International J.Math. Combin. Vol.2(2014), 116-121

Total Dominator Colorings in Caterpillars

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Abstract: Let G be a graph without isolated vertices. A total dominator coloring of a graph G is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper we determine the total dominator chromatic number in caterpillars.

Key Words: Total domination number, chromatic number and total dominator chromatic number, Smarandachely k-dominator coloring, Smarandachely k-dominator chromatic number.

AMS(2010): 05C15, 05C69

§1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [4].

Let $G = (V, E)$ be a graph of order n with minimum degree at least one. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined to be $\bigcup_{v\in S} N(v)$, and the closed neighborhood of S is $N[S] = N(S) \cup S$.

A subset S of V is called a total dominating set if every vertex in V is adjacent to some vertex in S. A total dominating set is minimal total dominating set if no proper subset of S is a total dominating set of G. The total domination number γ_t is the minimum cardinality taken over all minimal total dominating sets of G. A γ_t-set is any minimal total dominating set with cardinality γ_t.

A proper coloring of G is an assignment of colors to the vertices of G, such that adjacent vertices have different colors.

The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$. Let $V = \{u_1, u_2, u_3, \cdots, u_p\}$ and $C = \{C_1, C_2, C_3, \cdots, C_n\}$, $n \leqslant p$ be a collection of subsets $C_i \subset V$. A color represented in a vertex u is called a non-repeated color if there exists one color class $C_i \in C$ such that $C_i = \{u\}$.

A vertex v of degree 1 is called an end vertex or a pendant vertex of G and any vertex

[1]Received June 6, 2013, Accepted June 12, 2014.

which is adjacent to a pendant vertex is called a support.

A caterpillar is a tree with the additional property that the removal of all pendant vertices leaves a path. This path is called the spine of the caterpillar, and the vertices of the spine are called vertebrae. A vertebra which is not a support is called a zero string. In a caterpillar, consider the consecutive i zero string, called zero string of length i. A caterpillar which has no zero string of length at least 2 is said to be of class 1 and all other caterpillars are of class 2.

Let G be a graph without isolated vertices. For an integer $k \geqslant 1$, a Smarandachely k-dominator coloring of G is a proper coloring of G with the extra property that every vertex in G properly dominates a k-color classes and the smallest number of colors for which there exists a Smarandachely k-dominator coloring of G is called the Smarandachely k-dominator chromatic number of G and is denoted by $\chi_{td}^{S}(G)$. Let G be a graph without isolated vertices. A total dominator coloring of a graph G is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper we determine total dominator chromatic number in caterpillars.

Throughout this paper, we use the following notations.

Notation 1.1. Usually, the vertices of P_n are denoted by $u_1, u_2, \cdots, u_n$ in order. For $i < j$, we use the notation $\langle [i,j] \rangle$ for the sub path induced by $\langle u_i, u_{i+1}, \cdots, u_j \rangle$. For a given coloring C of P_n , $C/\langle [i,j] \rangle$ refers to the coloring C restricted to $\langle [i,j] \rangle$.

We have the following theorem from [1].

Theorem 1.2([1]) *Let G be any graph with $\delta(G) \geqslant 1$. Then* $\max\{\chi(G), \gamma_t(G)\} \leqslant \chi_{td}(G) \leqslant \chi(G) + \gamma_t(G)$.

From Theorem 1.2, $\chi_{td}(P_n) \in \{\gamma_t(P_n), \gamma_t(P_n)+1, \gamma_t(P_n)+2\}$. We call the integer n, good (respectively bad, very bad) if $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ (if respectively $\chi_{td}(P_n) = \gamma_t(P_n) + 1, \chi_{td}(P_n) = \gamma_t(P_n)$). First, we prove a result which shows that for large values of n, the behavior of $\chi_{td}(P_n)$ depends only on the residue class of n mod 4 [More precisely, if n is good, $m > n$ and $m \equiv n (\mathrm{mod}\ 4)$ then m is also good]. We then show that $n = 8, 13, 15, 22$ are the least good integers in their respective residue classes. This therefore classifies the good integers.

Fact 1.3 Let $1 < i < n$ and let C be a td-coloring of P_n. Then, if either u_i has a repeated color or u_{i+2} has a non-repeated color, $C/\langle [i+1, n] \rangle$ is also a td-coloring.

Theorem 1.4([2]) *Let n be a good integer. Then, there exists a minimum td-coloring for P_n with two n-d color classes.*

§2. Total Dominator Colorings in Caterpillars

After the classes of stars and paths, caterpillars are perhaps the simplest class of trees. For this reason, for any newly introduced parameter, we try to obtain the value for this class. In this paper, we give an upper bound for $\chi_{td}(T)$, where T is a caterpillar (with some restriction). First, we prove a theorem for a very simple type which however illustrates the ideas to be used in the general case.

Theorem 2.1 *Let G be a caterpillar such that*

(i) No two vertices of degree two are adjacent;
(ii) The end vertebrae have degree at least 3;
(iii) No vertex of degree 2 is a support vertex.

Then $\chi_{td}(G) \leqslant \left\lceil \frac{3r+2}{2} \right\rceil$.

Proof Let C be the spine of G. Let $u_1, u_2, \cdots, u_r$ be the support vertices and $u_{r+1}, u_{r+2}, \cdots, u_{2r-1}$ be the vertices of degree 2 in C. In a td-coloring of G, all support vertices receive a non-repeated color, say 1 to r and all pendant vertices receive the same repeated color say $r+1$ and the vertices u_{r+1} and u_{2r-1} receive a non-repeated color say $r+2$ and $r+3$ respectively. Consider the vertices $\{u_{r+2}, u_{r+3}, \cdots, u_{2r-2}\}$. We consider the following two cases.

Case 1 r is even.

In this case the vertices $u_{r+3}, u_{r+5}, \cdots, u_{r+(\frac{r}{2}-2)}, u_{r+\frac{r}{2}}, u_{r+(\frac{r}{2}+2)}, \cdots, u_{2r-3}$ receive the non-repeated colors say $r+4$ to $r+\left(\frac{r}{2}+1\right) = \frac{3r+2}{2}$ and the remaining vertices $u_{r+2}, u_{r+4}, \cdots, u_{2r-2}$ receive the already used repeated color $r+1$ respectively. Thus $\chi_{td}(G) \leqslant \frac{3r+2}{2}$.

Case 2 r is odd.

In this case the vertices $u_{r+3}, u_{r+5}, \cdots, u_{r+(\frac{r}{2}-2)}, u_{r+\frac{r}{2}}, u_{r+(\frac{r}{2}+2)}, \cdots, u_{2r-4}, u_{2r-2}$ receive the non-repeated colors say $r+4$ to $r+\left(\frac{r+3}{2}\right) = \frac{3r+3}{2}$ and the remaining vertices $u_{r+2}, u_{r+4}, \cdots, u_{2r-3}$ receive the already used repeated color $r+1$ respectively. Thus

$$\chi_{td}(G) \leqslant \frac{3r+3}{2} = \left\lceil \frac{3r+2}{2} \right\rceil. \qquad \square$$

Illustration 2.2 In Figures 1 and 2, we present 2 caterpillars holding with the upper bound of $\chi_{td}(G)$ in Theorem 2.1.

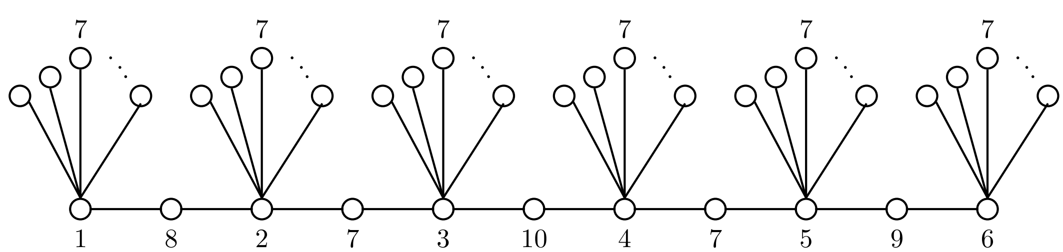

Figure 1

Clearly, $\chi_{td}(G) = 10 = \frac{3r+2}{2}$.

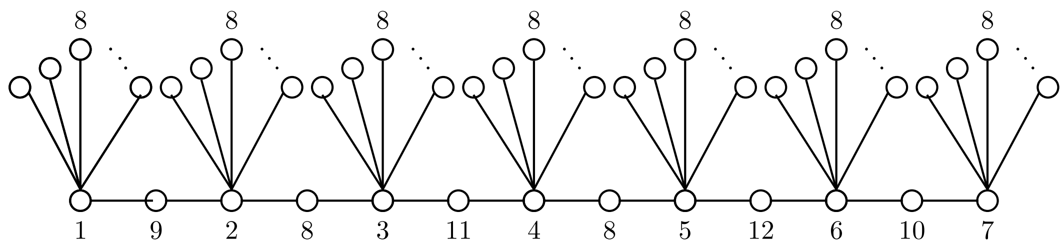

Figure 2

Clearly, $\chi_{td}(G) = 12 = \left\lceil \frac{3r+2}{2} \right\rceil$.

Remark 2.3 Let C be a minimal td-coloring of G. We call a color class in C, a non-dominated color class ($n-d$ color class) if it is not dominated by any vertex of G. These color classes are useful because we can add vertices to those color classes without affecting td-coloring.

Theorem 2.4 *Let G be a caterpillar of class 2 having exactly r vertices of degree at least 3 and r_i zero strings of length $i, 2 \leqslant i \leqslant m, m$ = maximum length of a zero string in G. Further suppose that $r_n \neq 0$ for some n, where $n-2$ is a good number and that end vertebrae are of degree at least 3. Then*

$$\chi_{td}(G) \leqslant 2(r+1) + \sum_{\substack{i=3 \\ i \equiv 1,2,3 (mod 4)}}^{m} r_i \left\lceil \frac{i-2}{2} \right\rceil + \sum_{\substack{i=4 \\ i \equiv 0 (mod 4)}}^{m} r_i \left(\left\lceil \frac{i-2}{2} \right\rceil + 1 \right).$$

Proof Let S be the spine of the caterpillar G and let $V(S) = \{u_1, u_2, \cdots, u_r\}$. We give the coloring of G as follows:

Vertices in S receive non-repeated colors, say from 1 to r. The set $N(u_j)$ is given the color $r+j, 1 \leqslant j \leqslant r$ (u_j is not adjacent to an end vertex of zero string of length 3 and if a vertex is adjacent to two supports, it is given one of the two possible colors). This coloring takes care of any zero string of length 1 or 2. Now, we have assumed $r_n \neq 0$ for some n, where $n-2$ is a good number. Hence there is a zero string of length n in G.

By Theorem 1.4, there is a minimum td-coloring of this path in which there are two $n-d$ colors. We give the sub path of length n this coloring with $n-d$ colors being denoted by $2r+1, 2r+2$. The idea is to use these two colors whenever $n-d$ colors occur in the coloring of zero strings. Next, consider a zero string of length 3, say

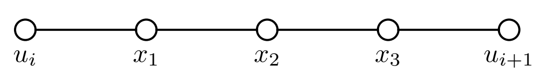

Figure 3

where u_i and u_{i+1} are vertices of degree at least 3 and we have denoted the vertices of the string of length 3 by x_1, x_2, x_3 for simplicity. Then, we give x_1 or x_3, say x_1 with a non-repeated color;

we give x_2 and x_3 the colors $2r+1$ and $2r+2$ respectively. Thus each zero string of length 3 introduces a new color and $\left\lceil \frac{3-2}{2} \right\rceil = 1$. Similarly, each zero string of length i introduces $\left\lceil \frac{i-2}{2} \right\rceil$ new colors when $i \equiv 1, 2, 3 (\mathrm{mod}\ 4)$. However, the proof in cases when $i > 3$ is different from case $i = 3$ (but are similar in all such cases in that we find a *td*-coloring involving two $n-d$ colors). e.g. a zero string of length 11.

We use the same notation as in case $i = 3$ with a slight difference:

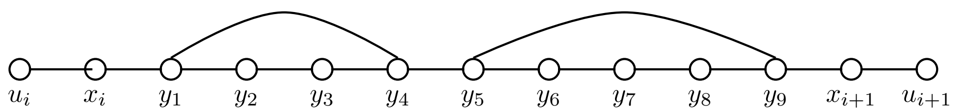

Figure 4

u_i and u_{i+1} being support vertices receive colors i and $i+1$. x_i and x_{i+1} receive $r+i$ and $r+i+1$ respectively. For the coloring of P_9 , we use the color classes $\{y_1, y_4\}, \{y_2\}, \{y_3\}, \{y_5, y_9\}, \{y_6\}$, $\{y_7\}, \{y_8\}$. We note that this is *not* a minimal *td*-coloring which usually has no $n-d$ color classes. This coloring has the advantage of having two $n-d$ color classes which can be given the class $2r+1$ and $2r+2$ and the remaining vertices being given non-repeated colors. In cases where i is a good integer, P_{i-2} requires $\left\lceil \frac{i-2}{2} \right\rceil + 2$ colors. However there will be two $n-d$ color classes for which $2r+1$ and $2r+2$ can be used. Thus each such zero string will require only $\left\lceil \frac{i-2}{2} \right\rceil$ new colors (except for the path containing the vertices we originally colored with $2r+1$ and $2r+2$). However, if $i \equiv 0(\mathrm{mod} 4), i-2 \equiv 2(\mathrm{mod} 4)$, and we will require $\left\lceil \frac{i-2}{2} \right\rceil + 1$ new colors. It is easily seen this coloring is a *td*-coloring. Hence the result. □

Illustration 2.5 In Figures 5 − 7, we present 3 caterpillars with minimum td-coloring.

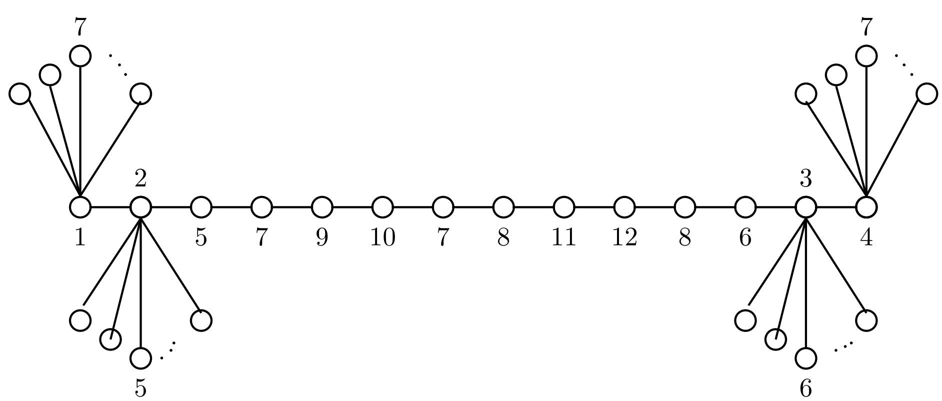

Figure 5

Then, $\chi_{td}(T) = 12 < 2(r+1) + r_{10} \left\lceil \frac{10-2}{2} \right\rceil$.

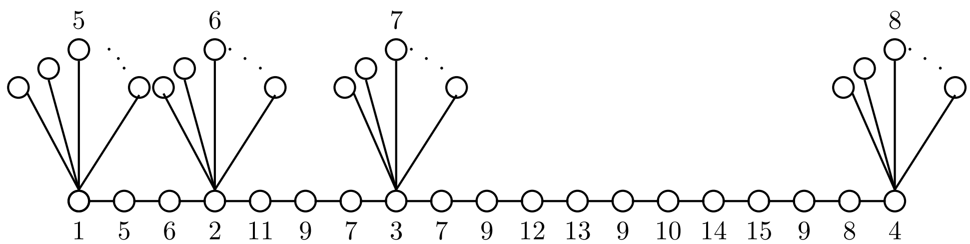

Figure 6

Then, $\chi_{td}(T_2) = 15 = 2(r+1) + r_3\left\lceil \frac{3-2}{2} \right\rceil + r_{10}\left\lceil \frac{10-2}{2} \right\rceil$

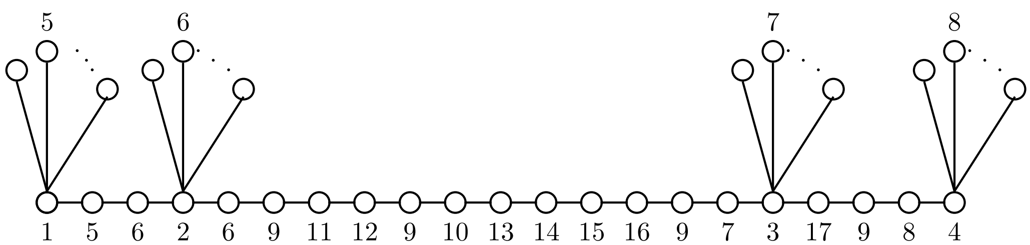

Figure 7

Then, $\chi_{td}(T_3) = 17 = 2(r+1) + r_3 + r_{12}\left(\left\lceil \frac{12-2}{2} \right\rceil + 1\right)$.

Remark 2.7 (1) The condition that end vertebrae are of degree at least 3 is adopted for the sake of simplicity. Otherwise the caterpillar 'begins' or 'ends' (or both) with a segment of a path and we have to add the χ_{td} -values for this (these) path(s).

(2) If in Theorem 2.1, we assume that all the vertices of degree at least 3 are adjacent (instead of (ii)), we get $\chi_{td}(G) = r + 1$.

(3) The bound in Theorem 2.4 does not appear to be tight. We feel that the correct bound will have $2r + 1$ on the right instead of $2r + 2$. There are graphs which attain this bound.

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