Non-Congruent Triangles with Equal Perimeters and Arias

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In [1] Professor I. Ivănescu from Craiova has proposed the following

Open problem

Construct, using a ruler and a compass, two non-congruent triangles, which have equal perimeters and arias.

In preparation for the proof of this problem we recall several notions and we prove a Lemma.

Definition

An A-ex-inscribed circle to a given triangle ABC is the tangent circle to the side (BC) and to the extended sides (AB), (AC).

The center of the A-ex-inscribed triangle is the intersection of the external bisectors of the angles B and C, which we note it with I_a and its radius with r_a.

Observation 1.

To a given triangle correspond three ex-inscribed circles. In figure 1 we represent the A-ex-inscribed circle to triangle ABC.

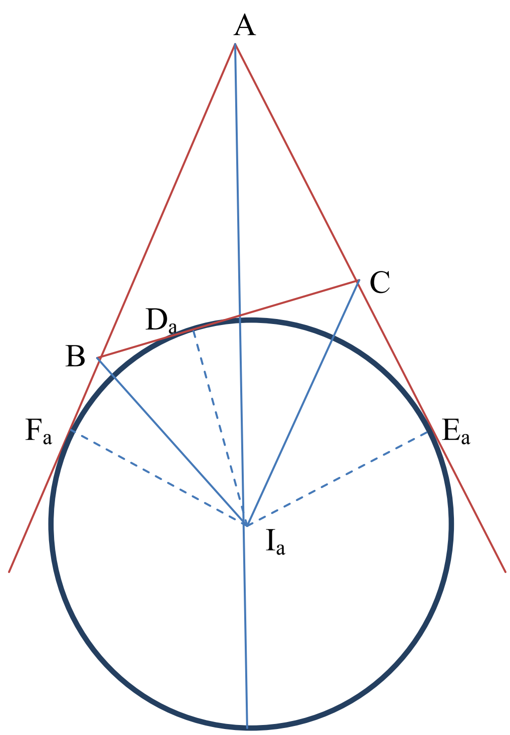

Fig. 1

Lemma 1

The length of the tangent constructed from one of the triangle's vertexes to the corresponding ex-inscribed circle is equal with the triangle's semi-perimeter.

Proof

Let D_a, E_a, F_a the points of contact of the A-ex-inscribed triangle with (BC), AC, AB. We have $AE_a = AF_a$, $BD_a = BF_a$, $CD_a = CE_a$ (the tangents constructed from a point to a circle are congruent). We note $BD_a = x$, $CD_a = y$ and we observe that $AE_a = AC + CE_a$, therefore $AE_a = b + y$, $AF_a = AB + BF_a$, it results that $AF_a = c + x$. We resolve the system:

$$\begin{cases} x + y = a \\ x + c = y + b \end{cases}$$

and we obtain

$$x = \frac{1}{2}(a + b - c)$$

$$y = \frac{1}{2}(a + c - b)$$

Taking into consideration that the semi-perimeter $p = \frac{1}{2}(a + b + c)$ we have $x = p - c$; $y = p - b$, and we obtain that $AF_a = AE_a = p$ thus the lemma is proved.

The proof of the open problem

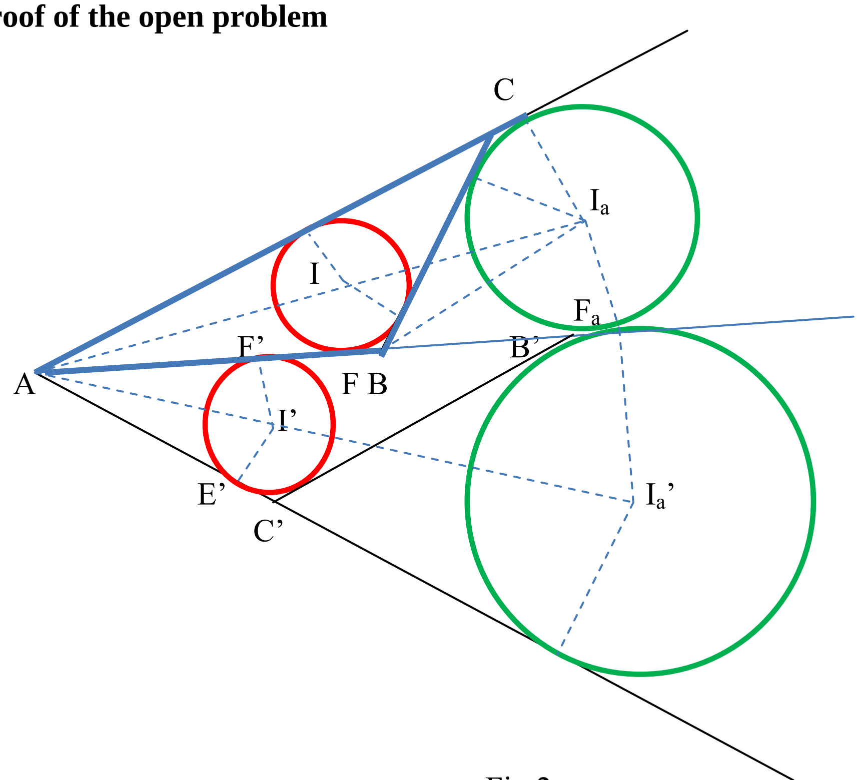

Fig.2

Let ABC a given triangle. We construct $\mathcal{C}(I,r)$ its inscribed circle and $\mathcal{C}(I_a,r_a)$ its A-ex-inscribed circle, see figure 2. In conformity with the Lemma we have that $AF_a = p$ - the semi-perimeter of triangle ABC.

We construct the point $F' \in (AF)$ and the circle of radius r tangent in F' to AB, that is $\mathcal{C}(I',r)$. It is easy to justify that angle *F'AI'* > angle *FAI* and therefore angle *F'AE'* > angle *A* (we noted *E'* the contact point with the circle $\mathcal{C}(I',r)$ of the tangent constructed from A). We note $I_a^{'}$ the intersection point of the lines AI', I_aF_a.

We construct the circle $\mathcal{C}(I_a^{'}I_a^{'}F_a)$ and then the internal common tangent to this circle and to the circle $\mathcal{C}(I',r)$; we note B', C' the intersections of this tangent with AB respectively with AE'. From these constructions it result that the circle $\mathcal{C}(I',r)$ is inscribed in the triangle $AB'C'$ and the circle $\mathcal{C}(I_a^{'}I_a^{'}F_a)$ ex-inscribed to this triangle.

The Lemma states that the semi-perimeter of the triangle $AB'C'$ is equal with AF_a therefore it is equal to p - the semi-perimeter of triangle ABC.

On the other side the inscribed circles in the triangles ABC and $AB'C'$ are congruent. Because the aria S of the triangle ABC is given by the formula $S = p \cdot r$, we obtain that also the aria of triangle $AB'C'$ is equal with S.

The constructions listed above can be executed with a ruler and a compass without difficulty, and the triangles ABC and $AB'C'$ are not congruent.

Indeed, our constructions are such that the angle *B'AC'* is greater than angle *BAC*. Also we can choose F' on (AF) such that $F'AI'$ is different of $\frac{1}{2}\mathrm{C}$ and of $\frac{1}{2}\mathrm{B}$. In this way the angle A of the triangle $AB'C'$ is not congruent with any angle of the triangle ABC.

Observation 2

We practically proved much more than the proposed problem asked, because we showed that for any given triangle ABC we can construct another triangle which will have the same aria and the same perimeter with the given triangle without being congruent with it.

Observation 3

In [2] the authors find two isosceles triangles in the conditions of the hypothesis.

Note

The authors thank to Professor Ştefan Brânzan from the National College "Fraţii Buzeşti" – Craiova for his suggestions, which made possible the enrichment of this article.

References

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