Rates of convergence of lognormal extremes under power normalization

JIANWEN HUANG[1] , SHOUQUAN CHEN[2]

1.School of Mathematics and Computational Science, Zunyi Normal College,

Zunyi Guizhou, 563002, China

E-mail: hjw1303987297@126.com

2.School of Mathematics and Statistics, Southwest University,

Chongqing, 400715, China

Abstract. Let $\{X_n,\ n \geq 1\}$ be an independent and identically distributed random sequence with common distribution F obeying the lognormal distribution. In this paper, we obtain the exact uniform convergence rate of the distribution of the maximum to its extreme value limit under power normalization.

Mathematics Subject Classification(2010). Primary 62E20, 60E05; Secondary 60F15, 60G15.

Keywords: P-max stable laws; Logarithm normal distribution; Maximum; Uniform convergence rate.

1 Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with common distribution function (df) F. Suppose that there exist normalizing constants $a_n > 0, b_n \in R$ and nondegenerate distribution function $G(x)$ such that

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x), \tag{1.1}$$

for all $x \in C(G)$, the set of all continuity points of G, where $M_n = \max_{i\leq n} X_i$ denote the largest of the first n. Then $G(x)$ must belong to one of the following three classes:

$$\Phi_\alpha(x) = \begin{cases} 0, & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\}, & \text{if } x \geq 0, \end{cases}$$

$$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$

$$\Lambda(x) = \exp\{-e^{-x}\},\ x \in R,$$

where α is positive parameter. We say that df F belongs to the max domain of attraction of G under linear normalization if (1.1) holds. We denote such a fact by $F \in D_l(G)$. Criteria for $F \in D_l(G)$ and the choice of normalization constants a_n and b_n can be found in de Haan[6], Galambos[7], Leadbetter et al.[10] and Resnick[20].

According to Mohan and Ravi[14], Pancheva[17], F is said to belong to the max domain of attraction of a df H under power normalization, standed for $F \in D_p(H)$ if there exist normalizing constants $\alpha_n > 0$ and $\beta_n > 0$, such that

$$\lim_{n\to\infty} P\left(\left|\frac{M_n}{\alpha_n}\right|^{\frac{1}{\beta_n}} \operatorname{sign}(M_n) \le x\right) = \lim_{n\to\infty} F^n(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) = H(x), \tag{1.2}$$

where $\operatorname{sign}(x) = -1, 0$ or 1 as $x < 0, x = 0, x > 0$. A df H is called power-max stable or p-max stable for short by Mohan and Ravi[14] if it satisfies the stability relation

$$H^n(\alpha_n |x|^{\beta_n} \operatorname{sign}(x)) = H(x), x \in R \textit{ and } n \in N,$$

for some constants $\alpha_n > 0$ and $\beta_n > 0$. Pancheva[17] showed that H is of the power type of one of the following six distributions:

$$\text{Type I: } H_{1,\alpha}(x) = \begin{cases} 0, & \text{if } x \le 1, \\ \exp\{-(\log x)^{-\alpha}\}, & \text{if } x > 1, \end{cases}$$

$$\text{Type II: } H_{2,\alpha}(x) = \begin{cases} 0, & \text{if } x \le 0, \\ \exp\{-(-\log x)^{\alpha}\}, & \text{if } 0 < x < 1, \\ 1, & \text{if } x \ge 1, \end{cases}$$

$$\text{Type III: } H_{3,\alpha}(x) = \begin{cases} 0, & \text{if } x \le -1, \\ \exp\{-(-\log(-x))^{-\alpha}\}, & \text{if } -1 < x < 0, \\ 1, & \text{if } x \ge 0, \end{cases}$$

$$\text{Type IV: } H_{4,\alpha}(x) = \begin{cases} \exp\{-(-\log(-x))^{\alpha}\}, & \text{if } x < -1, \\ 1, & \text{if } x \ge -1, \end{cases}$$

$$\text{Type V: } H_{5,\alpha}(x) = \Phi_1(x) = \begin{cases} 0, & \text{if } x \le 0, \\ \exp\{-x^{-1}\}, & \text{if } x > 0, \end{cases}$$

$$\text{Type VI: } H_{6,\alpha}(x) = \Psi_1(x) = \begin{cases} \exp\{x\}, & \text{if } x < 0, \\ 1, & \text{if } x \ge 0, \end{cases}$$

where α is a positive parameter. Necessary and sufficient conditions for F to satisfy (1.2) were given by Christoph and Fark[5], Mohan and Ravi[14], Mohan and Subramanya[15] and Subramanya[22].

It is popular nowadays in theoretical analysis and wide applications that the normal distribution is carried over to logarithmic normal one. Besides, the logarithm normal distribution (the lognormal distribution) is one of the most widely applied distributions in statistics, biology and some other disciplines. For applications of the field of Electronics, Astronomy and Physics, see Bergmann and Bill[2]. For applications of the field of Biological Sciences and Social Science, see Limpet et al.[12], Groňholm and Annila [9]. For applications of the field of Statistics, see Olsson [16], Burmaster and Hull[3], Parkin et al.[18], Bacry [1]. For the application of the field of Marine Ecology, see Gray [8]. For the application of the field of Environment, see Singh et al.[21]. The probability density function of the lognormal distribution is given by

$$F'(x) = \frac{x^{-1}}{\sqrt{2\pi}} \exp\left\{-\frac{(\log x)^2}{2}\right\}, \ x > 0.$$

Our interesting problem in extreme value theory is to estimate the rate of uniform convergence of $F^n(\cdot)$ to its extreme value distribution. For power normalization, Chen and Feng[4] proved the result that the uniform convergence rate of $F \sim$ STSD (STSD stands for the short-tailed symmetric distribution) to its exteme value limit is proportional to $1/\log n$. For linear normalization, Peng et al.[19] proved that $1/\log n$ is the most optimal convergence rate for the maximum of GED (GED stands for the general error distribution) random variables. Liu and Liu[13] proved a simialr result for the Maxwell distribution. Liao and Peng[11] derived the following results if df F is the lognormal distribution:

$$\frac{c_1}{(\log n)^{1/2}} < \sup_{x \in R} |F^n(a_n x + b_n) - \Lambda(x)| < \frac{c_2}{(\log n)^{1/2}},$$

for $n \geq 2$ and $0 < c_1 < c_2$, where norming constants a_n and b_n are given by:

$$2\pi(\log b_n)^2 \exp\left((\log b_n)^2\right) = n^2,$$

and

$$a_n = \frac{b_n}{\log b_n}.$$

The aim of this paper is to establish the more accurate uniform convergence rate of extreme value from the lognormal distribution under power normalization, which can be used to estimate the error committed by the replacement of the exact distribution of the extremes by that limiting form and data analysis.

This note is organized as follows: some auxiliary results are given in Section 2. In Section 3, we provide our main results. Proofs are deferred to Section 4.

2 Preliminaries

In order to derive the uniform convergence rate of extreme value from the lognormal distribution under power normalization, we cite some results from Liao and Peng[11], Mohan and Ravi[14].

In the sequel, let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with common distribution F which follows the lognormal distribution. As before let M_n represent the partial maximum of $\{X_k, 1 \leq k \leq n\}$. Liao and Peng[11] showed that:

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = \Lambda(x), \tag{2.1}$$

for all $x \in R$, where

$$a_n = \frac{\exp\left((2\log n)^{1/2}\right)}{(2\log n)^{1/2}}, \tag{2.2}$$

and

$$b_n = \left(\exp\left((2\log n)^{1/2}\right)\right)\left(1 - \frac{\log 4\pi + \log\log n}{2(2\log n)^{1/2}}\right). \tag{2.3}$$

From (2.1) we immediately obtain $F \in D_l(\Lambda)$. It follows from Liao and Peng[11] that

$$\frac{1 - F(x)}{F'(x)} \sim \frac{x}{\log x}, \tag{2.4}$$

as $x \to \infty$, where $F'(x)$ is of the density function of the lognormal distribution $F(x)$. It also follows from Liao and Peng[11] that

$$1 - F(x) = c(x)\exp\left(-\int_e^x \frac{g(t)}{\tilde{f}(t)}\,\mathrm{d}t\right),$$

for sufficiently large x, where $c(x) \to (2\pi e)^{-1/2}$ as $x \to \infty$, $g(x) = 1 + (\log x)^{-2}$ and

$$\tilde{f}(x) = \frac{x}{\log x}. \tag{2.5}$$

Noting that $\tilde{f}'(x) \to 0$, $g(x) \to 1$ as $x \to \infty$.

We will use the following properties of the lognormal distribution and Lemma 1 of Liao and Peng[11].

Lemma 2.1. *Let F denote the logarithm normal function. For $x > 1$, we have*

$$\begin{aligned} 1 - F(x) &= \frac{1}{\sqrt{2\pi}}(\log x)^{-1}\exp\left(-\frac{(\log x)^2}{2}\right) - \gamma(x) & (2.6)\\ &= \frac{1}{\sqrt{2\pi}}(\log x)^{-1}\exp\left(-\frac{(\log x)^2}{2}\right)\left(1 - (\log x)^{-2}\right) + \mathcal{S}(x), & (2.7)\end{aligned}$$

where

$$0 < \gamma(x) < \frac{1}{\sqrt{2\pi}}(\log x)^{-3}\exp\left(-\frac{(\log x)^2}{2}\right) \tag{2.8}$$

and

$$0 < \mathcal{S}(x) < \frac{3}{\sqrt{2\pi}}(\log x)^{-5}\exp\left(-\frac{(\log x)^2}{2}\right). \tag{2.9}$$

In order to obtain the main results, we need the following two lemmas.

Lemma 2.2. (Theorem 3.1(a)(ii), Mohan and Ravi[14])*Let F be a distribution function, if $F \in D_l(\Lambda)$ and $r(F) = \infty$, then*

$$F \in D_p(\Phi_1),$$

and the power normalizing constants

$$\alpha_n = b_n, \ \beta_n = \frac{a_n}{b_n},$$

where $r(F) = \sup\{x : F(x) < 1\}$.

Lemma 2.3. (Theorem 2.5, Mohan and Ravi[14])*Let F be a distribution function, if $F \in D_p(\Phi_1)$ if and only if*

(i)$r(F) > 0$, and

(ii) $\lim_{t\to r(F)} \frac{1-F(t\exp(yf(t)))}{1-F(t)} = \exp(-y)$, for some positive valued function f.

If (ii) holds for some f then $-\int_a^{r(F)}(1-F(x))/x\,\mathrm{d}x < \infty$ for $0 < a < r(F)$ and (ii) holds with the choice $f(t) = 1/(1-F(t))\int_t^{r(F)}(1-F(x))/x\,\mathrm{d}x$. The normalization constants may be chosen as $\alpha_n = F^{\leftarrow}(1-1/n)$ and $\beta_n = f(\alpha_n)$, where $F^{\leftarrow}(x) = \inf\{y : 1-F(y) \geq x\}$.

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of iid random variables with common distribution F which follows the lognormal distribution. Then $F \in D_p(\Phi_1)$ and the normalizing constants can be chosen as $\alpha_n^* = b_n$, $\beta_n^* = a_n/b_n$, where a_n and b_n are given by (2.2) and (2.3).*

Proof. Note that F follows the lognormal distribution, which implies $F \in D_p(\Phi_1)$ and $\alpha_n^* = b_n$, $\beta_n^* = a_n/b_n$, by (2.1) in Liao and Peng[11] and Lemma 2.2, where a_n and b_n are defined by (2.2) and (2.3). □

By Lemma 2.3 and (2.4), we may choose the norming constants α_n and β_n in such a way that α_n is the solution of the equation

$$\frac{1}{\sqrt{2\pi}}(\log\alpha_n)^{-1}\exp\left(-\frac{(\log\alpha_n)^2}{2}\right) = \frac{1}{n} \tag{2.10}$$

and

$$\beta_n = \frac{\tilde{f}(\alpha_n)}{\alpha_n} = \frac{1}{\log\alpha_n} \tag{2.11}$$

i.e.

$$(\log\alpha_n)\beta_n = 1,$$

where $\tilde{f}$ is given by (2.5). The solution of (2.10) may be expression as

$$\alpha_n = \left(\exp\left((2\log n)^{1/2}\right)\right)\left(1 - \frac{\log 4\pi + \log\log n}{2(2\log n)^{1/2}} + o(\frac{1}{(\log n)^{1/2}})\right) \tag{2.12}$$

and it is easy to see that

$$\beta_n \sim \frac{1}{(2\log n)^{1/2}}.$$

3 Main result

We provide two main results. Theorem 3.1 shows that the uniform convergence rate of $F(\alpha_n x^{\beta_n})$ to its extreme value limit is proportional to $1/\log n$. Theorem 3.2 shows that the pointwise rate of convergence of $|M_n/\alpha_n|^{1/\beta_n}\,\mathrm{sign}(M_n)$ to the extreme value distribution $\exp(-x^{-1})$ is of the order of $O(\exp(-x^{-1})x^{-1}\beta_n^2)$.

Theorem 3.1. *Let $\{X_n, n \geq 1\}$ denote a sequence of iid random variables with common distribution F which follows the lognormal distribution. Then there exist absolute constants $0 < \mathcal{C}_1 < \mathcal{C}_2$ such that*

$$\frac{\mathcal{C}_1}{\log n} < \sup_{x>0} |F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| < \frac{\mathcal{C}_2}{\log n}$$

for large $n > n_0$, where α_n and β_n defined by (2.10) and (2.11), respectively.

Theorem 3.2. *Let α_n and β_n given by (2.10) and (2.11). For $x > 0$, we have*

$$|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| \sim \exp\left\{-\frac{1}{x}\right\} \frac{1}{x}\beta_n^2$$

for large n.

4 Proofs

Firstly, Theorem of 3.2 is proved for it is relatively easy.

Proof of Theorem 3.2. By Lemma 2.1, we have

$$\begin{aligned}
1 - F(\alpha_n x^{\beta_n}) &= \frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1} \exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right) \\
&\quad \times \left(1 - (\log(\alpha_n x^{\beta_n}))^{-2}\right) + \mathcal{S}(\alpha_n x^{\beta_n}) \\
&=: T_1(x)T_2(x) + T_3(x)
\end{aligned}$$

for $x > 0$, where $T_1(x) = (\sqrt{2\pi})^{-1}(\log(\alpha_n x^{\beta_n}))^{-1} \exp\left(-(\log(\alpha_n x^{\beta_n}))^2/2\right)$, $T_2(x) = \left(1 - (\log(\alpha_n x^{\beta_n}))^{-2}\right)$ and $T_3(x) = \mathcal{S}(\alpha_n x^{\beta_n})$.

Firstly, we calculate the $T_1(x)$. By (2.10) and (2.11), we have

$$\begin{aligned}
T_1(x) &= \frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right)\\
&= \frac{1}{\sqrt{2\pi}}(\log\alpha_n)^{-1}\exp\left(-\frac{(\log\alpha_n)^2}{2}\right)(1+(\log\alpha_n)^{-1}\beta_n\log x)^{-1}\\
&\times\exp\left(-(\log\alpha_n)\beta_n\log x-\frac{\beta_n^2\log^2 x}{2}\right)\\
&= \frac{1}{nx}(1+\beta_n^2\log x)^{-1}\exp\left(-\frac{\beta_n^2\log^2 x}{2}\right)\\
&= \frac{1}{nx}(1-\beta_n^2\log x+O(\beta_n^4))\left(1-\frac{\beta_n^2\log^2 x}{2}+O(\beta_n^4)\right)\\
&= \frac{1}{nx}\left(1-\beta_n^2(1+\frac{1}{2}\log x)\log x+O(\beta_n^4)\right). \qquad (4.1)
\end{aligned}$$

Secondly, we estimate $T_2(x)$ and $T_3(x)$ for $x>0$. By (2.11), we derive

$$\begin{aligned}
T_2(x) &= 1-(\log(\alpha_n x^{\beta_n}))^{-2}\\
&= 1-(\log\alpha_n)^{-2}(1+(\log\alpha_n)^{-1}\beta_n\log x)^{-2}\\
&= 1-\beta_n^2(1+\beta_n^2\log x)^{-2}\\
&= 1-\beta_n^2(1-2\beta_n^2\log x+O(\beta_n^4))\\
&= 1-\beta_n^2+O(\beta_n^4), \qquad (4.2)
\end{aligned}$$

and by Lemma 2.1, we have

$$\begin{aligned}
T_3(x) &\le \frac{3}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-5}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right)\\
&= 3(\log(\alpha_n x^{\beta_n}))^{-4}T_1(x)\\
&= 3\beta_n^4(1+\beta_n^2\log x)^{-4}T_1(x)\\
&= O(\frac{1}{n}\beta_n^4). \qquad (4.3)
\end{aligned}$$

By (4.1), (4.2) and (4.3), we have

$$1-F^n(\alpha_n x^{\beta_n}) = \frac{1}{nx}\left(1-\beta_n^2\left(1+(1+\frac{1}{2}\log x)\log x\right)+O(\beta_n^4)\right).$$

Thus, we obtain

$$\begin{aligned}
&F^n(\alpha_n x^{\beta_n})-\Phi_1(x)\\
=&\left\{1-\frac{1}{nx}\left(1-\beta_n^2\left(1+(1+\frac{1}{2}\log x)\log x\right)+O(\beta_n^4)\right)\right\}^n-\exp(-\frac{1}{x})\\
=&\exp(-\frac{1}{x})\left\{\exp\left\{\frac{1}{x}\left(\beta_n^2\left(1+(1+\frac{1}{2}\log x)\log x\right)+O(\beta_n^4)\right)\right\}-1\right\}\\
=&\exp(-\frac{1}{x})\left\{\beta_n^2\frac{1}{x}\left(1+(1+\frac{1}{2}\log x)\log x\right)+O(\beta_n^4)\right\} \qquad (4.4)
\end{aligned}$$

for large n and $x > 0$. We immediately get the result of Theorem 3.2 by (4.4). $\square$

Proof of Theorem 3.1. (1)Firstly, we will estimate the lower bound for $x > 0$.

(i)Consider the case of $x > 1$. By Lemma 2.1, we obtain

$$\begin{aligned}
&F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)\\
=&\left\{1 - \frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right) + \gamma(\alpha_n x^{\beta_n})\right\}^n - \exp(-\frac{1}{x})\\
\geq&\left\{1 - \frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right)\right\}^n - \exp(-\frac{1}{x}).
\end{aligned}$$

By (4.1), we have

$$\begin{aligned}
&\left\{1 - \frac{1}{nx}\left\{1 - \beta_n^2(1 + \frac{1}{2}\log x)\log x + O(\beta_n^4)\right\}\right\}^n - \exp(-\frac{1}{x})\\
=&\exp(-\frac{1}{x})\left\{\exp\left\{\frac{1}{x}\beta_n^2(1 + \frac{1}{2}\log x)\log x + O(\beta_n^4)\right\} - 1\right\}\\
=&\exp(-\frac{1}{x})\frac{1}{x}\beta_n^2\left\{(1 + \frac{1}{2}\log x)\log x + O(\beta_n^2)\right\}\\
\geq&K\exp(-\frac{1}{x})\frac{1}{x}\beta_n^2(1 + \frac{1}{2}\log x)\log x
\end{aligned}$$

for large n, where K is a positive number and $0 < K < 1$. Hence, we have

$$\begin{aligned}
&\sup_{x>1}|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)|\\
\geq&K\sup_{x>1}\left|\exp(-\frac{1}{x})\frac{1}{x}\beta_n^2(1 + \frac{1}{2}\log x)\log x\right|\\
\geq&K\sup_{x>1}\left|e^{-1}\frac{1}{x}\beta_n^2(1 + \frac{1}{2}\log x)\log x\right|.
\end{aligned}$$

Let $h(x) = x^{-1}(1 + \frac{1}{2}\log x)\log x$, then $h'(x) = (1 - \frac{1}{2}\log^2 x)/x^2$. Put $h'(x) = 0$, we derive $\log x_0 = \sqrt{2}$.

Therefore, there exists $\tilde{c}_1$ such that

$$\begin{aligned}
&K\sup_{x>1}\left|e^{-1}\frac{1}{x}\beta_n^2(1 + \frac{1}{2}\log x)\log x\right|\\
\geq&Ke^{-1-\sqrt{2}}(1 + \sqrt{2})\beta_n^2\\
=&\frac{\tilde{c}_1}{\log n}.
\end{aligned}$$

(ii)Next, we will estimate the lower for the situation of $e^{-2} \leq x \leq 1$. Since

$$\begin{aligned}
&F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)\\
=&\exp(-\frac{1}{x})\left\{\beta_n^2\frac{1}{x}\left(1 + (1 + \frac{1}{2}\log x)\log x\right) + O(\beta_n^4)\right\}
\end{aligned}$$

and

$$1+(1+\frac{1}{2}\log x)\log x>0$$

for $e^{-2}\le x\le 1$. Thus, we have

$$\beta_n^2\frac{1}{x}\left(1+(1+\frac{1}{2}\log x)\log x\right)+O(\beta_n^4)>0$$

for large n. Hence, there exists $0<\epsilon<1$ such that

$$\begin{aligned}&\beta_n^2\frac{1}{x}\left(1+(1+\frac{1}{2}\log x)\log x\right)+O(\beta_n^4)\\ \ge&(1-\epsilon)\beta_n^2\frac{1}{x}\left(1+(1+\frac{1}{2}\log x)\log x\right).\end{aligned}$$

Therefore, there exists $\tilde{c}_2$ such that

$$\begin{aligned}&\sup_{e^{-2}\le x\le 1}|F^n(\alpha_n x^{\beta_n})-\Phi_1(x)|\\ \ge&\sup_{e^{-2}\le x\le 1}\left|(1-\epsilon)\exp(-\frac{1}{x})\beta_n^2\frac{1}{x}\left(1+(1+\frac{1}{2}\log x)\log x\right)\right|\\ =&\frac{\tilde{c}_2}{\log n}.\end{aligned}$$

(iii)Now we estimate the lower bound for $0<x<e^{-2}$. Note that

$$\begin{aligned}&\left\{1-\frac{1}{nx}\left\{1-\beta_n^2(1+\frac{1}{2}\log x)\log x+O(\beta_n^4)\right\}\right\}^n-\exp(-\frac{1}{x})\\ =&\exp(-\frac{1}{x})\frac{1}{x}\beta_n^2\left\{(1+\frac{1}{2}\log x)\log x+O(\beta_n^2)\right\}\\ \ge&\tilde{K}\exp(-\frac{1}{x})\frac{1}{x}\beta_n^2\left\{(1+\frac{1}{2}\log x)\log x\right\}>0,\end{aligned}$$

where $0<\tilde{K}<1$, there exists $\tilde{c}_3$, such that

$$\begin{aligned}&\sup_{0<x<e^{-2}}|F^n(\alpha_n x^{\beta_n})-\Phi_1(x)|\\ \ge&\sup_{0<x<e^{-2}}\tilde{K}\left|\exp(-\frac{1}{x})\beta_n^2\frac{1}{x}(1+\frac{1}{2}\log x)\log x\right|\\ =&\frac{\tilde{c}_3}{\log n}.\end{aligned}$$

(2) In order to obtain the upper bound for $x>0$, we need to prove

$$(a).\ \sup_{1\le x<\infty}|F^n(\alpha_n x^{\beta_n})-\Phi_1(x)|<d_1\beta_n^2, \tag{4.5}$$

$$(b).\ \sup_{c_n\le x<1}|F^n(\alpha_n x^{\beta_n})-\Phi_1(x)|<d_2\beta_n^2, \tag{4.6}$$

$$(c).\ \sup_{0<x<c_n}|F^n(\alpha_n x^{\beta_n})-\Phi_1(x)|<d_3\beta_n^2, \tag{4.7}$$

for $n > n_0$, where $d_i > 0,\ i = 1, 2, 3$ are absolute constants and

$$c_n = \frac{1}{2\log\log\alpha_n}$$

is positive for $n > n_0$. By (2.10) and (2.11), we have

$$0.4(2\log n)^{1/2} < \log\alpha_n < (2\log n)^{1/2}$$

and

$$\frac{1}{(2\log n)^{1/2}} < \beta_n < \frac{5}{2^{3/2}(\log n)^{1/2}}$$

for $n > n_0$.

(i)Firstly, consider the case of $x \geq c_n$. Set

$$R_n(x) = -[n\log F(\alpha_n x^{\beta_n}) + n\Psi_n(x)],\ B_n(x) = \exp(-R_n),\ A_n(x) = \exp(-n\Psi_n(x) + \frac{1}{x}),$$

where $\Psi_n(x) = 1 - F(\alpha_n x^{\beta_n})$ and $A_n(x) \to 1$, as $x \to \infty$. Since

$$\begin{aligned}
\Psi_n(x) \leq \Psi_n(c_n) &< \frac{1}{\sqrt{2\pi}}(\log(\alpha_n c_n^{\beta_n}))^{-1}\exp\left(-\frac{(\log(\alpha_n c_n^{\beta_n}))^2}{2}\right) \\
&= \frac{1}{n}(1+\beta_n^2\log c_n)^{-1}\exp\left(-\log c_n - \frac{\beta_n^2\log^2 c_n}{2}\right) \\
&< \frac{1}{n}(1+\beta_n^2\log c_n)^{-1}\exp\left(-\log c_n\right) \\
&= \frac{1}{n}(1+\beta_n^2\log c_n)^{-1}c_n^{-1} \\
&= \left(1 - \frac{\log(2\log\log\alpha_n)}{(\log\alpha_n)^2}\right)^{-1}\frac{2\log\log\alpha_n}{n} \\
&< \tilde{c}_4 < 1
\end{aligned}$$

for $n > n_0$.

So,

$$\inf_{x>c_n}(1 - \Psi_n(x)) > 1 - \tilde{c}_4 > 0.$$

Since

$$-x - \frac{x^2}{2(1-x)} < \log(1-x) < -x,$$

for $0 < x < 1$, we obtain

$$\begin{aligned} 0 < R_n(x) &\le \frac{n\Psi_n^2(x)}{2(1-\Psi_n(x))} < \frac{n\Psi_n^2(c_n)}{2(1-\Psi_n(x))} \\ &< \frac{n^{-1}(1+\beta_n^2\log c_n)^{-2}c_n^{-2}}{2(1-\Psi_n(x))} \\ &< \frac{n^{-1}(1+\beta_n^2\log c_n)^{-2}c_n^{-2}(\log\alpha_n)^2}{2(1-\tilde{c}_4)\beta_n^{-2}} \\ &= \frac{2}{\sqrt{2\pi}(1-\tilde{c}_4)}\left(1-\frac{\log(2\log\log\alpha_n)}{(\log\alpha_n)^2}\right)^{-2}\frac{(\log\log\alpha_n)^2\log\alpha_n}{\exp\left(\frac{(\log\alpha_n)^2}{2}\right)}\beta_n^2 \\ &< \tilde{c}_5\beta_n^2 \end{aligned}$$

for $n > n_0$.

Hence, we have

$$n^{-1}\beta_n^{-2}(1+\beta_n^2\log c_n)^{-2}c_n^{-2} < \tilde{c}_6$$

for $n > n_0$. Thus,

$$|B_n(x)-1| = |\exp(-R_n)-1| < R_n < \tilde{c}_5\beta_n^2, \tag{4.8}$$

for $n > n_0$.

By (4.8), we have

$$\begin{aligned} &|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| \\ &\le \Phi_1(x)B_n(x)|A_n(x)-1| + |B_n(x)-1| \\ &< \Phi_1(x)|A_n(x)-1| + \tilde{c}_5\beta_n^2 \end{aligned} \tag{4.9}$$

for $x \ge c_n$.

We now prove (4.5). By (2.10), (2.11) and the definition of $A_n(x)$, we have

$$\begin{aligned} A_n'(x) &= \left(\exp(-n\Psi_n(x)+\frac{1}{x})\right)(-n\Psi_n(x)+\frac{1}{x})' \\ &= A_n(x)\left(nF'(\alpha_n x^{\beta_n}) - \frac{1}{x^2}\right) \\ &= A_n(x)\left(\sqrt{2\pi}(\log\alpha_n)\left(\exp(\frac{(\log\alpha_n)^2}{2})\right)\frac{1}{\sqrt{2\pi}}\frac{1}{\alpha_n x^{\beta_n}}\left(\exp(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2})\right) - \frac{1}{x^2}\right) \\ &= A_n(x)\left(\frac{1}{x^{1+\beta_n}}\frac{\log\alpha_n}{\alpha_n}\exp\left(-\frac{1}{2}\beta_n^2\log^2 x\right) - \frac{1}{x^2}\right) \\ &< 0 \end{aligned}$$

for $x > 1$. Since

$$0 < n\gamma(\alpha_n) < (\log\alpha_n)^{-2} = \beta_n^2,$$

$$\exp(n\gamma(\alpha_n)) < \exp(\beta_n^2) < \exp\left(\frac{25}{8\log n}\right) < \exp\left(\frac{25}{8\log n_0}\right), \text{ for } n > n_0,$$

$$e^x - 1 \le xe^x, \text{ for } 0 \le x \le 1,$$

and (2.6), (2.10), we have

$$\begin{aligned}\sup_{x\ge1}|A_n(x)-1| &= |A_n(1)-1| \\ &= |\exp(n\gamma(\alpha_n))-1| \\ &\le n\gamma(\alpha_n)\exp(n\gamma(\alpha_n)) \\ &\le \tilde{c}_7\beta_n^2 \end{aligned} \tag{4.10}$$

for $n > n_0$.

Combine (4.9) with (4.10), we have

$$\begin{aligned}&\sup_{x\ge1}|F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| \\ &< (\tilde{c}_5+\tilde{c}_7)\beta_n^2.\end{aligned}$$

(ii)Secondly, consider the situation of $c_n \le x < 1$. By Lemma 2.1, we obtain

$$\begin{aligned}-n\Psi_n(x)+\frac{1}{x} &= -n(1-F(\alpha_n x^{\beta_n}))+\frac{1}{x} \\ &= -n\left(\frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right)-\gamma(\alpha_n x^{\beta_n})\right)+\frac{1}{x} \\ &= -n\Bigg(\frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-1}\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right) \\ &\quad -\frac{1}{\sqrt{2\pi}}(\log(\alpha_n x^{\beta_n}))^{-3}q_n(\alpha_n x^{\beta_n})\exp\left(-\frac{(\log(\alpha_n x^{\beta_n}))^2}{2}\right)\Bigg)+\frac{1}{x} \\ &= \frac{1}{x}(1+\beta_n^2\log x)^{-1}\big(-\big(1-(\log\alpha_n)^{-2}q_n(\alpha_n x^{\beta_n})(1+\beta_n^2\log x)^{-2}\big) \\ &\quad \times\exp\left(-\frac{1}{2}\beta_n^2\log^2 x\right)+1+\beta_n^2\log x\big) \\ &= \frac{1}{x}(1+\beta_n^2\log x)^{-1}Q_n(x),\end{aligned}$$

where $0 < q_n(x) < 1$ and

$$\begin{aligned}Q_n(x) &= -\big(1-(\log\alpha_n)^{-2}q_n(\alpha_n x^{\beta_n})(1+\beta_n^2\log x)^{-2}\big)\exp\left(-\frac{1}{2}\beta_n^2\log^2 x\right)+1+\beta_n^2\log x \\ &= -\big(1-\beta_n^2 q_n(\alpha_n x^{\beta_n})(1+\beta_n^2\log x)^{-2}\big)\exp\left(-\frac{1}{2}\beta_n^2\log^2 x\right)+1+\beta_n^2\log x.\end{aligned}$$

Since

$$e^{-x} > 1-x, \ x > 0,$$

we have

$$
\begin{aligned}
Q_n(x) &< -(1-\beta_n^2 q_n(\alpha_n x^{\beta_n})(1+\beta_n^2\log x)^{-2})(1-\frac{1}{2}\beta_n^2\log^2 x)+1+\beta_n^2\log x \\
&= -1+\beta_n^2 q_n(\alpha_n x^{\beta_n})(1+\beta_n^2\log x)^{-2}+\frac{1}{2}\beta_n^2\log^2 x-\frac{1}{2}\beta_n^4 q_n(\alpha_n x^{\beta_n})(1+\beta_n^2\log x)^{-2}\log^2 x \\
&+1+\beta_n^2\log x \\
&< \beta_n^2(1+\beta_n^2\log x)^{-2}+\frac{1}{2}\beta_n^2\log^2 x \\
&= \beta_n^2((1+\beta_n^2\log x)^{-2}+\frac{1}{2}\log^2 x).
\end{aligned}
$$

But

$$
\begin{aligned}
Q_n(x) &= -(1-\beta_n^2 q_n(\alpha_n x^{\beta_n})(1+\beta_n^2\log x)^{-2})\exp\left(-\frac{1}{2}\beta_n^2\log^2 x\right)+1+\beta_n^2\log x \\
&> -(1-\beta_n^2 q_n(\alpha_n x^{\beta_n})(1+\beta_n^2\log x)^{-2})+1+\beta_n^2\log x \\
&= \beta_n^2 q_n(\alpha_n x^{\beta_n})(1+\beta_n^2\log x)^{-2}+\beta_n^2\log x \\
&> \beta_n^2\log x.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
|Q_n(x)| &< \beta_n^2((1+\beta_n^2\log x)^{-2}+\frac{1}{2}\log^2 x+|\log x|) \\
&< \beta_n^2((1+\beta_n^2\log c_n)^{-2}+\frac{1}{2}\log^2 x+|\log x|) \\
&= \beta_n^2\left(\left(1-\frac{\log(2\log\log\alpha_n)}{\log^2\alpha_n}\right)^{-2}+\frac{1}{2}\log^2 x+|\log x|\right) \\
&< \beta_n^2(\tilde{c}_8+\frac{1}{2}\log^2 x+|\log x|)
\end{aligned}
$$

for $n \geq n_0$, where $c_n \leq x < 1$. Therefore,

$$
\begin{aligned}
|-n\Psi_n(x)+\frac{1}{x}| &< \beta_n^2(\tilde{c}_8+\frac{1}{2}\log^2 x+|\log x|)x^{-1}(1+\beta_n^2\log x)^{-1} \\
&< \beta_n^2(\tilde{c}_8+\frac{1}{2}\log^2 c_n+|\log c_n|)c_n^{-1}(1+\beta_n^2\log c_n)^{-1} \\
&< \tilde{c}_9
\end{aligned}
$$

for $n \geq n_0$. Thus, there exists $0 < \theta < 1$ such that

$$
\begin{aligned}
\Phi_1(x)|A_n(x)-1| &= \Phi_1(x)|\exp(-n\Psi_n(x)+\frac{1}{x})-1| \\
&< \Phi_1(x)\exp(\theta(-n\Psi_n(x)+\frac{1}{x}))|-n\Psi_n(x)+\frac{1}{x}| \\
&< \exp(\tilde{c}_9)\beta_n^2\sup_{c_n\leq x<1}|(\tilde{c}_8+\frac{1}{2}\log^2 x+|\log x|)x^{-1}|(1+\beta_n^2\log c_n)^{-1} \\
&< \tilde{c}_{10}\beta_n^2. \qquad (4.11)
\end{aligned}
$$

By (4.9) and (4.11), the proof of (4.6) is completed.

(iii) Thirdly ,consider the circumstance of $0 < x < c_n$. Note

$$\Phi_1(x) < \Phi_1(c_n) = \beta_n^2,$$

we have

$$\begin{aligned}&\sup_{0<x<c_n} |F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)|\\ =&F^n(\alpha_n c_n^{\beta_n}) + \Phi_1(c_n)\\ =&F^n(\alpha_n c_n^{\beta_n}) - \Phi_1(c_n) + 2\Phi_1(c_n)\\ =&\sup_{c_n<x<1} |F^n(\alpha_n x^{\beta_n}) - \Phi_1(x)| + 2\Phi_1(c_n)\\ <&(\tilde{c}_5 + \tilde{c}_{10})\beta_n^2 + \beta_n^2\\ <&\tilde{c}_{11}\beta_n^2.\end{aligned}$$

The proof of Theorem 3.1 is finished. □

References

[1] Bacry E, Kozhemyak A and Muzy J F, Log-Normal continuous cascades: aggregation properties and estimation, Quant, arXiv:0804.0185v1 [q-fin.ST] 1 Apr 2008

[2] Bergmann R B and Bill A, On the Origin of Logarithmic-Normal Distributions: An analytical derivation, and its application to nucleation and growth processes, *J. Cryst. Growth* **310** (2008) 3135-3138

[3] Burmaster D E and Hull D A, Using Lognormal Distributions and Lognormal Probability Plots in Probabilistic Risk Assessments, *Hum. Ecol. Ris. Assess.* **3** (1997) 235-255

[4] Chen S and Feng B, Rates of convergence of extreme for STSD under power normalization. *J. Korean Stat. Soc.* http://dx.doi.org/10.1016/j.jkss.2014.02.001 (2014)

[5] Christoph G and Falk M, A note on domains of attraction of p-max stable laws, *Statist. Prob. Lett.* **28** (1996) 279-284

[6] de Haan L, On regular variation and its applications to the weak convergence of sample, *M. C. Tracts* (32) (1970) 1-124

[7] Galambos J, The Asypmtotic Theory of Extreme Order Statistics, (1987) (New York: Wiley) pp. 51-87

[8] Gray J S, Use and misuse of the log-normal plotting method for detection of effects of pollution-a reply to Shaw et al, *Mar. Ecol.-Prog. Ser.* (1983) 203-204

[9] Groňholm T and Annila A, Natural distribution, *Math. Biosci.* **210** (2007) 659 – 667

[10] Leadbetter M R, Lindgren G and Rootzén H, Extremes and Related Properties of Random Sequences and Processes, (1983) (New York: Springer-Verlag) pp. 5-19

[11] Liao X and Peng Z, Convergence rates of limit distribution of maxima of lognormal samples, *J. Math. Anal. Appl.* **395** (2012) 643-653

[12] Limpert E, Stahel W A and Abbt M, Log-normal Distributions across the Sciences: Keys and Clues, *BioScience* **51** (2001) 341-352

[13] Liu C and Liu B, Convergence rate of extremes from Maxwell sample, *J. Inequal. Appl.* Available on-line at

http://www.journalofinequalitiesandapplications.com/content/2013/1/477

[14] Mohan N R and Ravi S, Max domains of attraction of univariate and multivariate p-max stable laws, *Theory Prob. Appl.* **37** (1993) 632-643

[15] Mohan N R and Subramanya U R, Proceedings of the Symposium on Distribution Theory, (1991) (India: Kochi Kerala)

[16] Ollson U, Confident intervals for the mean of a lognormal distribution, *J. Statist. Edu.* **13** (2005) 1-8

[17] Pancheva E, Limit theorems for extreme order statistics under nonliear normalization, *Lecture Notes in Mathematics* **1155** (1985) 284-309

[18] Parkin T B, Chester S T and Robinson J A, Calculating Confidence Intervals for the Mean of a Lognormally Distributed Variable, *Soil Sci. Soc. Am. J.* **54** (1990) 321-326

[19] Peng Z, Nadarajah S and Lin F, Convergence rate of extremes for the general error distribution, *J. Appl. Probab.* **47** (2010) 668-679

[20] Resnick S I, Extreme value, Regular Variation and Point Processes, (New York: Springer-Verlag) pp. 38-67

[21] Singh A K, Singh A and Engelhardt M, The Lognormal Distribution in Environmental Applications, Technology Support Center Issue, EPA/600/S-97/006 (1997)

[22] Subramanya U R, On max domains of attraction of univariate p-max stable laws, *Statist. Prob. Lett.* **19** (1994) 271-279