f(v_{i,2}) = (n+1)(i-1)+1$. For $1 \leq i \leq k$ and $1 \leq j \leq \dfrac{n+2}{2}$, let $f(v_{i,j}) = i(n+1) - j + 2$. For $1 \leq i \leq k$ and $\dfrac{n+4}{2} \leq j \leq n$, let $f(v_{i,j}) = i(n+1) - j$ if i is odd, and $i(n+1) - j + 2$ if i is even.

Case 2. k is even.

In this case, for $1 \leq i \leq k$ and $1 \leq j \leq \dfrac{n-2}{2}$,let $f(v_{i,j}) = i(n+1) - j - 2$, for $1 \leq i \leq k$ and $\dfrac{n}{2} \leq j \leq n$, let $f(v_{i,j}) = i(n+1) - j - 2$ if i is odd, and $i(n+1) - j$ if i is even.

The above described function satisfies all the conditions of odd sequential labeling. That is, path union graph of even cycle C_n is an odd sequential graph.

Illustration 2.12 The following Figure 6 shows an odd sequential labeling of 4 copies of cycle C_8.

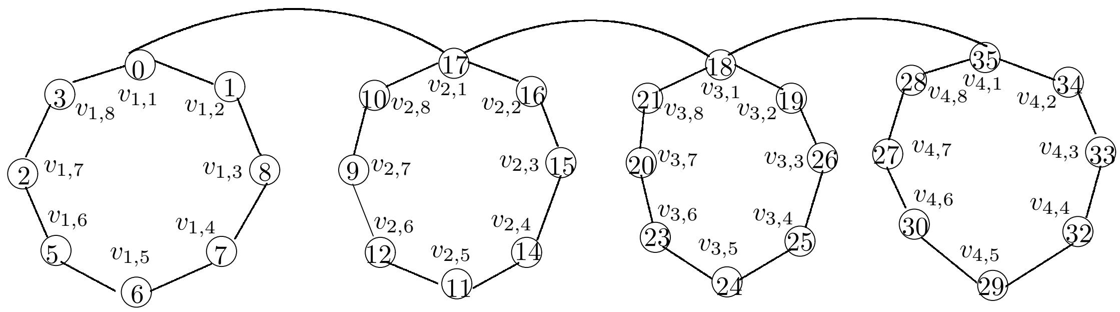

Figure 6

§3. Concluding Remarks

This paper presents 6 families of odd sequential graphs which are generated by some graph operations on some standard graphs. To investigate similar results for other graph families in the context of different labeling techniques is an open area of research.

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