A rigorous procedure for generating a well-ordered Set of Reals without use of Axiom of Choice / Well-Ordering Theorem

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Abstract

Well-ordering of the Reals presents a major challenge in Set theory. Under the standard Zermelo Fraenkel Set theory (ZF) with the Axiom of Choice (ZFC), a well-ordering of the Reals is indeed possible. However the Axiom of Choice (AC) had to be introduced to the original ZF theory which is then shown equivalent to the well-ordering theorem. Despite the result however, no way has still been found of actually constructing a well-ordered Set of Reals. In this paper the author attempts to generate a well ordered Set of Reals without using the AC i.e. under ZF theory itself using the Axiom of the Power Set as the guiding principle.

Introduction:
In this paper, the author attempts a well-ordering of the Reals. Specifically the well-ordering in achieved in the closed interval $[0,1]$. This does not in any way loose generality as the Set of Reals in the open interval $(0,1)$ is equinumerous with the Set of Reals in $(-\infty,\infty)$ via the tangent function $\tan(\pi x-\pi/2)$.

As stated, well-ordering of the Reals presents a major challenge in Set theory. The most popular version of Axiomatic Set theory is the Zermelo Fraenkel Set theory (ZF) with the Axiom of Choice (ZFC). Under this theory a well-ordering of the Reals is indeed possible. However, a new Axiom, the Axiom of Choice (AC) had to be appended to the original axioms of ZF theory for this purpose. The Axiom of Choice is shown to be equivalent to Zorn's lemma and the well-ordering theorem [1]. The well-ordering theorem simply states that every Set can be well-ordered. The Axiom of Choice, though largely accepted by most mathematicians still retains a few detractors as the AC can establish the existence of certain Sets without actually specifying any way of constructing them. Even for those whose unquestionably accept the AC, any proof given using just ZF theory is considered in some sense 'superior' to the same proof given using ZFC. Coming back to the problem at hand, it is to be realized that although a well-ordering of the Reals is possible in ZFC, this remains very much an in-principle concept. No way has still been found of actually constructing a well-ordered Set of Reals [2][3][4].

In this paper, the author makes two bold claims. Not only can a well-ordering of the Reals be achieved in ZF theory i.e. without using AC but further that a mechanical / procedural method is elucidated of actually constructing this elusive Set. The author attempts this feat using the Power Set Axiom. The Axiom of Power Set simply states that for any Set X , there is a Set Y that contains every subset of X . To the author, this is a powerful Axiom, whose functionality to prove results has not been fully appreciated in the Set theoretic community.

Tabular Power Set construction:
We begin by first describing what will henceforth be referred to as the construction of the 'Power Set table'. Construction of the Power Set table is a systematic / procedural method of generating Power Sets for any given Set. The method involves constructing a table in which the columns represent the elements of the Set (whose Power Set is to be constructed) and the rows represent the elements of the Power Set so constructed. In this context, the word construction is to be taken not just as in ordinary English but also a rigorous method of assembling of individual elements to from a Set.

To understand the steps involved, we first take an example of a finite Set say with four elements a , b , c , d . There is a systematic way [5] of constructing the Power Set table in which the elements of the Set are listed in the first row and below each element is written the number 1 or 0 to indicate whether it is or not included in the corresponding subset, for e.g.,

a	b	c	d	$subsets$
0	0	0	0	$\{\}$
1	0	0	0	$\{a\}$
0	1	0	0	$\{b\}$
1	1	0	0	$\{a, b\}$
0	0	1	0	$\{c\}$
1	0	1	0	$\{a, c\}$
0	1	1	0	$\{b, c\}$
1	1	1	0	$\{a, b, c\}$
0	0	0	1	$\{d\}$
1	0	0	1	$\{a, d\}$
0	1	0	1	$\{b, d\}$
1	1	0	1	$\{a, b, d\}$
0	0	1	1	$\{c, d\}$
1	0	1	1	$\{a, c, d\}$
0	1	1	1	$\{b, c, d\}$
1	1	1	1	$\{a, b, c, d\}$

The Set of elements a, b, c, d is given in the first to fourth columns. Below each column, a digit 0 or 1 is placed. The corresponding subset in the fifth column will contain the element if a 1 is indicated and will not contain the element if a 0 is indicated. The fifth column lists all possible subsets based on this exclusion / inclusion $0/1$ rule. It is to be realized that by all possible combinations of 0 and 1's, one can generate all possible subsets of a given Set. A rigorous way to construct this table is to consider the 0's and 1's as binary digits and keep incrementing them each succeeding row. This way all combinations of 0's and 1's are systematically exhausted.

A point of note: I am incrementing the binary digits in reverse i.e. binary addition is done from left to right rather than the traditional right to left, a variation which is useful when dealing with infinite Sets.

The rules for incrementing are simple; $0+0=0$, $0+1=1$, $1+1=0$ with 1 carried over to the element to the immediate right. One key realization is that the first element is the empty Set corresponding to all 0's (in this case 0000) and the last element of our table is generated when all the elements of the Set are included giving all 1's (in this case 1111).

No. of columns of the table $=$ No. of elements of the Set $= 4$

No. of rows of the table $=$ No. of all possible subsets (i.e. no. of elements of the Power Set) $= 2^4 = 16$

To the author, this represents an elegant and rigorous method of generating all subsets of any given Set.

We now extend the same procedure to the Set of Naturals [5]. We will construct a similar table as that for the finite Set but now enumerate the entire Set of Naturals.

1	2	3	4	5	6	...	*subsets*
0	0	0	0	0	0	...	$\{\}$
1	0	0	0	0	0	...	$\{1\}$
0	1	0	0	0	0	...	$\{2\}$
1	1	0	0	0	0	...	$\{1,2\}$
0	0	1	0	0	0	...	$\{3\}$
1	0	1	0	0	0	...	$\{1,3\}$
0	1	1	0	0	0	...	$\{2,3\}$
1	1	1	0	0	0	...	$\{1,2,3\}$
0	0	0	1	0	0	...	$\{4\}$
...	...	...	...	...	...	...	...
1	1	1	1	1	1	...	$\{1,2,3,4,...\}$

No. of elements of the Set of N = $\aleph_0$

No. of rows of the table $=$ No. of all possible subsets of N (i.e. the Power Set of N) $= 2^{\aleph_0}$ $=$ the Set of Reals

Examining the table, we derive a Set X consisting of each row to get…

$X = \{$
$0_1 0_2 0_3 0_4 \ldots,$
$1_1 0_2 0_3 0_4 \ldots,$
$0_1 1_2 0_3 0_4 \ldots,$
$1_1 1_2 0_3 0_4 \ldots,$
$0_1 0_2 1_3 0_4 \ldots,$
$\ldots,$
$1_1 1_2 1_3 1_4 \ldots$
$\}$

(The subscripts are retained for clarity)

The Set X is constructed vide the following rules:

1. The first element is all zero's i.e. $0_1 0_2 0_3 0_4 \ldots$
2. Each succeeding element is the previous element incremented by binary addition. The binary addition is done from left to right rather than the traditional right to left. The rules being simple; $0+0=0$, $0+1=1$, $1+1=0$ with 1 carried over to the element to the immediate right

We can slightly modify Set X into Set Y by first dropping the subscripts and then by appending a decimal point (more appropriately, called a binary point) before each of the binary sequences to get all the Real numbers.

$Y = \{$
$0.0000\ldots,$
$0.1000\ldots,$
$0.0100\ldots,$
$0.1100\ldots,$
$0.0010\ldots,$
$\ldots,$
$0.1111\ldots$
$\}$

A few points to note:

- The numbers are in base 2 i.e. binary
- $0.0000\ldots = 0$

- $0.1111... = 0.\overline{1} = 1$
- All elements of Set Y are unique
- The Set Y contains possible combinations of 0's and 1's which can be put in a bijection with all the subsets of the Naturals, thus ensuring that all Reals in $[0,1]$ are covered

In addition, the Set's X and Y are ordered as for any two elements a and b it can be established that one and only one of the two conditions is fulfilled either $a \prec b$ i.e. a precedes b or $a \succ b$ that is a succeeds b. The reader should take note that the ordering defined by '$\prec$' is not the same as '$<$' as $0.1 \prec 0.01$ but $0.1 > 0.01$.

Further the Sets are also well-ordered as every one of its non-empty subsets contains a first element thus achieving our two-fold aim of generating a well-ordered Set of Reals using a mechanical procedure involving a Power Set table and further without using of any 'Choice' function, staying within ZF theory only.

Conclusion:

A well-ordering of the Reals is indeed possible using ZFC and associated theorems based on the AC, specifically the well-ordering theorem. However, no way has still been found of actually constructing a well-ordered Set of Reals. In this paper, the author establishes by the Power Set Axiom that there is indeed a way of generating a well-ordered Set of Reals using ZF theory only i.e. without use of the AC or the well-ordering theorem. Further a systematic and mechanical procedure, by construction of a Power Set table, is given for generating the same. If restrictions on the non-denumerablity of the Power Set columns are imposed, then at the very least an in-principle method of well-ordering of the Reals is established using ZF theory only i.e. without use of the AC or the well-ordering theorem.

A few possible objections answered:

The author anticipates a few queries or reservations about his work which he will try to pre-answer. Please find below the same, put in a question / answer format…

Q1. The columns of the Power Set table continue on indefinitely. Is this allowed?

A1. To the best of the author's knowledge, in both Cantorian and Axiomatic Set theory, this is indeed allowed. Indeed, Cantor envisaged infinity as an actual realized entity rather than a potential one [6], as was perceived earlier by Gauss and others. For e.g., Cantor, who originally discovered the ordinal numbers, used them to extend the finite counting numbers. A subset of the ordinals is given below:

$\{1, 2, 3, ...; \omega, \omega+1, \omega+2, ..., \omega 2, ...\}$

In the same vein the columns of the Power Set table continue on till the Set so formed has cardinality $\aleph_0$ and if we restrict ourselves to the smallest transfinite ordinal, till the Set so formed had ordinal number ω.

Q2. Even if we accept that the number of columns being denumerable is OK, the number of rows of the Power Set table are non denumerable. So how is the Power Set table actually ever completed in its entirety?

A2. Any non denumerable Set can never be enumerated; that much is obvious… Therefore, a complete enumeration of the Power Set table is not possible. But what has to be realized is that this is an inherent property of the Reals itself. It has no brook with any attempted well ordering. For e.g., if the query is just to write down (read enumerate) all Reals without any precondition of say well-ordering or any other requirement, it is still not possible to enumerate this Set because of its inherent non denumerablity. However, even if the author accepts the query as being a genuine problem with this method, still this paper represents a significant advancement in Set theory. The best way to illustrate this point is to make a comparison table, given below:

	Earlier work…	Author's work…
1.	Consider ZFC theory which includes the Power Set Axiom	Consider ZF theory which includes the Power Set Axiom
2.	ZFC theory also includes the AC (with AC implying the well-ordering theorem)	ZF theory does not include the AC and therefore we cannot use the well-ordering theorem
3.	Using the well-ordering theorem, a well-ordering of the Reals is possible. However, it remains an in-principle possibility as no way has still been found of actually constructing the Set	Using the Power Set Axiom, a well-ordering of the Reals can be similarly achieved by an in-principle construction of the above Power Set table. Although completion of the table is not possible because of the non-denumerablity of the Reals

In essence, at the very least, an in-principle procedure for well-ordering of the Reals is achieved under ZF theory i.e. without using the AC and / or the well-ordering theorem.

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