Infinite Series and Integral Representations for Bessel Function of the first kind $J_v(z)$

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Therefore also now, saith the LORD, turn ye even to me with all your heart, and with fasting, and with weeping, and with mourning. - Joel 2:12.

Abstract. I prove two expansions of infinite series and some integral representations for Bessel function of the first kind.

1. Introduction

In this paper, I demonstrated the expansions of infinite power series for Bessel function of the first kind:

$$\frac{2^v\sqrt{\pi}}{z^v}J_v(z)=\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(v+k+1)(2k)!}z^{2k}$$

and

$$J_v(z)=\frac{z^v}{2^v\Gamma(v+1)}{}_1F_2\left(v+\frac{1}{2};\frac{1}{2},v+1;-\frac{z^2}{4}\right).$$

I, too, prove the news integral representations for Bessel function of the first kind:

$$2\int_0^1\frac{t^{2v}\cos\left(z\sqrt{1-t^2}\right)}{\sqrt{1-t^2}}\mathrm{dt}=\frac{2^v\sqrt{\pi}}{z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z),$$

$$\int_0^{\pi/2}\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi=\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{4\Gamma(v+1)}-\frac{2^v\sqrt{\pi}}{4z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z),$$

$$\int_0^{\pi/2}\cos(z\sin\varphi)\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi$$
$$=-\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{8\Gamma(v+1)}+\frac{2^v\sqrt{\pi}}{4z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z)-\frac{\sqrt{\pi}}{8z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(2z),$$

$$\int_0^{\pi/2}\cos^2(z\sin\varphi)\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi$$
$$=\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{8\Gamma(v+1)}-\frac{2^v3\sqrt{\pi}}{16\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z)$$
$$+\frac{\sqrt{\pi}}{8\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(2z)-\frac{2^v\sqrt{\pi}}{16\cdot 3^v\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(3z)$$

and

$$2\int_0^{\pi/2}\cos(z\cos\varphi)\cos^{2v}\varphi d\varphi=\frac{2^v\sqrt{\pi}}{z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z).$$

2. Acceleration of Infinite Serie for Bessel Function of the first kind

Lemma 1. *For $z\in\mathbb{R}$, then*

$$e^{-z}=\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}\left[1-\frac{z}{2k+1}\right],$$

where e^z denotes the exponential function and $k!$ denotes the factorial function.

Proof. I easily know that

$$e^{-z}=\sum_{k=0}^{\infty}\frac{\cos\left(\frac{\pi k}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}z^{k/2}. \tag{1}$$

On the other hand, I calculate

$$\cos\left(\frac{\pi k}{2}\right)=\begin{cases}1, k=0,4,8,12,16,20,...\\0, k=1,3,5,7,9,11,13,15,17,19,...\\-1, k=2,6,10,14,18,...\end{cases} \tag{2}$$

From (1) and (2), it follows that

$$e^{-z}=\sum_{k=0}^{\infty}\frac{\cos\left(\frac{\pi k}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)}z^{k/2}=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+1)}-\sum_{k=0}^{\infty}\frac{z^{2k+1}}{\Gamma(2k+2)}$$
$$=\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}-\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}=\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}\left[1-\frac{z}{2k+1}\right],$$

which is the desired result. □

Theorem 2. *For* $\mathrm{Re}(v)>-\frac{1}{2}$ *and* $z\in\mathbb{R}$*, then*

$$\frac{2^v\sqrt{\pi}}{z^v}J_v(z)=\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(v+k+1)(2k)!}z^{2k},$$

where $J_v(z)$ denotes the Bessel function of the first kind, $\Gamma(v)$ denotes the gamma function and $k!$ denotes the factorial function.

Proof. Let $z\to izt$ in Lemma 1, therefore,

$$e^{-izt}=\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!}\left[t^{2k}-iz\frac{t^{2k+1}}{2k+1}\right]. \tag{3}$$

I multiply the Eq. (3) by $(1-t^2)^{v-\frac{1}{2}}$ and integrate from -1 at 1 with respect to t, thus

$$\int_{-1}^{1}(1-t^2)^{v-\frac{1}{2}}e^{-izt}\mathrm{dt}=\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k)!}\left[\int_{-1}^{1}(1-t^2)^{v-\frac{1}{2}}t^{2k}\mathrm{dt}\right.$$
$$\left.-\frac{iz}{2k+1}\int_{-1}^{1}(1-t^2)^{v-\frac{1}{2}}t^{2k+1}\mathrm{dt}\right]=\Gamma\left(v+\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(v+k+1)(2k)!}z^{2k}. \tag{4}$$

But, I know [1, p. 48] that

$$\int_{-1}^{1}(1-t^2)^{v-\frac{1}{2}}e^{-izt}\mathrm{dt}=\frac{2^v\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{z^v}J_v(z). \tag{5}$$

From (4) and (5), it follows that

$$\frac{2^v\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{z^v}J_v(z)=\Gamma\left(v+\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(v+k+1)(2k)!}z^{2k}.$$

Eliminate $\Gamma\left(v+\frac{1}{2}\right)$ in both the members of the equation above; this completes the proof. □

3. CONSEQUENCES OF INFINITE SERIE FOR BESSEL FUNCTION OF THE FIRST KIND

3.1. Integral Representations for Bessel Function of the first kind.

Theorem 3. *For* $\mathrm{Re}(v) > -\frac{1}{2}$ *and* $z \in \mathbb{R}$*, then*

$$(i)\frac{2^v\sqrt{\pi}}{z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) = 2\int_0^{\pi/2}\cos(z\sin\varphi)\cos^{2v}\varphi\, d\varphi,$$
$$(ii)\frac{2^v\sqrt{\pi}}{z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) = 2\int_0^{\pi/2}\cos(z\cos\varphi)\sin^{2v}\varphi\, d\varphi,$$
$$(iii)\frac{2^v\sqrt{\pi}}{z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) = 2\int_0^{1}\cos(zt)(1-t^2)^{v-\frac{1}{2}}\,\mathrm{dt},$$
$$(iv)\frac{2^v\sqrt{\pi}}{z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) = 2\int_0^{1}\frac{t^{2v}\cos\left(z\sqrt{1-t^2}\right)}{\sqrt{1-t^2}}\,\mathrm{dt},$$

where $\Gamma(v)$ *denotes the gamma function,* $J_v(z)$ *denotes the Bessel function of the first kind,* $\cos(\varphi)$ *denotes the cosine function and* $\sin(\varphi)$ *denotes the sine function.*

Proof. Multiply both members of the Theorem 2 by $\Gamma\left(v+\frac{1}{2}\right)$

$$\frac{2^v\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{z^v}J_v(z) = \Gamma\left(v+\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(v+k+1)(2k)!}z^{2k}. \tag{6}$$

I know [2, page 899, formula 8.384.1] that

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = B(b,a), \tag{7}$$

for $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$.

Setting $a = k+\frac{1}{2}$ and $b = v+\frac{1}{2}$ in (7), I encounter

$$B\left(k+\frac{1}{2}, v+\frac{1}{2}\right) = \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(v+\frac{1}{2}\right)}{\Gamma(v+k+1)} = B\left(v+\frac{1}{2}, k+\frac{1}{2}\right). \tag{8}$$

From (6) and (8), I obtain

$$\frac{2^v\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{z^v}J_v(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}B\left(k+\frac{1}{2}, v+\frac{1}{2}\right)z^{2k} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}B\left(v+\frac{1}{2}, k+\frac{1}{2}\right)z^{2k}. \tag{9}$$

I know [2, page 898, formula 8.380.2] that

$$B(a,b) = 2\int_0^{\pi/2}\sin^{2a-1}\varphi\cos^{2b-1}\varphi\, d\varphi. \tag{10}$$

From (9) and (10), I get

$$\frac{2^v\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{z^v}J_v(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left[2\int_0^{\pi/2}\sin^{2k}\varphi\cos^{2v}\varphi\, d\varphi\right]z^{2k}$$
$$=2\int_0^{\pi/2}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\sin^{2k}\varphi\, z^{2k}\right]\cos^{2v}\varphi\, d\varphi = 2\int_0^{\pi/2}\cos(z\sin\varphi)\cos^{2v}\varphi\, d\varphi$$

and

$$\frac{2^v\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{z^v}J_v(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left[2\int_0^{\pi/2}\sin^{2v}\varphi\cos^{2k}\varphi\, d\varphi\right]z^{2k}$$
$$=2\int_0^{\pi/2}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\cos^{2k}\varphi\, z^{2k}\right]\sin^{2v}\varphi\, d\varphi = 2\int_0^{\pi/2}\cos(z\cos\varphi)\sin^{2v}\varphi\, d\varphi.$$

I know [2, page 898, formula 8.380.1] that

$$B(a,b) = 2\int_0^{1}t^{2a-1}(1-t^2)^{b-1}\,\mathrm{dt}. \tag{11}$$

From (9) and (11), I find

$$\frac{2^v\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{z^v}J_v(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left[2\int_0^1 t^{2k}\left(1-t^2\right)^{v-\frac{1}{2}}\mathrm{dt}\right]z^{2k}$$

$$=2\int_0^1\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k}z^{2k}\right]\left(1-t^2\right)^{v-\frac{1}{2}}\mathrm{dt}=2\int_0^1\cos(zt)\left(1-t^2\right)^{v-\frac{1}{2}}\mathrm{dt}$$

and

$$\frac{2^v\Gamma\left(v+\frac{1}{2}\right)\sqrt{\pi}}{z^v}J_v(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left[2\int_0^1 t^{2v}\left(1-t^2\right)^{k-\frac{1}{2}}\mathrm{dt}\right]z^{2k}$$

$$=2\int_0^1\frac{t^{2v}}{\sqrt{1-t^2}}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}(1-t^2)^k z^{2k}\right]\mathrm{dt}=2\int_0^1\frac{t^{2v}\cos\left(z\sqrt{1-t^2}\right)}{\sqrt{1-t^2}}\mathrm{dt}.$$

This completes the proof. □

3.2. Integral Representations for Sine Function.

Corollary 4. *For* $z\in\mathbb{R}$, *then*

$$(i)\frac{\sin(z)}{z}=\int_0^{\pi/2}\cos(z\sin\varphi)\cos\varphi\, d\varphi,$$
$$(ii)\frac{\sin(z)}{z}=\int_0^{\pi/2}\cos(z\cos\varphi)\sin\varphi\, d\varphi,$$
$$(iii)\frac{\sin(z)}{z}=\int_0^1\cos(zt)\,\mathrm{dt},$$
$$(iv)\frac{\sin(z)}{z}=\int_0^1\frac{t\cos\left(z\sqrt{1-t^2}\right)}{\sqrt{1-t^2}}\mathrm{dt},$$

where $\cos(z)$ *denotes the cosine function and* $\sin(z)$ *denotes the sine function.*

Proof. I know [2, page 914, formula 8.464.1] that

$$\sqrt{\frac{2\pi}{z}}J_{1/2}(z)=\frac{2\sin(z)}{z}. \tag{12}$$

On the other hand, I put v into $1/2$ in Theorem 3, Eqs. (i), (ii), (iii), (iv), and obtain

$$\begin{aligned}\sqrt{\frac{2\pi}{z}}J_{1/2}(z)&=2\int_0^{\pi/2}\cos(z\sin\varphi)\cos\varphi\, d\varphi,\\ \sqrt{\frac{2\pi}{z}}J_{1/2}(z)&=2\int_0^{\pi/2}\cos(z\cos\varphi)\sin\varphi\, d\varphi,\\ \sqrt{\frac{2\pi}{z}}J_{1/2}(z)&=2\int_0^1\cos(zt)\,\mathrm{dt},\\ \sqrt{\frac{2\pi}{z}}J_{1/2}(z)&=2\int_0^1\frac{t\cos\left(z\sqrt{1-t^2}\right)}{\sqrt{1-t^2}}\mathrm{dt}.\end{aligned} \tag{13}$$

I substitute (12) into (13) and eliminate 2 in both members; this concludes the proof. □

3.3. Integral Representations Arising out of a Recurrence Relation.

Theorem 5. *For* $\mathrm{Re}(v)>-\frac{1}{2}$ *and* $n\in\mathbb{N}_{>2}$, *then*

$$(i)\int_0^{\pi/2}\cos(\sin\varphi)\cos[(n-1)\sin\varphi]\cos^{2v}\varphi\, d\varphi=2^{v-1}\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)\left[\frac{J_v(n)}{n^v}+\frac{J_v(n-2)}{(n-2)^v}\right],$$
$$(ii)\int_0^{\pi/2}\cos(\cos\varphi)\cos[(n-1)\cos\varphi]\sin^{2v}\varphi\, d\varphi=2^{v-1}\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)\left[\frac{J_v(n)}{n^v}+\frac{J_v(n-2)}{(n-2)^v}\right],$$

where $\Gamma(v)$ *denotes the gamma function,* $J_v(z)$ *denotes the Bessel function of the first kind,* $\cos(\varphi)$ *denotes the cosine function and* $\sin(\varphi)$ *denotes the sine function.*

Proof. It is well know that

$$\cos n\theta = 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta. \tag{14}$$

Let $z\to n$ in Theorem Eqs. (i) and (ii), therefore

$$\begin{aligned} (i)\frac{2^v\sqrt{\pi}}{n^v}\Gamma\left(v+\frac{1}{2}\right)J_v(n) &= 2\int_0^{\pi/2}\cos(n\sin\varphi)\cos^{2v}\varphi d\varphi, \\ (ii)\frac{2^v\sqrt{\pi}}{n^v}\Gamma\left(v+\frac{1}{2}\right)J_v(n) &= 2\int_0^{\pi/2}\cos(n\cos\varphi)\sin^{2v}\varphi d\varphi. \end{aligned} \tag{15}$$

Now, I set $\theta\to\sin\varphi$ in (14), multiply by $2\cos^{2v}\varphi$ and integrate from 0 at $\pi/2$ with respect to φ, thus

$$\begin{aligned} 2\int_0^{\pi/2}\cos(n\sin\varphi)\cos^{2v}\varphi d\varphi &= 2\int_0^{\pi/2}\cos(\sin\varphi)\cos[(n-1)\sin\varphi]\cos^{2v}\varphi d\varphi \\ &\quad -2\int_0^{\pi/2}\cos[(n-2)\sin\varphi]\cos^{2v}\varphi d\varphi, \end{aligned} \tag{16}$$

consequently, from $(15.i)$ and (16), I find

$$\int_0^{\pi/2}\cos(\sin\varphi)\cos[(n-1)\sin\varphi]\cos^{2v}\varphi d\varphi = 2^{v-1}\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)\left[\frac{J_v(n)}{n^v}+\frac{J_v(n-2)}{(n-2)^v}\right].$$

Again, I set $\theta\to\cos\varphi$ in (14), multiply by $2\sin^{2v}\varphi$ and integrate from 0 at $\pi/2$ with respect to φ, thus

$$\begin{aligned} 2\int_0^{\pi/2}\cos(n\cos\varphi)\sin^{2v}\varphi d\varphi &= 2\int_0^{\pi/2}\cos(\cos\varphi)\cos[(n-1)\cos\varphi]\sin^{2v}\varphi d\varphi \\ &\quad -2\int_0^{\pi/2}\cos[(n-2)\cos\varphi]\sin^{2v}\varphi d\varphi, \end{aligned} \tag{17}$$

consequently, from $(15.ii)$ and (17), I encounter

$$\int_0^{\pi/2}\cos(\cos\varphi)\cos[(n-1)\cos\varphi]\sin^{2v}\varphi d\varphi = 2^{v-1}\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)\left[\frac{J_v(n)}{n^v}+\frac{J_v(n-2)}{(n-2)^v}\right].$$

This concludes the proof. □

3.4. Special Cases.

Theorem 6. *For* $\mathrm{Re}(v) > -\frac{1}{2}$ *and* $z\in\mathbb{R}$*, then*

$$\int_0^{\pi/2}\cos^2(\sin\varphi)\cos^{2v}\varphi d\varphi = \frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{4}J_v(2) + \frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{4\Gamma(v+1)}$$

and

$$\int_0^{\pi/2}\cos(z\sin\varphi)\cos(2\varphi)d\varphi = \frac{\pi}{z}J_1(z) - \frac{\pi}{2}J_0(z),$$

where $\Gamma(v)$ *denotes the gamma function,* $J_v(z)$ *denotes the Bessel function of the first kind,* $\cos(\varphi)$ *denotes the cosine function and* $\sin(\varphi)$ *denotes the sine function.*

Proof. In [2, page 34, formula 1.335.1], I have

$$\cos(2x) = 2\cos^2 x - 1. \tag{18}$$

Change x into $\sin\varphi$ in (18), multiply by $2\cos^{2v}\varphi$ and integrate from 0 at $\pi/2$ with respect to φ, thus

$$2\int_0^{\pi/2}\cos(2\sin\varphi)\cos^{2v}\varphi d\varphi=4\int_0^{\pi/2}\cos^2(\sin\varphi)\cos^{2v}\varphi d\varphi-2\int_0^{\pi/2}\cos^{2v}\varphi d\varphi, \tag{19}$$

therefrom, I easily calculate that

$$\begin{aligned}&\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)J_v(2)=4\int_0^{\pi/2}\cos^2(\sin\varphi)\cos^{2v}\varphi d\varphi-\sqrt{\pi}\frac{\Gamma\left(v+\frac{1}{2}\right)}{\Gamma(v+1)}\\ &\Rightarrow\int_0^{\pi/2}\cos^2(\sin\varphi)\cos^{2v}\varphi d\varphi=\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{4}J_v(2)+\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{4\Gamma(v+1)}.\end{aligned} \tag{20}$$

Let $x=\varphi$ in (18), multiply by $2\cos(z\sin\varphi)$ and integrate from 0 at $\pi/2$ with respect to φ, thus

$$2\int_0^{\pi/2}\cos(z\sin\varphi)\cos(2\varphi)d\varphi=4\int_0^{\pi/2}\cos(z\sin\varphi)\cos^2\varphi d\varphi-2\int_0^{\pi/2}\cos(z\sin\varphi)d\varphi, \tag{21}$$

thereafter, I easily calculate

$$\int_0^{\pi/2}\cos(z\sin\varphi)\cos(2\varphi)d\varphi=\frac{\pi}{z}J_1(z)-\frac{\pi}{2}J_0(z),$$

which is the desired result. □

3.5. More Integral Representations and Power Series.

Theorem 7. *For* $\mathrm{Re}(v)>-\frac{1}{2}$ *and* $z\in\mathbb{R}$, *then*

$$\int_0^{\pi/2}\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi=\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{4\Gamma(v+1)}-\frac{2^v\sqrt{\pi}}{4z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z),$$

$$\begin{aligned}&\int_0^{\pi/2}\cos(z\sin\varphi)\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi\\ &=-\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{8\Gamma(v+1)}+\frac{2^v\sqrt{\pi}}{4z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z)-\frac{\sqrt{\pi}}{8z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(2z),\end{aligned}$$

$$\begin{aligned}&\int_0^{\pi/2}\cos^2(z\sin\varphi)\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi\\ &=\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{8\Gamma(v+1)}-\frac{2^v3\sqrt{\pi}}{16\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z)\\ &+\frac{\sqrt{\pi}}{8\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(2z)-\frac{2^v\sqrt{\pi}}{16\cdot 3^v\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(3z),\end{aligned}$$

$$2\int_0^{\pi/2}\cos(z\cos\varphi)\cos^{2v}\varphi d\varphi=\frac{2^v\sqrt{\pi}}{z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z)$$

and

$$J_v(z)=\frac{z^v}{2^v\Gamma(v+1)}{}_1F_2\left(v+\frac{1}{2};\frac{1}{2},v+1;-\frac{z^2}{4}\right),$$

where $\Gamma(v)$ *denotes the gamma function,* $J_v(z)$ *denotes the Bessel function of the first kind,* $\cos(\varphi)$ *denotes the cosine function,* $\sin(\varphi)$ *denotes the sine function and* ${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$ *denotes the generalized hypergeometric series.*

Proof. I know that

$$\sin^2\left(\frac{x}{2}\right)=\frac{1}{2}[1-\cos(x)], \tag{22}$$

$$\cos(x)\sin^2\left(\frac{x}{2}\right)=\frac{1}{4}[-1+2\cos(x)-\cos(2x)] \tag{23}$$

and

$$\cos^2(x)\sin^2\left(\frac{x}{2}\right)=\frac{1}{8}[2-3\cos(x)+2\cos(2x)-\cos(3x)]. \tag{24}$$

Let $x = z\sin\varphi$ in (22)-(24), multiplity by $2\cos^{2v}\varphi$ and integrate from 0 at $\pi/2$ with respect to φ, thus

$$2\int_0^{\pi/2}\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi = \int_0^{\pi/2}\cos^{2v}\varphi d\varphi - \int_0^{\pi/2}\cos(z\sin\varphi)\cos^{2v}\varphi d\varphi, \tag{25}$$

$$\begin{aligned} &4\int_0^{\pi/2}\cos(z\sin\varphi)\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi \\ =&-\int_0^{\pi/2}\cos^{2v}\varphi d\varphi + 2\int_0^{\pi/2}\cos(z\sin\varphi)\cos^{2v}\varphi d\varphi - \int_0^{\pi/2}\cos(2z\sin\varphi)\cos^{2v}\varphi d\varphi \end{aligned} \tag{26}$$

and

$$\begin{aligned} &8\int_0^{\pi/2}\cos^2(z\sin\varphi)\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi \\ =&2\int_0^{\pi/2}\cos^{2v}\varphi d\varphi - 3\int_0^{\pi/2}\cos(z\sin\varphi)\cos^{2v}\varphi d\varphi \\ &+2\int_0^{\pi/2}\cos(2z\sin\varphi)\cos^{2v}\varphi d\varphi - \int_0^{\pi/2}\cos(3z\sin\varphi)\cos^{2v}\varphi d\varphi \end{aligned} \tag{27}$$

From Theorem 3 and (25) at (27), I obtain easily

$$\int_0^{\pi/2}\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi = \frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{4\Gamma(v+1)} - \frac{2^v\sqrt{\pi}}{4z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z), \tag{28}$$

$$\begin{aligned} &\int_0^{\pi/2}\cos(z\sin\varphi)\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi \\ =&-\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{8\Gamma(v+1)} + \frac{2^v\sqrt{\pi}}{4z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) - \frac{\sqrt{\pi}}{8z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(2z) \end{aligned} \tag{29}$$

and

$$\begin{aligned} &8\int_0^{\pi/2}\cos^2(z\sin\varphi)\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi \\ =&\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{\Gamma(v+1)} - \frac{2^v3\sqrt{\pi}}{2\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) \\ &+\frac{\sqrt{\pi}}{z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(2z) - \frac{2^v\sqrt{\pi}}{2\cdot 3^v\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(3z) \\ \Rightarrow&\int_0^{\pi/2}\cos^2(z\sin\varphi)\sin^2\left(\frac{z\sin\varphi}{2}\right)\cos^{2v}\varphi d\varphi \\ =&\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{8\Gamma(v+1)} - \frac{2^v3\sqrt{\pi}}{16\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) \\ &+\frac{\sqrt{\pi}}{8\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(2z) - \frac{2^v\sqrt{\pi}}{16\cdot 3^v\cdot z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(3z) \end{aligned} \tag{30}$$

The expansion of $\sin^2\left(\frac{z\sin\varphi}{2}\right)$ in the left hand side of (28) give me

$$\begin{aligned} &\frac{1}{2}\int_0^{\pi/2}[1-\cos(z\cos\varphi)]\cos^{2v}\varphi d\varphi = \frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{4\Gamma(v+1)} - \frac{2^v\sqrt{\pi}}{4z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) \\ \Rightarrow&\frac{1}{2}\int_0^{\pi/2}\cos^{2v}\varphi d\varphi - \frac{1}{2}\int_0^{\pi/2}\cos(z\cos\varphi)\cos^{2v}\varphi d\varphi = \frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{4\Gamma(v+1)} - \frac{2^v\sqrt{\pi}}{4z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) \\ \Rightarrow&\frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{2\Gamma(v+1)} - \int_0^{\pi/2}\cos(z\cos\varphi)\cos^{2v}\varphi d\varphi = \frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{2\Gamma(v+1)} - \frac{2^v\sqrt{\pi}}{2z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) \\ \Rightarrow&2\int_0^{\pi/2}\cos(z\cos\varphi)\cos^{2v}\varphi d\varphi = \frac{2^v\sqrt{\pi}}{z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z). \end{aligned} \tag{31}$$

On the other hand, I calculate that

$$\int_0^{\pi/2} \cos(z\cos\varphi)\cos^{2v}\varphi d\varphi = \frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{2\Gamma(v+1)} {}_1F_2\left(v+\frac{1}{2};\frac{1}{2},v+1;-\frac{z^2}{4}\right). \tag{32}$$

From (31) and (32), it follows that

$$\frac{2^v\sqrt{\pi}}{z^v}\Gamma\left(v+\frac{1}{2}\right)J_v(z) = \frac{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}{\Gamma(v+1)} {}_1F_2\left(v+\frac{1}{2},\frac{1}{2};v+1;-\frac{z^2}{4}\right)$$
$$\Rightarrow J_v(z) = \frac{z^v}{2^v\Gamma(v+1)} {}_1F_2\left(v+\frac{1}{2};\frac{1}{2},v+1;-\frac{z^2}{4}\right),$$

and this completes the proof. □

Exercise 1. I left to reader the following questions: a) Prove that

$$J_v(z) = \frac{-v\sqrt{z}}{2^{v-1}\Gamma(v+1)} s_{v-\frac{3}{2},v+\frac{1}{2}}(z),$$

valid for $z \neq 0$ and $v > \frac{1}{2}$, where $s_{\mu,\nu}(z)$ denotes the first Lommel's function.

b) Prove that

$$J_v(z) = \frac{-\sqrt{z}}{2^{v+1}\Gamma(v+1)}\left[4vS_{v-\frac{3}{2},v+\frac{1}{2}}(z) + v2^{v+\frac{1}{2}}\sqrt{\pi}\Gamma(v)Y_{v+\frac{1}{2}}(z)\right],$$

valid for $z \neq 0$ and $v > \frac{1}{2}$, where $S_{\mu,\nu}(z)$ denotes the second Lommel's function.

c) Prove that

$$J_v(z) = \frac{z^v\sqrt{\pi}}{2^v\Gamma\left(v+\frac{1}{2}\right)} G^{1,1}_{-,2}\left(\begin{matrix}\frac{1}{2}-v; -\\ 0; \frac{1}{2}, -v\end{matrix}\middle| \frac{z^2}{4}\right),$$

valid for z and $v \in \mathbb{R}$, where $G^{m,n}_{p,q}\left(\begin{matrix}a_1,\dots,a_n; a_{n+1},\dots,a_p\\ b_1,\dots,b_m; b_{m+1},\dots,b_q\end{matrix}\middle| z\right)$ denotes the Meijer G function.

Hint. Use the differential equation for Bessel function of the first kind.

REFERENCES

[1] Watson, G. N., *A Treatise on the Theory of the Bessel Functions*, Merchant Books, 2008.

[2] Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series and Products*, Academic Press, 2000.