About the replacement of metric tensor

Sangwha-Yi

Department of Math , Taejon University 300-716

ABSTRACT

In the general relativity theory, study the replacement of the metric tensor in the Einstein gravity field equation.

PACS Number:**04,04.90+e**

Key words:The general relativity theory,

The metric tensor,

The co-variant differentiation

e-mail address:sangwha1**@nate.com**

Tel:051-624-3953

1.Introduction

In the general relativity, we study the replacement of the metric tensor.

The gravity field equation is

$$R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} R + \Lambda g_{\mu\upsilon} = -\frac{8\pi G}{c^4} T_{\mu\upsilon} \tag{1}$$

In this time, we study the replacement of the cosmological term $\Lambda g_{\mu\upsilon}$.

We look the co-invariant differentiation of scalars $g_{;\mu} \cdot h_{;\upsilon}$.

In this time, $g \neq \det(g_{\mu\upsilon})$, g, h are scalars.

2.The replacement of the metric tensor

$$g_{;\mu} = \frac{\partial g}{\partial x^{\mu}} \quad , \quad h_{;\upsilon} = \frac{\partial h}{\partial x^{\upsilon}} \tag{2}$$

The co-invariant differentiation of the metric tensor $g_{\mu\upsilon}$ is

$$g_{\mu\upsilon;\lambda} = 0 \tag{3}$$

Co-invariant differentiation of scalars $g_{;\mu} \cdot h_{;\upsilon}$. is

$$\begin{aligned}(g_{;\mu} \cdot h_{;\upsilon})_{;\alpha} &= g_{;\mu;\alpha} \cdot h_{;\upsilon} + g_{;\mu} \cdot h_{;\upsilon;\alpha} \\ &= [\frac{\partial}{\partial x^{\alpha}}(\frac{\partial g}{\partial x^{\mu}}) - \Gamma^{\lambda}{}_{\alpha\mu}(\frac{\partial g}{\partial x^{\lambda}})]\frac{\partial h}{\partial x^{\upsilon}} + [\frac{\partial}{\partial x^{\alpha}}(\frac{\partial h}{\partial x^{\upsilon}}) - \Gamma^{\lambda}{}_{\alpha\upsilon}(\frac{\partial h}{\partial x^{\lambda}})]\frac{\partial g}{\partial x^{\mu}} \\ &= \frac{\partial}{\partial x^{\alpha}}(\frac{\partial g}{\partial x^{\mu}} \cdot \frac{\partial h}{\partial x^{\upsilon}}) - \Gamma^{\lambda}{}_{\alpha\mu}\frac{\partial g}{\partial x^{\lambda}}\frac{\partial h}{\partial x^{\upsilon}} - \Gamma^{\lambda}{}_{\alpha\upsilon}\frac{\partial g}{\partial x^{\lambda}}\frac{\partial h}{\partial x^{\mu}}\end{aligned} \tag{4}$$

$$g_{\mu\upsilon;\alpha} = \frac{\partial g_{\mu\upsilon}}{\partial x^{\alpha}} - \Gamma^{\lambda}{}_{\alpha\mu} g_{\lambda\upsilon} - \Gamma^{\lambda}{}_{\alpha\upsilon} g_{\lambda\mu} \tag{5}$$

In this time, if suppose $g_{;\mu} \cdot h_{;\upsilon} = g_{\mu\upsilon}, g^{;\mu} \cdot h^{;\upsilon} = g^{\mu\upsilon}$ (6)

$$\begin{aligned}\Gamma^{\lambda}{}_{\mu\upsilon} &= \frac{1}{2} g^{\lambda\varepsilon}(\frac{\partial g_{\mu\varepsilon}}{\partial x^{\upsilon}} + \frac{\partial g_{\upsilon\varepsilon}}{\partial x^{\mu}} - \frac{\partial g_{\mu\upsilon}}{\partial x^{\varepsilon}}) \\ &= \frac{1}{2}\frac{\partial g}{\partial x_{\lambda}}\frac{\partial h}{\partial x_{\varepsilon}}[\frac{\partial}{\partial x^{\upsilon}}(\frac{\partial g}{\partial x^{\mu}}\frac{\partial h}{\partial x^{\varepsilon}}) + \frac{\partial}{\partial x^{\mu}}(\frac{\partial g}{\partial x^{\upsilon}}\frac{\partial h}{\partial x^{\varepsilon}}) - \frac{\partial}{\partial x^{\varepsilon}}(\frac{\partial g}{\partial x^{\mu}}\frac{\partial h}{\partial x^{\upsilon}})]\end{aligned} \tag{7}$$

Therefore, Eq(4) is

$$g_{\mu\upsilon;\alpha} = (g_{;\mu}\cdot h_{;\upsilon})_{;\alpha} = g_{;\mu;\alpha}\cdot h_{;\upsilon} + g_{;\mu}\cdot h_{;\upsilon;\alpha}$$

$$\begin{aligned}
&= \frac{\partial}{\partial x^{\alpha}}(\frac{\partial g}{\partial x^{\mu}}\cdot\frac{\partial h}{\partial x^{\upsilon}}) - \Gamma^{\lambda}{}_{\alpha\mu}\frac{\partial g}{\partial x^{\lambda}}\frac{\partial h}{\partial x^{\upsilon}} - \Gamma^{\lambda}{}_{\alpha\upsilon}\frac{\partial g}{\partial x^{\lambda}}\frac{\partial h}{\partial x^{\mu}} \\
&= \frac{\partial}{\partial x^{\alpha}}(\frac{\partial g}{\partial x^{\mu}}\cdot\frac{\partial h}{\partial x^{\upsilon}}) - \frac{1}{2}g^{\lambda\varepsilon}(\frac{\partial g_{\varepsilon\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\varepsilon\mu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\mu}}{\partial x^{\varepsilon}})\frac{\partial g}{\partial x^{\lambda}}\frac{\partial h}{\partial x^{\upsilon}} \\
&\qquad - \frac{1}{2}g^{\lambda\varepsilon}(\frac{\partial g_{\alpha\varepsilon}}{\partial x^{\upsilon}} + \frac{\partial g_{\upsilon\varepsilon}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\upsilon}}{\partial x^{\varepsilon}})\frac{\partial g}{\partial x^{\lambda}}\frac{\partial h}{\partial x^{\mu}} \\
&= \frac{\partial}{\partial x^{\alpha}}(\frac{\partial g}{\partial x^{\mu}}\cdot\frac{\partial h}{\partial x^{\upsilon}}) - \frac{1}{2}(\frac{\partial g}{\partial x_{\lambda}}\frac{\partial h}{\partial x_{\varepsilon}})[\frac{\partial}{\partial x^{\mu}}(\frac{\partial g}{\partial x^{\varepsilon}}\frac{\partial h}{\partial x^{\alpha}}) \\
&\qquad + \frac{\partial}{\partial x^{\alpha}}(\frac{\partial g}{\partial x^{\varepsilon}}\frac{\partial h}{\partial x^{\mu}}) - \frac{\partial}{\partial x^{\varepsilon}}(\frac{\partial g}{\partial x^{\alpha}}\frac{\partial h}{\partial x^{\mu}})]\frac{\partial g}{\partial x^{\lambda}}\frac{\partial h}{\partial x^{\upsilon}} \\
&\quad - \frac{1}{2}(\frac{\partial g}{\partial x_{\lambda}}\frac{\partial h}{\partial x_{\varepsilon}})[\frac{\partial}{\partial x^{\upsilon}}(\frac{\partial g}{\partial x^{\alpha}}\frac{\partial h}{\partial x^{\varepsilon}}) \\
&\qquad + \frac{\partial}{\partial x^{\alpha}}(\frac{\partial g}{\partial x^{\upsilon}}\frac{\partial h}{\partial x^{\varepsilon}}) - \frac{\partial}{\partial x^{\varepsilon}}(\frac{\partial g}{\partial x^{\alpha}}\frac{\partial h}{\partial x^{\upsilon}})]\frac{\partial g}{\partial x^{\lambda}}\frac{\partial h}{\partial x^{\mu}} \\
&= \frac{\partial^2 g}{\partial x^{\alpha}\partial x^{\mu}}\frac{\partial h}{\partial x^{\upsilon}} + \frac{\partial g}{\partial x^{\mu}}\frac{\partial^2 h}{\partial x^{\alpha}\partial x^{\upsilon}} \\
&- \frac{1}{2}(\frac{\partial g}{\partial x_{\lambda}}\frac{\partial h}{\partial x_{\varepsilon}})(\frac{\partial g}{\partial x^{\lambda}}\frac{\partial h}{\partial x^{\upsilon}})[\frac{\partial^2 g}{\partial x^{\mu}\partial x^{\varepsilon}}\frac{\partial h}{\partial x^{\alpha}} + \frac{\partial g}{\partial x^{\varepsilon}}\frac{\partial^2 h}{\partial x^{\mu}\partial x^{\alpha}} \\
&+ \frac{\partial^2 g}{\partial x^{\alpha}\partial x^{\varepsilon}}\frac{\partial h}{\partial x^{\mu}} + \frac{\partial g}{\partial x^{\varepsilon}}\frac{\partial^2 h}{\partial x^{\alpha}\partial x^{\mu}} - \frac{\partial^2 g}{\partial x^{\varepsilon}\partial x^{\alpha}}\frac{\partial h}{\partial x^{\mu}} - \frac{\partial g}{\partial x^{\alpha}}\frac{\partial^2 h}{\partial x^{\varepsilon}\partial x^{\mu}}] \\
&\quad - \frac{1}{2}(\frac{\partial g}{\partial x_{\lambda}}\frac{\partial h}{\partial x_{\varepsilon}})(\frac{\partial g}{\partial x^{\lambda}}\frac{\partial h}{\partial x^{\mu}})[\frac{\partial^2 g}{\partial x^{\upsilon}\partial x^{\alpha}}\frac{\partial h}{\partial x^{\varepsilon}} + \frac{\partial g}{\partial x^{\alpha}}\frac{\partial^2 h}{\partial x^{\upsilon}\partial x^{\varepsilon}} \\
&+ \frac{\partial^2 g}{\partial x^{\alpha}\partial x^{\upsilon}}\frac{\partial h}{\partial x^{\varepsilon}} + \frac{\partial g}{\partial x^{\upsilon}}\frac{\partial^2 h}{\partial x^{\alpha}\partial x^{\varepsilon}} - \frac{\partial^2 g}{\partial x^{\varepsilon}\partial x^{\alpha}}\frac{\partial h}{\partial x^{\upsilon}} - \frac{\partial g}{\partial x^{\alpha}}\frac{\partial^2 h}{\partial x^{\varepsilon}\partial x^{\upsilon}}] \qquad (8)
\end{aligned}$$

In this time,

$$g_{\mu\upsilon} = \frac{\partial g}{\partial x^{\mu}}\frac{\partial h}{\partial x^{\upsilon}} = g_{\upsilon\mu} = \frac{\partial h}{\partial x^{\mu}}\frac{\partial g}{\partial x^{\upsilon}} \rightarrow \quad g \leftrightarrow h \text{ (According to } \mu,\upsilon\text{)}$$

$$g^{\mu\upsilon}g_{\mu\upsilon} = (\frac{\partial g}{\partial x_{\mu}}\frac{\partial h}{\partial x_{\upsilon}})(\frac{\partial g}{\partial x^{\mu}}\frac{\partial h}{\partial x^{\upsilon}}) = 1 \qquad (9)$$

Hence, if we calculates Eq(8),

$$g_{\mu\upsilon;\alpha} = (g_{;\mu}\cdot h_{;\upsilon})_{;\alpha} = g_{;\mu;\alpha}\cdot h_{;\upsilon} + g_{;\mu}\cdot h_{;\upsilon;\alpha}$$

$$= \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\upsilon} + \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\upsilon}$$

$$- \frac{1}{2} (\frac{\partial g}{\partial x_\lambda} \frac{\partial h}{\partial x_\varepsilon})(\frac{\partial g}{\partial x^\lambda} \frac{\partial h}{\partial x^\upsilon}) [2 \frac{\partial g}{\partial x^\varepsilon} \frac{\partial^2 h}{\partial x^\alpha \partial x^\mu}]$$

$$- \frac{1}{2} (\frac{\partial g}{\partial x_\lambda} \frac{\partial h}{\partial x_\varepsilon})(\frac{\partial g}{\partial x^\lambda} \frac{\partial h}{\partial x^\mu}) [2 \frac{\partial^2 g}{\partial x^\upsilon \partial x^\alpha} \frac{\partial h}{\partial x^\varepsilon}]$$

$$= \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\upsilon} + \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\upsilon}$$

$$- (\frac{\partial g}{\partial x_\lambda} \frac{\partial h}{\partial x_\varepsilon})(\frac{\partial g}{\partial x^\lambda} \frac{\partial h}{\partial x^\upsilon}) \frac{\partial h}{\partial x^\varepsilon} \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu}$$

$$- (\frac{\partial h}{\partial x_\lambda} \frac{\partial g}{\partial x_\varepsilon})(\frac{\partial h}{\partial x^\lambda} \frac{\partial g}{\partial x^\mu}) \frac{\partial^2 h}{\partial x^\upsilon \partial x^\alpha} \frac{\partial g}{\partial x^\varepsilon}$$

$$= \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\upsilon} + \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\upsilon}$$

$$- (\frac{\partial g}{\partial x_\lambda} \frac{\partial h}{\partial x_\varepsilon})(\frac{\partial g}{\partial x^\lambda} \frac{\partial h}{\partial x^\varepsilon}) \frac{\partial h}{\partial x^\upsilon} \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu}$$

$$- (\frac{\partial h}{\partial x_\lambda} \frac{\partial g}{\partial x_\varepsilon})(\frac{\partial h}{\partial x^\lambda} \frac{\partial g}{\partial x^\varepsilon}) \frac{\partial^2 h}{\partial x^\upsilon \partial x^\alpha} \frac{\partial g}{\partial x^\mu}$$

$$= \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} \frac{\partial h}{\partial x^\upsilon} + \frac{\partial g}{\partial x^\mu} \frac{\partial^2 h}{\partial x^\alpha \partial x^\upsilon} - \frac{\partial h}{\partial x^\upsilon} \frac{\partial^2 g}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 h}{\partial x^\upsilon \partial x^\alpha} \frac{\partial g}{\partial x^\mu} = 0 \qquad (10)$$

3.The replacement of the matric tensor in the Minkowski spacetime

$$ds^2 = \eta_{\mu\upsilon} dx^\mu dx^\upsilon = \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\upsilon} dx^\mu dx^\upsilon$$

$$= \frac{\partial g}{\partial x^\mu} \frac{\partial h}{\partial x^\upsilon} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\upsilon}{\partial x'^\beta} dx'^\alpha dx'^\beta$$

$$= \frac{\partial g}{\partial x'^\alpha} \frac{\partial h}{\partial x'^\beta} dx'^\alpha dx'^\beta = \eta'_{\alpha\beta} dx'^\alpha dx'^\beta \qquad (11)$$

If we act the Lorentz transformation in Eq(11),

$$cdt = \gamma(cdt' + \frac{v}{c} dx'), dx = \gamma(dx' + vdt'), \ dy = dy', dz = dz'$$

$$\frac{1}{c}\frac{\partial}{\partial t} = \gamma(\frac{1}{c}\frac{\partial}{\partial t'} - \frac{v}{c}\frac{\partial}{\partial x'}), \frac{\partial}{\partial x} = \gamma(\frac{\partial}{\partial x'} - \frac{v}{c}\frac{1}{c}\frac{\partial}{\partial t'}), \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

$$\gamma = 1/\sqrt{1 - v^2/c^2}$$

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{00} c^2 dt^2 + \eta_{11} dx^2 + \eta_{22} dy^2 + \eta_{33} dz^2$$

$$= \frac{\partial g}{c\partial t}\frac{\partial h}{c\partial t}c^2dt^2 + \frac{\partial g}{\partial x}\frac{\partial h}{\partial x}dx^2 + \frac{\partial g}{\partial y}\frac{\partial h}{\partial y}dy^2 + \frac{\partial g}{\partial z}\frac{\partial h}{\partial z}dz^2$$

$$= \gamma(\frac{\partial g}{c\partial t'} - \frac{v}{c}\frac{\partial g}{\partial x'})\gamma(\frac{\partial h}{c\partial t'} - \frac{v}{c}\frac{\partial h}{\partial x'})\gamma^2(cdt' + \frac{v}{c}dx')^2$$

$$+ \gamma(\frac{\partial g}{\partial x'} - \frac{v}{c}\frac{\partial g}{c\partial t'})\gamma(\frac{\partial h}{\partial x'} - \frac{v}{c}\frac{\partial h}{c\partial t'})\gamma^2(dx' + vdt')^2$$

$$+ \frac{\partial g}{\partial y'}\frac{\partial h}{\partial y'}dy'^2 + \frac{\partial g}{\partial z'}\frac{\partial h}{\partial z'}dz'^2 \qquad (12)$$

In this time,

$$ds^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 = \eta'_{00}c^2dt'^2 + \eta'_{11}dx'^2 + \eta'_{22}dy'^2 + \eta'_{33}dz'^2$$

$$= \frac{\partial g}{c\partial t'}\frac{\partial h}{c\partial t'}c^2dt'^2 + \frac{\partial g}{\partial x'}\frac{\partial h}{\partial x'}dx'^2 + \frac{\partial g}{\partial y'}\frac{\partial h}{\partial y'}dy'^2 + \frac{\partial g}{\partial z'}\frac{\partial h}{\partial z'}dz'^2 \qquad (13)$$

Eq(12) is

$$ds^2 = \gamma^2(\frac{\partial g}{c\partial t'}\frac{\partial h}{c\partial t'} + \frac{v^2}{c^2}\frac{\partial g}{\partial x'}\frac{\partial h}{\partial x'})\gamma^2(cdt' + \frac{v}{c}dx')^2$$

$$+ \gamma^2(\frac{\partial g}{\partial x'}\frac{\partial h}{\partial x'} + \frac{v^2}{c^2}\frac{\partial g}{c\partial t'}\frac{\partial h}{c\partial t'})\gamma^2(dx' + vdt')^2 + \frac{\partial g}{\partial y'}\frac{\partial h}{\partial y'}dy'^2 + \frac{\partial g}{\partial z'}\frac{\partial h}{\partial z'}dz'^2$$

$$= \gamma^2(-1 + \frac{v^2}{c^2})\gamma^2(cdt' + \frac{v}{c}dx')^2 + \gamma^2(1 - \frac{v^2}{c^2})\gamma^2(dx' + vdt')^2 + dy'^2 + dz'^2$$

$$= -\gamma^2(c^2dt'^2 + 2dt'vdx' + \frac{v^2}{c^2}dx'^2) + \gamma^2(dx'^2 + 2dx'vdt' + v^2dt'^2) + dy'^2 + dz'^2$$

$$= -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 = \eta'_{00}c^2dt'^2 + \eta'_{11}dx'^2 + \eta'_{22}dy'^2 + \eta'_{33}dz'^2 \qquad (14)$$

4.Conclusion

Therefore, the gravity field equation is

$$R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R + \Lambda g_{\mu\upsilon}$$

$$= R_{\mu\upsilon} - \frac{1}{2}g_{\mu\upsilon}R + \Lambda g_{;\mu}h_{;\upsilon} = -\frac{8\pi G}{c^4}T_{\mu\upsilon}$$

$$g_{;\mu}\cdot h_{;\upsilon} = g_{\mu\upsilon},\, g^{;\mu}\cdot h^{;\upsilon} = g^{\mu\upsilon} \qquad (15)$$

For example, the metric tensor of Schwarzschild solution is

$$g_{00} = -1 + \frac{2GM}{rc^2} = \frac{\partial g}{c\partial t}\frac{\partial h}{c\partial t} \ , \qquad g_{11} = \frac{1}{1 - \frac{2GM}{rc^2}} = \frac{\partial g}{\partial r}\frac{\partial h}{\partial r}$$

$$g_{22} = r^2 = \frac{\partial g}{\partial \theta}\frac{\partial h}{\partial \theta} \quad , \quad g_{33} = r^2 \sin^2\theta = \frac{\partial g}{\partial \phi}\frac{\partial h}{\partial \phi} \tag{16}$$

*Important caution

$$g_{01} = 0 = \frac{\partial g}{c\partial t}\frac{\partial h}{\partial r} \to (\frac{\partial g}{c\partial t} \neq 0, \frac{\partial h}{\partial r} \neq 0 \to g_{00} \neq 0, g_{11} \neq 0)$$

$$g_{10} = 0 = \frac{\partial g}{\partial r}\frac{\partial h}{c\partial t} \to (\frac{\partial g}{\partial r} \neq 0, \frac{\partial h}{c\partial t} \neq 0 \to g_{00} \neq 0, g_{11} \neq 0) \tag{17}$$

The important point is

$$ds^2 = g_{\mu\upsilon} dx^\mu dx^\upsilon = \frac{\partial g}{\partial x^\mu}\frac{\partial h}{\partial x^\upsilon} dx^\mu dx^\upsilon , ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \frac{\partial g}{\partial x^\alpha}\frac{\partial h}{\partial x^\beta} dx^\alpha dx^\beta$$

$$ds^4 = \frac{\partial g}{\partial x^\mu}\frac{\partial h}{\partial x^\upsilon} dx^\mu dx^\upsilon \frac{\partial g}{\partial x^\alpha}\frac{\partial h}{\partial x^\beta} dx^\alpha dx^\beta = g_{\mu\upsilon} g_{\alpha\beta} dx^\mu dx^\upsilon dx^\alpha dx^\beta$$

$$= \frac{\partial g}{\partial x^\mu}\frac{\partial h}{\partial x^\beta} dx^\mu dx^\beta \frac{\partial g}{\partial x^\alpha}\frac{\partial h}{\partial x^\upsilon} dx^\alpha dx^\upsilon = g_{\mu\beta} g_{\alpha\upsilon} dx^\mu dx^\beta dx^\alpha dx^\upsilon$$

$$= \frac{\partial g}{\partial x^\alpha}\frac{\partial h}{\partial x^\upsilon} dx^\alpha dx^\upsilon \frac{\partial g}{\partial x^\mu}\frac{\partial h}{\partial x^\beta} dx^\mu dx^\beta = g_{\alpha\upsilon} g_{\mu\beta} dx^\alpha dx^\upsilon dx^\mu dx^\beta \tag{18}$$

Hence,

$$g_{\mu\upsilon} g_{\alpha\beta} = g_{\mu\beta} g_{\alpha\upsilon} = g_{\alpha\upsilon} g_{\mu\beta} \tag{19}$$

Reference

[1]S.Weinberg, Gravitation and Cosmology (John wiley & Sons,Inc,1972)

[2]A.Miller, Albert Einstein's Special Theory of Relativity(Addison-Wesley Publishing Co., Inc., 1981)

[3]W. Rindler, Special Relativity(2nd ed., Oliver and Boyd, Edinburg, 1966)

[4]P.Bergman, Introduction to the Theory of Relativity(Dover Pub. Co.,Inc., New York,1976),Chapter V