*International J.Math. Combin. Vol.2(2014), 89-95*

# On Pathos Adjacency Cut Vertex Jump Graph of a Tree

Nagesh.H.M

(Department of Science and Humanities, PES Institute of Technology, Bangalore - 560 100, India)

R.Chandrasekhar

(Department of Mathematics, Atria Institute of Technology, Bangalore - 560 024, India)

E-mail: nageshhm@pes.edu, dr.chandri@gmail.com

**Abstract**: In this paper the concept of pathos adjacency cut vertex jump graph $PJC(T)$ of a tree $T$ is introduced. We also present a characterization of graphs whose pathos adjacency cut vertex jump graphs are planar, outerplanar, minimally non-outerplanar, Eulerian and Hamiltonian.

**Key Words**: Jump graph $J(G)$, pathos, Smarandache pathos-cut jump graph, crossing number cr$(G)$, outerplanar, minimally non-outerplanar, inner vertex number $i(G)$.

**AMS(2010)**: 05C78

## §1. Introduction

For standard terminology and notation in graph theory, not specifically defined in this paper, the reader is referred to Harary [2]. The concept of *pathos* of a graph $G$ was introduced by Harary [3], as a collection of minimum number of edge disjoint open paths whose union is $G$. The *path number* of a graph $G$ is the number of paths in any pathos. The path number of a tree $T$ is equal to $k$, where $2k$ is the number of odd degree vertices of $T$. A *pathos vertex* is a vertex corresponding to a path $P$ in any pathos of $T$.

The *line graph* of a graph $G$, written $L(G)$, is the graph whose vertices are the edges of $G$, with two vertices of $L(G)$ adjacent whenever the corresponding edges of $G$ are adjacent.

The *jump graph* of a graph $G$ ([1]), written $J(G)$, is the graph whose vertices are the edges of $G$, with two vertices of $J(G)$ adjacent whenever the corresponding edges of $G$ are not adjacent. Clearly, the jump graph $J(G)$ is the complement of the line graph $L(G)$ of $G$.

The *pathos jump graph* of a tree $T$ [5], written $J_P(T)$, is the graph whose vertices are the edges and paths of pathos of $T$, with two vertices of $J_P(T)$ adjacent whenever the corresponding edges of $T$ are not adjacent and the edges that lie on the corresponding path $P_i$ of pathos of $T$.

The *cut vertex jump graph* of a graph $G$ ([6]), written $JC(G)$, is the graph whose vertices are the edges and cut vertices of $G$, with two vertices of $JC(G)$ adjacent whenever the corresponding edges of $G$ are not adjacent and the edges incident to the cut vertex of $G$.

[1]Received December 11, 2013, Accepted June 5, 2014.

The *edge degree* of an edge $pq$ of a tree $T$ is the sum of the degrees of $p$ and $q$. A graph $G$ is *planar* if it can be drawn on the plane in such a way that no two of its edges intersect. If all the vertices of a planar graph $G$ lie in the exterior region, then $G$ is said to be an outerplanar.

An outerplanar graph $G$ is *maximal outerplanar* if no edge can be added without losing its outer planarity. For a planar graph $G$, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of $G$ in the plane. A graph $G$ is said to be *minimally non-outerplanar* if the inner vertex number $i(G)=1$ ([4]).

The least number of edge-crossings of a graph $G$, among all planar embeddings of $G$, is called the *crossing number* of $G$ and is denoted by $\mathrm{cr}(G)$.

A *wheel graph* $W_n$ is a graph obtained by taking the join of a cycle and a single vertex. The *Dutch windmill* graph $D_3^{(m)}$, also called a *friendship graph*, is the graph obtained by taking $m$ copies of the cycle graph $C_3$ with a vertex in common, and therefore corresponds to the usual *windmill graph* $W_n^{(m)}$. It is therefore natural to extend the definition to $D_n^{(m)}$, consisting of $m$ copies of $C_n$.

A *Smarandache pathos-cut jump graph* of a tree $T$ on subtree $T_1 < T$, written $SPJC(T_1)$, is the graph whose vertices are the edges, paths of pathos and cut vertices of $T_1$ and vertices $V(T)-V(T_1)$, with two vertices of $SPJC(T_1)$ adjacent whenever the corresponding edges of $T_1$ are not adjacent, edges that lie on the corresponding path $P_i$ of pathos, the edges incident to the cut vertex of $T_1$ and edges in $E(T)\setminus E(T_1)$. Particularly, if $T_1 = T$, such a graph is called *pathos adjacency cut vertex jump graph* and denoted by $PJC(T)$. Two distinct pathos vertices $P_m$ and $P_n$ are adjacent in $PJC(T)$ whenever the corresponding paths of pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex, say $v_c$ in $T$.

Since the pattern of pathos for a tree is not unique, the corresponding pathos adjacency cut vertex jump graph is also not unique.

In the following, Fig.1 shows a tree $T$ and Fig.2 is its corresponding $PJC(T)$.

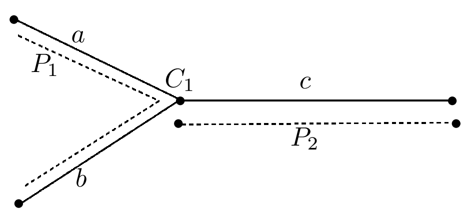


**Fig.1** Tree $T$

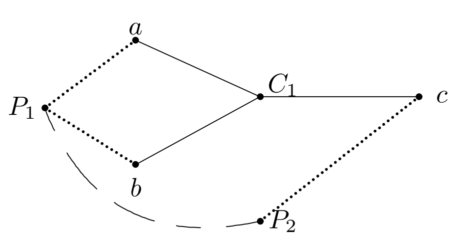


**Fig.2** $PJC(T)$

The following existing result is required to prove further results.

**Theorem A**([2]) *A connected graph $G$ is Eulerian if and only if each vertex in $G$ has even degree.*

Some preliminary results which satisfies for any $PJC(T)$ are listed following.

**Remark 1** For any tree $T$ with $n \geq 3$ vertices, $J(T) \subseteq J_P(T)$ and
$J(T) \subseteq JC(T) \subseteq PJC(T)$. Here $\subseteq$ is the subgraph notation.

**Remark 2** If the edge degree of an edge $pq$ in a tree $T$ is even(odd) and $p$ and $q$ are the cut vertices, then the degree of the corresponding vertex $pq$ in $PJC(T)$ is even.

**Remark 3** If the edge degree of a pendant edge $pq$ in $T$ is even(odd), then the degree of the corresponding vertex $pq$ in $PJC(T)$ is even.

**Remark 4** If $T$ is a tree with $p$ vertices and $q$ edges, then the number of edges in $J(T)$ is

$$\frac{q(q+1) - \sum_{i=1}^{p} d_i^2}{2},$$

where $d_i$ is the degree of vertices of $T$.

**Remark 5** Let $T$ be a tree(except star graph). Then the number of edges whose end vertices are the pathos vertices in $PJC(T)$ is $(k-1)$, where $k$ is the path number of $T$.

**Remark 6** If $T$ is a star graph $K_{1,n}$ on $n \geq 3$ vertices, then the number of edges whose end vertices are the pathos vertices in $PJC(T)$ is $\frac{k(k-1)}{2}$, where $k$ is the path number of $T$. For example, the edge $P_1P_2$ in Fig.2.

## §2. Calculations

In this section, we determine the number of vertices and edges in $PJC(T)$.

**Lemma** 2.1 *Let $T$ be a tree(except star graph) on $p$ vertices and $q$ edges such that $d_i$ and $C_j$ are the degrees of vertices and cut vertices $C$ of $T$, respectively. Then $PJC(T)$ has $(q+k+C)$ vertices and*

$$\frac{q(q+1) - \sum_{i=1}^{p} d_i^2}{2} + \sum_{j=1}^{C} C_j + q + (k-1)$$

*edges, where $k$ is the path number of $T$.*

*Proof* Let $T$ be a tree(except star graph) on $p$ vertices and $q$ edges. The number vertices of $PJC(T)$ equals the sum of edges, paths of pathos and cut vertices $C$ of $T$. Hence $PJC(T)$ has $(q+k+C)$ vertices. The number of edges of $PJC(T)$ equals the sum of edges in $J(T)$, degree of cut vertices, edges that lie on the corresponding path $P_i$ of pathos of $T$ and the number of edges whose end vertices are the pathos vertices. By Remark 4 and 5, the number of edges in $PJC(T)$ is given by

$$\frac{q(q+1) - \sum_{i=1}^{p} d_i^2}{2} + \sum_{j=1}^{C} C_j + q + (k-1). \qquad \square$$

**Lemma** 2.2 *If $T$ is a star graph $K_{1,n}$ on $n \geq 3$ vertices and $m$ edges, then $PJC(T)$ has $(m+k+1)$ vertices and $\frac{4m+k(k-1)}{2}$ edges, where $k$ is the path number of $T$.*

*Proof* Let $T$ be a star graph $K_{1,n}$ on $n \geq 3$ vertices and $m$ edges. By definition, $PJC(T)$ has $(m+k+1)$ vertices. Also, for a star graph, the number of edges of $PJC(T)$ equals the sum of edges in $J(T)$, i.e., zero, twice the number of edges of $T$ and the number of edges whose end vertices are the pathos vertices. By Remark 6, the number of edges in $PJC(T)$ is given by

$$2m + \frac{k(k-1)}{2} \Rightarrow \frac{4m+k(k-1)}{2}. \qquad \square$$

## §3. Main Results

**Theorem** 3.1 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree $T$ is planar if and only if the following conditions hold:*

*(i) $T$ is a path $P_n$ on n=3 and 4 vertices;*
*(ii) $T$ is a star graph $K_{1,n}$, on n =3,4,5 and 6 vertices.*

*Proof* (i) Suppose $PJC(T)$ is planar. Assume that $T$ is a path $P_n$ on $n \geq 5$ vertices. Let $T$ be a path $P_5$ and let the edge set $E(P_5) = \{e_1, e_2, e_3, e_4\}$. Then the jump graph $J(T)$ is the path $P_4 = \{e_3, e_1, e_4, e_2\}$. Since the path number of $T$ is exactly one, $J_P(T)$ is $W_n - e$, where $W_n$ is the join of a cycle with the vertices corresponding to edges of $T$ and a single vertex corresponding to pathos vertex $P$, and $e$ is an edge between any two vertices corresponding to arcs of $T$ in $W_n$. Let $\{C_1, C_2, C_3\}$ be the cut vertex set of $T$. Then the edges joining to $J(T)$ from the corresponding cut vertices gives $PJC(T)$ such that the crossing number of $PJC(T)$ is one, i.e., $\mathrm{cr}(PJC(T)) = 1$, a contradiction.

For sufficiency, we consider the following two cases.

**Case 1** If $T$ is a path $P_3$, then $PJC(T)$ is cycle $C_4$, which is planar.

**Case 2** Let $T$ be a path $P_4$ and let $E(P_4) = \{e_1, e_2, e_3\}$. Also, the path number of $T$ is exactly one, i.e., $P$. Then $J_P(T)$ is $K_{1,3} + e$, where $P$ is the vertex of degree three, and $e$ is an edge between any two vertices corresponding to edges of $T$ in $K_{1,3}$. Let $\{C_1, C_2\}$ be the cut vertex set of $T$. Then the edges joining to $J(T)$ from the corresponding cut vertices gives $PJC(T) = W_n - \{a, b\}$, where $W_n$ is join of a cycle with the vertices corresponding to edges and cut vertices of $T$ and a single vertex corresponding to pathos vertex $P$, and $\{a, b\}$ are the edges between pathos vertex $P$ and cut vertices $C_1$ and $C_2$ of $W_n$. Clearly, $\mathrm{cr}(PJC(T)) = 0$. Hence $PJC(T)$ is planar.

($ii$) Suppose that $PJC(T)$ is planar. Let $T$ be a star graph $K_{1,n}$ on $n \geq 7$ vertices. If $T$ is $K_{1,7}$, then $J(T)$ is a null graph of order seven. Since each edge in $T$ lies on exactly one cut vertex $C$, $JC(T)$ is a star graph $K_{1,7}$. Furthermore, the path number of $T$ is exactly four. Hence $PJC(T)$ is $D_4^{(4)} - v$, where $v$ is a vertex at distance one from the common vertex in $D_4^{(4)}$. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in $T$, by Remark 6, $\text{cr}(PJC(T)) = 1$, a contradiction.

Conversely, suppose that $T$ is a star graph $K_{1,n}$ on $n$=3,4,5 and 6 vertices. For $n$=3,4,5 and 6 vertices, $J(T)$ is a null graph of order $n$. Since each edge in $T$ lies on exactly one cut vertex $C$, $JC(T)$ is a star graph of order $n+1$. The path number of $T$ is at most 3. Now, for $n$=4, $PJC(T)$ is the join of two copies of cycle $C_4$ with a common vertex and for $n$=6, $PJC(T)$ is the join of three copies of cycle $C_4$ with a common vertex. Next, for $n$=3, $PJC(T)$ is $D_4^{(2)} - v$, and $n$=5, $PJC(T)$ is $D_4^{(3)}$, respectively, where $v$ is the vertex at distance one from the common vertex. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in $T$, by Remark 6, $\text{cr}(PJC(T)) = 0$. Hence $PJC(T)$ is planar. □

**Theorem** 3.2 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree $T$ is an outerplanar if and only if $T$ is a path $P_3$.*

*Proof* Suppose that $PJC(T)$ is an outerplanar. By Theorem 3.1, $PJC(T)$ is planar if and only if $T$ is a path $P_3$ and $P_4$. Hence it is enough to verify the necessary part of the Theorem for a path $P_4$. Assume that $T$ is a path $P_4$ and the edge set $E(P_4)$= $e_i$, where $e_i$=$(v_i, v_{i+1})$, for all $i = 1, 2, 3$. Then the jump graph $J(T)$ is a disconnected graph with two connected components, namely $K_1$ and $K_2$, where $K_1 = e_2$ and $K_2 = (e_1, e_3)$. Let $\{C_1, C_2\}$ be the cut vertex set of $T$. Hence $JC(T)$ is the cycle $C_5 = \{C_1, e_1, e_3, C_2, e_2, C_1\}$. Furthermore, the path number of $T$ is exactly one. Then the edges joining to $J(T)$ from the corresponding pathos vertex gives $PJC(T)$ such that the inner vertex number of $PJC(T)$ is non-zero, i.e., $i(PJC(T)) \neq 0$, a contradiction.

Conversely, if $T$ is a path $P_3$, then $PJC(T)$ is a cycle $C_4$, which is an outerplanar. □

**Theorem** 3.3 *For any tree $T$, $PJC(T)$ is not maximal outerplanar.*

*Proof* By Theorem 3.2, $PJC(T)$ is an outerplanar if and only if $T$ is a path $P_3$. Moreover, for a path $P_3$, $PJC(T)$ is a cycle $C_4$, which is not maximal outerplanar, since the addition of an edge between any two vertices of cycle $C_4$ does not affect the outerplanarity of $C_4$. Hence for any tree $T$, $PJC(T)$ is not maximal outerplanar. □

**Theorem** 3.4 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree $T$ is minimally non-outerplanar if and only if $T$ is (i) a star graph $K_{1,3}$, and (ii) a path $P_4$.*

*Proof* ($i$) Suppose that $PJC(T)$ is minimally non-outerplanar. If $T$ is a star graph $K_{1,n}$ on $n \geq 7$ vertices, by Theorem 3.1, $PJC(T)$ is nonplanar, a contadiction. Let $T$ be a star graph $K_{1,n}$ on $n$=4,5 and 6 vertices. Now, for $n$=4, $PJC(T)$ is the join of two copies of cycle $C_4$ with a common vertex and for $n$=6, $PJC(T)$ is the join of three copies of cycle $C_4$ with a common vertex. For $n$=5, $PJC(T)$ is $D_4^{(3)} - v$. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in $T$, the inner vertex number of $PJC(T)$ is more than one, i.e., $i(PJC(T)) > 1$, a contradiction.

Conversely, suppose that $T$ is a star graph $K_{1,3}$. Then $J(T)$ is a null graph of order three. Since edge in $T$ lies on exactly one cut vertex $C$, $JC(T)$ is a star graph $K_{1,3}$. The path number of $T$ is exactly two. By definition, $PJC(T)$ is $D_4^{(2)} - v$. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in $T$, the inner vertex number of $PJC(T)$ is exactly one, i.e., $i(PJC(T)) = 1$. Hence $PJC(T)$ is minimally non-outerplanar.

$(ii)$ Suppose $PJC(T)$ is minimally non-outerplanar. Assume that $T$ is a path on $n \geq 5$ vertices. If $T$ is a path $P_5$, by Theorem 3.1, $PJC(T)$ is nonplanar, a contradiction.

Conversely, if $T$ is a path $P_4$, by Case 2 of sufficiency part of Theorem 3.1, $PJC(T)$ is $W_n - \{a, b\}$. Clearly, $i(PJC(T)) = 1$. Hence $PJC(T)$ is minimally non-outerplanar. □

**Theorem** 3.5 *The pathos adjacency cut vertex jump graph $PJC(T)$ of a tree $T$ is Eulerian if and only if the following conditions hold:*

$(i)$ *$T$ is a path $P_n$ on $n = 2i+1$ vertices, for all $i = 1, 2, \cdots$;*

$(ii)$ *$T$ is a star graph $K_{1,n}$ on $n = 4j+2$ vertices, for all $j = 0, 1, 2, \cdots$.*

*Proof* $(i)$ Suppose that $PJC(T)$ is Eulerian. If $T$ is a path $P_n$ on $n = 2(i+1)$ vertices, for all $i = 1, 2, \cdots$, then the number of vertices in $J(T)$ is $(2i+1)$, which is always odd. Since the path number of $T$ is exactly one, by definition, the degree of the corresponding pathos vertex in $PJC(T)$ is odd. By Theorem [A], $PJC(T)$ is non-Eulerian, a contradiction.

For sufficiency, we consider the following two cases.

**Case 1** If $T$ is a path $P_3$, then $PJC(T)$ is a cycle $C_4$, which is Eulerian.

**Case 2** Suppose that $T$ is a path $P_n$ on $n = 2i+1$ vertices, for all $i = 2, 3, \cdots$. Let $\{e_1, e_2, \cdots e_{n-1}\}$ be the edge set of $T$. Then $d(e_1)$ and $d(e_{n-1})$ in $J(T)$ is even and degree of the remaining vertices $e_2, e_3, \cdots, e_{n-2}$ is odd. The number of cut vertices in $T$ is $(n-2)$. By definition, in $JC(T)$ the degree of even and odd degree vertices of $J(T)$ will be incremented by one and two, respectively. Hence the degree of every vertex of $JC(T)$ except cut vertices is odd. Furthermore, the path number of $T$ is exactly one and the corresponding pathos vertex is adjacent to every vertex of $J(T)$. Clearly, every vertex of $PJC(T)$ has an even degree. By Theorem A, $PJC(T)$ is Eulerian.

$(ii)$ Suppose that $PJC(T)$ is Eulerian. We consider the following two cases.

**Case 1** Suppose that $T$ is a star graph $K_{1,n}$ on $n = 2j+1$ vertices, for all $j = 1, 2, \cdots$. Then $J(T)$ is a null graph of order $n$. Since each edge in $T$ lies on exactly one cut vertex $C$, $JC(T)$ is a star graph $K_{1,n}$ in which $d(C)$ is odd. Moreover, since the degree of a cut vertex $C$ does not change in $PJC(T)$, it is easy to observe that the vertex $C$ remains as an odd degree vertex in $PJC(T)$. By Theorem A, $PJC(T)$ is non-Eulerian, a contradiction.

**Case 2** Suppose that $T$ is a star graph $K_{1,n}$ on $n = 4j$ vertices, for all $j = 1, 2, \cdots$. Then $J(T)$ is a null graph of order $n$. Since each edge in $T$ lies on exactly one cut vertex $C$, $JC(T)$

is a star graph $K_{1,n}$ in which $d(C)$ is even. Since the path number of $T$ is $[\frac{n}{2}]$, by definition, $PJC(T)$ is the join of at least two copies of cycle $C_4$ with a common vertex. Hence for every $v \in PJC(T)$, $d(v)$ is even. Finally, on embedding $PJC(T)$ in any plane for the adjacency of pathos vertices corresponding to paths of pathos in $T$, there exists at least one pathos vertex, say $P_m$ of odd degree in $PJC(T)$. By Theorem [A], $PJC(T)$ is non-Eulerian, a contradiction.

For sufficiency, we consider the following two cases.

**Case 1** For a star graph $K_{1,2}$, $T$ is a path $P_3$. Then $PJC(T)$ is a cycle $C_4$, which is Eulerian.

**Case 2** Suppose that $T$ is a star graph $K_{1,n}$ on $n = 4j + 2$ vertices, for all $j = 1, 2, \cdots$. Then the jump graph $J(T)$ is a null graph of order $n$. Since each edge in $T$ lies on exactly one cut vertex $C$, $JC(T)$ is a star graph $K_{1,n}$ in which $d(C)$ is even. The path number of $T$ is $[\frac{n}{2}]$. By definition, $PJC(T)$ is the join of at least three copies of cycle $C_4$ with a common vertex. Hence for every $v \in PJC(T)$, $d(v)$ is even. Finally, on embedding $PJC(T)$ in any plane for the the adjacency of pathos vertices corresponding to paths of pathos in $T$, degree of every vertex of $PJC(T)$ is also even. By Theorem A, $PJC(T)$ is Eulerian. □

**Theorem** 3.6 *For any path $P_n$ on $n \geq 3$ vertices, $PJC(T)$ is Hamiltonian.*

*Proof* Suppose that $T$ is a path $P_n$ on $n \geq 3$ vertices with $\{v_1, v_2, \cdots v_n\} \in V(T)$ and $\{e_1, e_2, \cdots e_{n-1}\} \in E(T)$. Let $\{C_1, C_2, \cdots C_{n-2}\}$ be the cut vertex set of $T$. Also, the path number of $T$ is exactly one and let it be $P$.

By definition $\{e_1, e_2, \cdots e_{n-1}\} \cup \{C_1, C_2, \cdots C_{n-2}\} \cup P$ form the vertex set in $PJC(T)$. In forming $PJC(T)$, the pathos $P$ becomes a vertex adjacent to every vertex of $\{e_1, e_2, \cdots, e_{n-1}\}$ in $J(T)$. Also, the cut vertices $C_j$, for all $j = 1, 2, \cdots, (n-2)$ are adjacent to $(e_i, e_{i+1})$ for all $i = 1, 2, \cdots, (n-1)$ of $J_P(T)$. Clearly, there exist a cycle $(P, e_1, C_1, e_2, C_2, \cdots e_{n-1}, C_{n-2}, e_{n-1}, P)$ containing all the vertices of $PJC(T)$. Hence $PJC(T)$ is Hamiltonian. □

## References


[1] Chartrand H.Hevia and Jarrett M.Schultz, Subgraph distances in graphs defined by edge transfers, *Discrete Math.*, 170, pp-63-79.

[2] Harary.F., *Graph Theory*, Addison-Wesley, Reading, Mass (1969).

[3] Harary.F., Converging and packing in graphs-I, *Annals of New York Academy of Science*, 175(1970), pp-198-205.

[4] Kulli.V.R.,On minimally non outerplanar graphs, *Proc.Indian Nat.Sci.Acad.*, 41 (1975), pp-276-280.

[5] Venkanagouda.M.Goudar, Nagendrappa E. and Murali R., Pathos jump graph of a tree *International Journal of Mathematics Research*, Vol.4, 4(2012), 333-338.

[6] Venkanagouda.M.Goudar and Nagendrappa G., Cut vertex jump graph, *International Journal of Publications of Applications and Problems in Engineering Research Paper*, Vol.2(2011), 129-133.